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A DICHOTOMY FOR A CLASS OF POSITIVE DEFINITE FUNCTIONS

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We study under which conditions certain positive definite functions on discrete free groups, are weakly associated with the left regular representation.

1. Introduction. In this paper we study some properties of a class of positive definite functions on free groups introduced in [3]. These functions resemble in many respects Riesz products on abelian groups and have been used to investigate properties of the Fourier-Stieltjes algebra of a free group. Let G be a discrete group, and F a free subset of G, namely a set with no relations among its elements. For every $x \in [F]$, the group generated by F, the length of x, with respect to F, is the number of factors in $F \cup F^{-1}$ which are needed to write x as a reduced word in the free generators of F. We denote by $|x|$ the length of x. We recall the following definition given in [3].

DEFINITION. A Haagerup function on G is a function u such that:

(i) u is zero on $G \backslash [F]$ and $|u(x)| \leqq 1$ for every $x \in [F]$,
(ii) $u(1) = 1$,
(iii) $\overline{u(x)} = u(x^{-1})$,
(iv) $u(xy) = u(x)u(y)$ if $|xy| = |x| + |y|$.

The following result, which is similar to Zygmund's dichotomy theorem [11] on Riesz products, was proved in [3]:

THEOREM. *If*

$$\sum_{x \in F} |u(x)|^2 = +\infty$$

then u belongs to the orthogonal complement, in the Fourier-Stieltjes algebra, of $B_\lambda(G)$, the space of all coefficients of the representations weakly contained in the left regular representation of G.

If

$$\sum_{x \in F} |u(x)|^2 < 1/2$$

then $u \in A(G)$, the space of coefficients of the regular representation.

In this paper we fill the gap between the above conditions, proving that, if $\sum_{x \in F} |u(x)|^2 < +\infty$, then u is not in the orthogonal complement of $A(G)$ in the Fourier-Stieltjes algebra of G.

Moreover a direct computation shows that, unlike classical case, no necessary and sufficient condition can be given in terms of the L^2-norm of $u|_F$ in order that the function u belongs to $B_\lambda(G)$. Therefore the above result is, in some sense, the best possible. Finally we prove that, as for Riesz products, the spectrum of some Haagerup functions is strictly larger than the range.

2. Let $B(G)$ be the Fourier-Stieltjes algebra of G, consisting of all linear combinations of positive definite functions. Following [6] we recall that $B(G)$ is a Banach algebra under the norm which makes it the dual space of the group C^*-algebra $C^*(G)$ whose universal W^*-enveloping algebra is denoted by $W^*(G)$.

The Fourier algebra $A(G)$, and $B_\lambda(G)$, are closed ideals of $B(G)$. The space $B_\lambda(G)$ may be identified with the dual space of $C^*_\lambda(G)$, the completion of $L^1(G)$ with respect to the convolution norm. It follows from [5, Cor. 3, p. 42] and [9] that $B_\lambda(G)$ is complemented in $B(G)$, i.e., $B(G) = B_\lambda(G) \oplus B_\lambda^\perp(G)$ where $B_\lambda^\perp(G)$ is a closed subspace of $B(G)$, invariant under translation by elements of G. We shall say that a positive definite function u is orthogonal to $B_\lambda(G)$, if $u \in B_\lambda^\perp(G)$. Similarly $A(G)$ is complemented in $B(G)$ and $A^\perp(G)$ will denote its orthogonal complement (see also [10], p. 22 and Prop. 1, p. 33). Finally we recall that every positive definite function ϕ defines [4, p. 256] a Hilbert seminorm on the space of finitely supported functions:

$$(1) \qquad \|f\|_\phi^2 = \langle f^* * f, \phi\rangle = \sum_{x,z\in G} \phi(x^{-1}z)\overline{f(x)}f(z) .$$

A more convenient form for this seminorm, if ϕ is a Haagerup function, is provided by the following lemma:

LEMMA 1. *Let F be a free subset of G and u a Haagerup function such that $u(x) \neq 0$ for $x \in F$.*

Then for every finitely supported function f, with $\operatorname{supp} f \subseteq [F]$:

$$(2) \qquad \|f\|_u^2 = |\langle u, f\rangle|^2 + \sum_{0<|z|} (|u(z)|^{-2} - |u(z^\sim)|^{-2})|\langle u, f\chi_{A(z)}\rangle|^2$$

where

$$|z^\sim| = |z| - 1 \quad and \quad |z^{\sim -1}z| = 1$$
$$A(z) = \{x \in [F]: |z^{-1}x| = |x| - |z|\}$$

and $\chi_{A(z)}$ is the characteristic function of $A(z)$.

