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# AN ARITHMETIC POISSON FORMULA

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**Let $a(n)$ denote any arithmetic function. Since most arithmetic functions that are encountered are defined on only the nonnegative integers, we define $a(-n) = a(n)$ for each positive integer $n$. Our objective is to develop a Poisson type formula for $\sum_{n=-\infty}^{\infty} a(n)f(n)$, where $f$ belongs to a suitable class of functions. We conclude the paper with several applications of our arithmetic Poisson formula.**

In another paper [2], the author has used a new technique in contour integration to examine infinite series of the type $\sum_{n=0}^{\infty} a(n)f(n)$, where now $f$ is a suitable rational function. The author's method has been generalized and improved by Krishnaiah and Sita Rama Chandra Rao [5].

Let $b(n)$ be an arithmetic function, and let $S$ denote any subset of the natural numbers. Define, for each positive integer $m$,

$$(1)\qquad a(m, n; S) = \sum_{\substack{d \leqq m \\ d|n \\ d \in S}} b(d) \quad \text{and} \quad a(n; S) = \sum_{\substack{d|n \\ d \in S}} b(d) .$$

Clearly, if $m \geqq n$, $a(m, n; S) = a(n; S)$. If $S$ is the set of all positive integers, we write $a(n; S) = a(n)$. Alternatively, given an arithmetic function $a(n)$, we could define $b(n)$ by

$$b(n) = \sum_{d|n} \mu(d)a(n/d) ,$$

where $\mu$ denotes the Möbius function.

There exist several formulations of Poisson's summation formula

$$(2)\qquad \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{2\pi inx}dx$$

in the literature, where here and in the sequel all doubley infinite sums are interpreted symmetrically, i.e., as $\lim_{N\to\infty} \sum_{n=-N}^{N}$. We shall choose the setting (with slightly stronger hypotheses) from Bellman's book [1, p. 8]. Thus, let $f(x)$ be continuous on $(-\infty, \infty)$, suppose that $f \in L^1(-\infty, \infty)$, and assume that the left and right sides of (2) converge absolutely. Then (2) holds.

THEOREM. *Suppose that $f$ satisfies the conditions of Poisson's formula* (2) *as specified above. Let $a(n)$ and $b(n)$ denote arithmetic functions as related above. Assume also that $\sum_{n=-\infty}^{\infty} a(n)f(n)$ is absolutely convergent. Then*

$$(3) \qquad \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a(n; S)f(n) = \sum_{\substack{d=1 \\ d\in S}}^{\infty} \frac{b(d)}{d}\left\{\sum_{r=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi i r x/d}dx - f(0)d\right\} .$$

*Proof.* For each natural number $d$ and real number $x$, define

$$\mathscr{S}(d; x) = \sum_{j=0}^{d-1} e^{2\pi i j x/d} .$$

Thus, if $n$ is an integer,

$$(4) \qquad \mathscr{S}(d; n) = \begin{cases} d , & \text{if } d\,|\,n , \\ 0 , & \text{otherwise} . \end{cases}$$

Define, for each natural number $m$,

$$B_m(x; S) = \sum_{\substack{d=1 \\ d\in S}}^{m} \frac{b(d)}{d}\mathscr{S}(d; x) .$$

Hence, by (1) and (4), if $n \neq 0$,

$$(5) \qquad B_m(n; S) = a(m, n; S) ,$$

while if $n = 0$,

$$(6) \qquad B_m(0; S) = \sum_{\substack{d=1 \\ d\in S}}^{m} b(d) .$$

Now put $F(x) = B_m(x; S)f(x)$. From the hypotheses of our theorem, it is easily seen that the real and imaginary parts of $F(x)$ satisfy the conditions of the ordinary Poisson summation formula (2). Hence, using (5) and (6), we find that

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a(m, n; S)f(n) + \sum_{\substack{d=1 \\ d\in S}}^{m} b(d)f(0) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\sum_{\substack{d=1 \\ d\in S}}^{m}\frac{b(d)}{d}\sum_{j=0}^{d-1} e^{2\pi i(n+j/d)x}dx .$$

Setting $r = nd + j$, we find that

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a(m, n; S)f(n) = \sum_{\substack{d=1 \\ d\in S}}^{m} \frac{b(d)}{d}\left\{\sum_{r=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi i r x/d}dx - f(0)d\right\} .$$

Letting $m$ tend to $\infty$ on both sides above, we deduce (3). Taking the limit under the summation sign on the left side is justified by a lemma of Krishnaiah and Sita Rama Chandra Rao [5] that requires the absolute convergence of $\sum_{n=-\infty}^{\infty} a(n)f(n)$.

In each of the examples below we tacitly assume that $a(n)$ and $b(n)$ satisfy the hypotheses of our theorem.

EXAMPLE 1. Let $f(x) = 1/(x^2 + a^2)$, where $a^2 \neq n^2$ if $n$ is an integer. Now if $r$ is any integer [4, p. 406],

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i r x/d}}{x^2 + a^2} dx = \frac{\pi}{2} e^{-2\pi r a/d} .$$

After an elementary computation, the theorem yields

$$\sum_{n=1}^{\infty} \frac{a(n; S)}{n^2 + a^2} = \frac{1}{2a^2} \sum_{\substack{d=1 \\ d \in S}}^{\infty} b(d) \left\{ \frac{\pi a}{d} \coth\left(\frac{\pi a}{d}\right) - 1 \right\} .$$

For some specific examples of the above formula, see [2] and [5].

A formula for $\sum_{n=1}^{\infty} (-1)^n a(n; S)/(n^2 + a^2)$ may be obtained by setting $f(x) = \cos(\pi x)/(x^2 + a^2)$ in the theorem. See [5] for such a formula.

EXAMPLE 2. Let $f(x) = x^2/(x^4 + a^4)$, where $a^4 \neq -n^4$ if $n$ is an integer. Then for any integer $r$ [4, pp. 409, 42],

$$\begin{aligned} &\sum_{r=1}^{\infty} \int_0^{\infty} \frac{x^2 \cos(2\pi r x/d)}{x^4 + a^4} dx \\ &= \frac{\pi\sqrt{2}}{4a} \sum_{r=1}^{\infty} e^{-\sqrt{2}\pi r a/d} \{\cos(\sqrt{2}\pi r a/d) - \sin(\sqrt{2}\pi r a/d)\} \\ &= \frac{\pi\sqrt{2}}{8a} \left\{ \frac{\sinh(\sqrt{2}\pi a/d)}{\cosh(\sqrt{2}\pi a/d) - \cos(\sqrt{2}\pi a/d)} \right. \\ &\quad \left. - 1 - \frac{\sin(\sqrt{2}\pi a/d)}{\cosh(\sqrt{2}\pi a/d) - \cos(\sqrt{2}\pi a/d)} \right\} . \end{aligned}$$

Hence, using also the integral evaluation for $r = 0$, we deduce that

$$\sum_{n=1}^{\infty} \frac{a(n; S) n^2}{n^4 + a^4} = \frac{\pi\sqrt{2}}{4a} \sum_{\substack{d=1 \\ d \in S}}^{\infty} \frac{b(d)}{d} \left\{ \frac{\sinh(\sqrt{2}\pi a/d) - \sin(\sqrt{2}\pi a/d)}{\cosh(\sqrt{2}\pi a/d) - \cos(\sqrt{2}\pi a/d)} \right\} .$$

In particular, if we let $a(n) = 1$, $n = 1$, and $a(n) = 0$, $n > 1$, then $b(n) = \mu(n)$, and so

$$\frac{1}{1 + a^4} = \frac{\pi\sqrt{2}}{4a} \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \left\{ \frac{\sinh(\sqrt{2}\pi a/d) - \sin(\sqrt{2}\pi a/d)}{\cosh(\sqrt{2}\pi a/d) - \cos(\sqrt{2}\pi a/d)} \right\} .$$

EXAMPLE 3. Let $f(x) = \operatorname{sech}(\pi x)$. Then for each integer $r$ [4, p. 503],

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi r x/d)}{\cosh(\pi x)} dx = \operatorname{sech}(\pi r/d) .$$

Hence,

$$\sum_{n=1}^{\infty} a(n; S) \operatorname{sech}(\pi n) = \frac{1}{2} \sum_{\substack{d=1 \\ d \in S}}^{\infty} \frac{b(d)}{d} \left\{ 2 \sum_{r=1}^{\infty} \operatorname{sech}(\pi r/d) + 1 - d \right\} .$$

As an example, let $b(n)$ be Ramanujan's trigonometric sum $c_n(m)$, the sum of the $m$th powers of the primitive $n$th roots of unity. Then $a(n) = n$ or 0 according as $n | m$ or $n \nmid m$. Then the above formula yields

$$\sum_{n|m} n \operatorname{sech}(\pi n) = \frac{1}{2} \sum_{d=1}^{\infty} \frac{c_d(m)}{d} \left\{ 2 \sum_{r=1}^{\infty} \operatorname{sech}(\pi r/d) - d \right\} ,$$

since $\sum_{d=1}^{\infty} c_d(m)/d = 0$.