Proof. As the right hand side of (2) can be written in the form:

$$\sum_{x,y\in[F]} v(x,\,y)\overline{f(x)}f(y)$$

we only need to prove that $v(x,\,y) = u(x^{-1}y)$.

Let $|x^{-1}y| = |x| + |y|$, then x and y can not belong to the same $A(z)$ for every $|z| \geqq 1$. Since the term $\bar{f}(x)f(y)$ appears in $|\langle u,\, f\chi_{A(z)}\rangle|^2$ if and only if $x,\, y\in A(z)$, then it follows that $v(x,\,y) = u(x^{-1})u(y) = u(x^{-1}y)$. More generally, suppose x and y satisfy the following condition: $x = z_0x_0$, $y = z_0y_0$ with $|z_0| \geqq 1$ and $|x| = |z_0| + |x_0|$, $|y| = |z_0| + |y_0|$, $|x^{-1}y| = |x_0| + |y_0|$. Let $B_{z_0} = \{z\in|F|: 1 \leqq |z| \leqq |z_0|,\, |z^{-1}z_0| = |z_0| - |z|\}$; then, by the same argument as above, we obtain:

$$\begin{aligned} v(x,\,y) &= u(x^{-1})u(y) + \sum_{z\in B_{z_0}} (|u(z)|^{-2} - |u(z^{\sim})|^{-2})u(x^{-1})u(y) \\ &= u(x^{-1})u(y)|u(z_0)|^{-2} = u(x^{-1}y)\ . \end{aligned}$$

REMARKS. 1. The above formula gives a new, direct proof that Haagerup functions are positive definite.

2. If $u(x) = 0$ for some $x\in F$, a similar but more complicated formula holds.

LEMMA 2. *Let F be a free subset of G, $\delta_1 = \chi_{\{1\}}$ and u an Haagerup function such that*:

$$0 < |u(x)| < 1 \quad \textit{for every} \quad x\in F\ .$$

If

$$\sum_{x\in F} |u(x)|^2 < +\infty$$

then for every finitely supported function f with $f(1) = 0$:

$$\|f - \delta_1\|_u^2 > c \tag{3}$$

for some positive constant c.

Proof.

$$\begin{aligned} \|f - \delta_1\|_u^2 &= 1 - 2\,\mathrm{Re}\,\langle u,\, f\rangle + \|f\|_u^2 \\ \text{by (2)} \qquad &\geqq |1 - \langle u,\, f\rangle|^2 + \sum_{|z|=1} (|u(z)|^{-2} - 1)|\langle u,\, f\chi_{A(z)}\rangle|^2 \\ (4) \qquad &\geqq |1 - \langle u,\, f\rangle|^2 + c'\sum_{|z|=1} |\langle u,\, f\chi_{A(z)}\rangle|^2|u(z)|^{-2}\ , \end{aligned}$$

where

$$c' = \operatorname*{Inf}_{|z|=1}(1 - |u(z)|^2)$$

is greater than zero.

Let

$$A = \{z\colon |z| = 1 \text{ and } |\langle u,\, f\chi_{A(z)}\rangle| \geqq (1/4)|\langle u,\, f\rangle||u(z)|^2\|u\chi_F\|_2^{-2}\}$$

then

$$\sum_{z\in A} |\langle u, f\chi_{A(z)}\rangle| \geqq (1/2)|\langle u, f\rangle| .$$

Therefore:

$$(5)\qquad \sum_{|z|=1} |\langle u, f\chi_{A(z)}\rangle|^2 |u(z)|^{-2} \geqq \sum_{z\in A} |\langle u, f\chi_{A(z)}\rangle|^2 |u(z)|^{-2}$$
$$\geqq (1/2)|\langle u, f\rangle| \, \|u\chi_F\|_2^{-2} \sum_{z\in A} |\langle u, f\chi_{A(z)}\rangle|$$
$$\geqq (1/4)|\langle u, f\rangle|^2 \|u\chi_F\|_2^{-2} .$$

From (4) and (5) we obtain

$$\|f - \delta_1\|_u^2 \geqq |1 - \langle u, f\rangle|^2 + c''|\langle u, f\rangle|^2 \geqq c ,$$

where

$$c = \min(1/2, c''/2) .$$

THEOREM. *Let F be a free subset of G, u an Haagerup function such that*

$$0 < |u(x)| < 1 \colon x \in [F]$$

and

$$\sum_{x\in F} |u(x)|^2 < +\infty .$$

Then u does not belong to $A^{\perp}(G)$.

Proof. Let Φ_u be the positive linear functional on $C^*(G)$ canonically associated with u. We first prove that condition (3) in Lemma 2 implies Φ_u is not orthogonal to Φ_{δ_1}.

Assume the contrary: then there exists an hermitian projection P in $W^*(G)$ such that

$$\langle \Phi_u, P\rangle = 1 \quad \text{and} \quad \langle \Phi_{\delta_1}, P\rangle = 0 \qquad [4,\ 12.3.1.(\mathrm{i})] .$$

By Kaplansky's density theorem [5, p. 43], for any $\varepsilon > 0$ we can find an hermitian element $P' \in C^*(G)$ such that

$$\|P'\|_{C^*} \leqq 1 , \qquad |\langle P', \Phi_u\rangle - 1| < \varepsilon , \qquad |\langle P', \Phi_{\delta_1}\rangle| < \varepsilon .$$

However P' can be approximated with a finitely supported function f on $[F]$ such that $f(1) = 0$ and:

$$(6)\qquad \|f\|_{C^*} \leqq 1 + \varepsilon$$

$$(7)\qquad |\langle u, f\rangle - 1| < 3\varepsilon , \qquad |\langle u, f^*\rangle - 1| < 3\varepsilon .$$

For such a function, (6) and (7) imply:

$$\begin{aligned}\|f - \delta_1\|_u &= \langle u, f^* * f\rangle - \langle u, f\rangle - \langle u, f^*\rangle + 1 \\ &\leqq \langle u, f^* * f\rangle - 1 + 6\varepsilon \\ &\leqq (1 + \varepsilon)^2 - 1 + 6\varepsilon \leqq 9\varepsilon ,\end{aligned}$$

but this contradicts condition (3). By Lemma 2, Φ_u is not orthogonal to Φ_{δ_1} and this implies

$$A_{\pi_u} \cap A(G) \neq \{0\} \tag{8}$$

where A_{π_u} is the closed, translation invariant subspace of $B(G)$ generated by u (see for example [1]).

Since $A(G)^\perp$ is closed and translation invariant, $u \in A(G)^\perp$ implies $A_{\pi_u} \subseteqq A(G)^\perp$ contradicting (8).

REMARK. It is easy to see that if $u(x) = 1$ for some $x \in F$, then $u \in A(G)^\perp$; moreover if $u(x_1) = u(x_2) = 1$ for some $x_1 \neq x_2$ in F, then $u \in B_\lambda(G)^\perp$. It will be proved in the following section that, if $u(x')=1$, then $u \in B_\lambda(G)$ if and only if $u(x) = 0$ for every $x \in F$, $x \neq x'$.

3. In this section we prove two further properties of Haagerup functions; the second one shows another analogy with the classical Riesz products.

From [8, Cor. 3.2] a function $\phi(x) = \exp(-t|x|)$ on the free group G with finitely many generators $a_1, \cdots, a_N$, is in $B_\lambda(G)$ if and only if

$$\|\phi\chi_F\|_2^2 = \sum_{i=1}^{N} |\phi(a_i)|^2 \leqq N(2N-1)^{-1} . \tag{9}$$

A restriction argument shows that the condition in [3, Th. 2] is in some sense the best possible in the case of infinitely many generators.

One may wonder if an L^2-condition similar to (9) still holds for general Haagerup functions. The following proposition shows that this is not the case, even for the free group F_2 on two generators a and b.

PROPOSITION 1. *Let u be a Haagerup function on F_2 and*

$$\beta = |u(a)|^2 + 3|u(a)|^2|u(b)|^2 + |u(b)|^2 .$$

Then $u \in B_\lambda(G)$ if and only if $\beta \leqq 1$. Moreover $u \in A(G)$ if $\beta < 1$.