EXAMPLE 4. Let $f(x) = x \operatorname{csch}(\pi x)$. Then if $r$ is any integer [4, p. 511],

$$\int_0^{\infty} \frac{x \cos(2\pi r x/d)}{\sinh(\pi x)} dx = \frac{1}{4} \operatorname{sech}^2(\pi r/d) .$$

Hence, our theorem gives

$$\sum_{n=1}^{\infty} a(n; S) n \operatorname{csch}(\pi n) = \sum_{\substack{d=1 \\ d \in S}}^{\infty} \frac{b(d)}{d} \left\{ \frac{1}{2} \sum_{r=1}^{\infty} \operatorname{sech}^2(\pi r/d) + \frac{1}{4} - \frac{d}{2\pi} \right\} .$$

In particular, if we let $a(n) = \sigma_{-1}(n) = \sum_{d|n} 1/d$, the above formula reduces to

$$\sum_{n=1}^{\infty} \sigma_1(n) \operatorname{csch}(\pi n) = \frac{\pi^2}{24} + \frac{1}{2} \sum_{d=1}^{\infty} \frac{1}{d^2} \left\{ \sum_{r=1}^{\infty} \operatorname{sech}^2(\pi r/d) - \frac{d}{\pi} \right\} .$$

EXAMPLE 5. Let $f(t) = (x/|t|)^{1/2} J_1(2\pi\sqrt{|t|x})$, $x > 0$, where $J_k$ denotes the ordinary Bessel function of order $k$. From the definition of $J_1$ it is easily seen that $f(0) = \pi x$. Since $J_0'(u) = -J_1(u)$, we find that

$$\int_0^{\infty} (x/t)^{1/2} J_1(2\pi\sqrt{tx}) dt = \frac{1}{\pi} \int_0^{\infty} J_1(u) du = \frac{1}{\pi} .$$

Next, for each nonzero integer $r$, [4, p. 742],

$$\int_0^{\infty} (x/t)^{1/2} J_1(2\pi\sqrt{tx}) \cos(2\pi r t/d) dt$$
$$= \frac{1}{\pi} \int_0^{\infty} J_1(u) \cos\left(\frac{r u^2}{2\pi x d}\right) du = \frac{2}{\pi} \sin^2\left(\frac{\pi x d}{4r}\right) .$$

Put $a(n) = r(n)$, the number of representations of $n$ as the sum of two squares. Then $r(n) = 4 \sum_{d|n} \chi(d)$, where $\chi$ denotes the primitive character of modulus 4. Thus, the theorem yields

$$\sum_{n=1}^{\infty} r(n)(x/n)^{1/2}J_1(2\pi\sqrt{nx})$$
$$= 4\sum_{d=1}^{\infty}\frac{\chi(d)}{d}\left\{\frac{4}{\pi}\sum_{r=1}^{\infty}\sin^2\left(\frac{\pi xd}{4r}\right)+\frac{1}{\pi}-\frac{\pi xd}{2}\right\} .$$

Now Voronoï [6] has shown that

$$\sum_{n\leq x}{}' r(n) = \pi x - 1 + \sum_{n=1}^{\infty} r(n)(x/n)^{1/2}J_1(2\pi\sqrt{nx}) ,$$

where the dash $'$ on the summation sign on the left side indicates that if $x$ is an integer, only $r(x)/2$ is counted. Combining the last two equalities and using the fact that

$$\sum_{a=1}^{\infty}\chi(d)/d = \pi/4 ,$$

we deduce that

$$\sum_{n\leq x}{}' r(n) = \pi x + 4\sum_{d=1}^{\infty}\frac{\chi(d)}{d}\left\{\frac{4}{\pi}\sum_{r=1}^{\infty}\sin^2\left(\frac{\pi xd}{4r}\right)-\frac{\pi xd}{2}\right\} .$$

Unfortunately, this does not appear to be of any use in the "circle problem." For related material, see [3, §8] and the references contained therein.

For additional examples, see [2] and, especially, [5].

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