Proof. By [8, Th. 3.1(2)] it is enough to evaluate $\|u\chi_n\|_2$, where χ_n is the characteristic function of the set

$$E_n = \{x \in G: |x| = n\} .$$

We shall only sketch the lengthy computation involved. Computing the number of elements in E_n respectively of the form

$$a^{\pm\varepsilon_1}b^{\pm\sigma_1}\cdots a^{\pm\varepsilon_s}b^{\pm\sigma_s}$$
$$b^{\pm\sigma_1}a^{\pm\varepsilon_1}\cdots a^{\pm\varepsilon_s}b^{\pm\sigma_{s+1}}$$
$$a^{\pm\varepsilon_1}b^{\pm\sigma_1}\cdots b^{\pm\sigma_{s-1}}a^{\pm\varepsilon_s}$$

with

$$\sum \varepsilon_i = k\ , \qquad \sum \sigma_i = n-k\ , \qquad \varepsilon_i, \sigma_i \geqq 1\ ,$$

one obtains

$$\|u\chi_n\|_2^2 = \sum_{k=0}^{n} R(n,k)|u(a)|^{2k}|u(b)|^{2(n-k)}$$

where, for n large enough:

$$\begin{aligned} R(n,k) &= \sum_{s=1}^{\min(k,n-k+1)} 2^{2s-1}\binom{k-1}{s-1}\left\{4\binom{n-k}{s}+\binom{n-k-1}{s-2}\right\} \\ &= \sum_s \xi(s,k,n)\ . \end{aligned}$$

By Stirling's formula

$$\gamma g(s,k,n) \leqq \xi(s,k,n) \leqq n^{\delta} g(s,k,n)$$

where $\gamma, \delta > 0$ are independent of n and k, and $g(s,k,n) = k^k(n-k)^{n-k}(s/2)^{-2s}(k-s)^{s-k}(n-k-s)^{k+s-n}$. Letting s be continuous, $g(s,k,n)$ takes its maximum value $g(s',k,n)$ at the point $s' = (2/3)(n-(n^2-3kn+3k^2)^{1/2})$. Trivially

$$\gamma g(s',k,n) \leqq R(n,k) \leqq n^{\delta'} g(s',k,n)$$

and if we put $k = \alpha n$,

$$\gamma'(\max P(\alpha))^n \leqq \|u\chi_n\|_2^2 \leqq n^{\delta''}(\max P(\alpha))^n$$

where

$$\begin{aligned} P(\alpha) &= \alpha^\alpha(1-\alpha)^{1-\alpha}\left\{\alpha - \frac{2}{3}(1-(1-3\alpha+3\alpha^2)^{1/2})\right\}^{-\alpha} \\ &\quad\times \left\{1-\alpha-\frac{2}{3}(1-(1-3\alpha+3\alpha^2)^{1/2})\right\}^{\alpha-1}|u(a)|^{2\alpha}|u(b)|^{2(1-\alpha)}\ . \end{aligned}$$

It turns out that the maximum of $P(\alpha)$ is attained at

$$\alpha' = (1/2)(1+(r-1)(r^2+14r+1)^{-1/2})\ , \quad r = |u(a)|^2|u(b)|^{-2}$$

and we finally obtain:

$$\gamma'' \leqq \|u\chi_n\|_2^2\{(1/2)|u(b)|^2(1+r+(r^2+14r+1)^{1/2})\}^{-n} \leqq n^{\delta'''}\ . \tag{10}$$

Because the two following relations are equivalent

$$(1/2)|u(b)|^2(1 + r + (r^2 + 14r + 1)^{1/2}) \leqslant 1$$
$$|u(a)|^2 + 3|u(a)|^2|u(b)|^2 + |u(b)|^2 \leqslant 1$$

it follows from (10) and [8, Th. 3.1(2)] that $u \in A(G)$ for $\beta < 1$ and $u \notin B_\lambda(G)$ if $\beta > 1$. If $\beta = 1$, set $u_t = e^{-t|x|}u$. For every $t > 0$ $\|u_t\| = 1$ and $u_t \in A(G)$ by the previous result. Then $u \in B_\lambda(G)$ because $u_t(x) \to u(x)$ as $t \to 0$ for every $x \in G$.

PROPOSITION 2. *Let u an Haagerup function which assumes the constant value A on the infinite free set F. Then if $|A| < 1$, A is not isolated in the Gelfand spectrum of u.*

Proof. Let $\hat{u}$ be the Gelfand transform of u, and $\mathscr{M}$ the maximal ideal space of $B(G)$. Suppose A isolated in the spectrum of u, then the set $H = \{y \in \mathscr{M}: \hat{u}(y) = A\}$ is an open compact set in $\mathscr{M}$. By Gelfand's operational calculus [7, § 14], there would exist a function $v \in B(G)$, such that v is identically one on H and zero on $\mathscr{M} \backslash H$. Then uv is supported by F and there this function assumes the constant value A. Because any function of $B(G)$ supported on a free set must vanish at infinity, see, for example, [2], we have a contradiction.

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