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OPERATORS SIMILAR TO UNITARY OR SELFADJOINT ONES

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Let T be a bounded linear operator on a Hilbert space. General necessary and sufficient conditions are given in order that VTV^{-1} is unitary for some bounded linear operator V with bounded everywhere defined inverse. Similarly let B be a closed and densely defined linear operator in a Hilbert space. General necessary and sufficient conditions are given in order that VBV^{-1} is selfadjoint for some bounded linear operator V with bounded everywhere defined inverse.

1. Introduction and some preliminaries. Throughout this paper $\mathbf{H}$ is a complex Hilbert space with inner-product $(\ ,\ )$. Let A and B be linear operators with domain and range in $\mathbf{H}$. Then A and B are said to be *similar* if there exists a continuous linear operator V: $\mathbf{H} \to \mathbf{H}$ with bounded everywhere defined inverse such that $VA = BV$. Let T: $\mathbf{H} \to \mathbf{H}$ be a bounded linear operator. Then T is said to be *power bounded* if $\sup\{\|T^n\|: n \in \mathbf{N}\}$ is finite. Again let T: $\mathbf{H} \to \mathbf{H}$ be a continuous linear operator and suppose that its spectrum is contained in the circumference of the closed unit disc. The problem which poses itself is to find conditions on the resolvent family $\{(\lambda I - T)^{-1}: |\lambda| \neq 1\}$ which guarantee that T is similar to a unitary operator. Next let A be a closed linear operator with domain and range in $\mathbf{H}$. Suppose that its spectrum is a subset of $\mathbf{R}$. Find necessary and sufficient conditions on the resolvent family $\{(\lambda I - iA)^{-1}: \operatorname{Re}\lambda \neq 0\}$ in order that A is similar to a selfadjoint operator. The main tool we use is what might be called an operator valued Poisson kernel. If the spectrum of T is a subset of $\{\lambda \in \mathbf{C}: |\lambda| = 1\}$ and if T has inverse S, the corresponding Poisson kernel is given by

$$(1 - r^2)(I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1}, \qquad 0 \leq r \leq 1, -\pi \leq \theta \leq +\pi.$$

If A is a closed linear operator in $\mathbf{H}$ the spectrum of which is a subset of $\mathbf{R}$, then the corresponding Poisson kernel is given by

$$\omega\left(\omega^2 I + (\xi I - A)^2\right)^{-1} = \omega\left(((\omega + i\xi)I - iA)^{-1}((\omega - i\xi)I + iA)^{-1}\right), \qquad \omega > 0, \xi \in \mathbf{R}.$$

The present results generalize Theorems 1 and 2 in Van Casteren [**9**], where more related references can be found too. They are also closely related to a problem posed by Sz.-Nagy in [**3**, p. 585]. See also Sz.-Nagy and Foiaş [**8**, Chapitre IX, p. 334]. Another closely related paper is Stampfli [**5**]. This reference should also have been given in [**9**].

First we shall prove some lemmas which will be useful in the sequel.

1.1. LEMMA (a) *Let* $(\alpha_k:\ k \in \mathbf{N})$ *be a sequence of nonnegative real numbers. The following inequalities hold*:

$$e^{-1}\sup\left\{(n+1)^{-1}\sum_{k=0}^{n}\alpha_k : n \in \mathbf{N}\right\} \leq \sup\left\{(1-r)\sum_{k=0}^{\infty} r^k\alpha_k : 0 \leq r < 1\right\}$$
$$\leq \sup\left\{(n+1)^{-1}\sum_{k=0}^{n}\alpha_k : n \in \mathbf{N}\right\}.$$

(b) *Let* $\alpha\colon [0,\infty) \to [0,\infty)$ *be a Borel measurable function. Then*

$$e^{-1}\sup\left\{t^{-1}\int_0^t \alpha(s)\,ds : t > 0\right\} \leq \sup\left\{\omega\int_0^\infty e^{-\omega s}\alpha(s)\,ds : \omega > 0\right\}$$
$$\leq \sup\left\{t^{-1}\int_0^t \alpha(s)\,ds : t > 0\right\}.$$

Proof. (a) The following (in-) equalities are readily verified:

$$(1-r)r^n\sum_{k=0}^{n}\alpha_k \leq (1-r)\sum_{k=0}^{\infty} r^k\alpha_k = (1-r)^2\sum_{k=0}^{\infty} r^k\sum_{j=0}^{k}\alpha_j$$
$$\leq (1-r)^2\left(\sup_{k\in\mathbf{N}}(k+1)^{-1}\sum_{j=0}^{k}\alpha_j\right)\sum_{k=0}^{\infty}(k+1)r^k$$
$$= \sup_{k\in\mathbf{N}}(k+1)^{-1}\sum_{j=0}^{k}\alpha_j, \qquad 0 \leq r < 1,\ n \in \mathbf{N}.$$

From this (a) follows with $r = n(n+1)^{-1}$.
(b) As in the proof of (a) we have

$$e^{-1}\omega\int_0^{\omega^{-1}}\alpha(s)\,ds \leq \omega\int_0^\infty e^{-\omega s}\alpha(s)\,ds = \omega^2\int_0^\infty e^{-\omega t}\left(\int_0^t \alpha(s)\,ds\right)dt.$$

The following lemma will be needed for the proof of Theorem 3.1.

1.2. LEMMA. *Let* h *be a complex valued harmonic function on the right half plane for which*

$$M := \sup\left\{\int_{-\infty}^{\infty}|h(\omega,\xi)|\,d\xi : \omega > 0\right\}$$

is finite. The following assertions hold true.

(a) *The function h satisfies the following inequality*:

$$3\pi\omega|h(\omega, \xi)| \le 4M, \qquad \omega > 0, \xi \in \mathbf{R}.$$

(b) *There exists a complex measure μ on* $\mathbf{R}$, *which is of bounded total variation, such that*

$$h(\omega, \xi) = \frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + (\xi - \eta)^2}\, d\mu(\eta), \qquad \omega > 0, \xi \in \mathbf{R}.$$

(c) *Suppose that h is of the form*

$$h(\omega, \xi) = F(\omega + i\xi) + G(\omega - i\xi), \qquad \omega > 0, \xi \in \mathbf{R},$$

where F and G are holomorphic and where

$$\sup\{\omega|F(\omega)|: \omega > 0\}$$

is finite. Then

$$F(\lambda) = (2\pi)^{-1}\int_{-\infty}^{\infty} (\lambda - i\eta)^{-1}\, d\mu(\eta), \qquad \operatorname{Re}\lambda > 0,$$

and

$$G(\lambda) = (2\pi)^{-1}\int_{-\infty}^{\infty} (\lambda + i\eta)^{-1}\, d\mu(\eta), \qquad \operatorname{Re}\lambda > 0.$$

Proof. (a) The reader is referred to Duren [**1**, Lemma 1, p. 188].

(b) We refer the reader to Stein [**6**, Theorem 2, Corollary, p. 200].

(c) Let λ in $\mathbf{C}$ be such that $\operatorname{Re}\lambda > 0$. Then

$$F(\lambda) - (2\pi)^{-1}\int_{-\infty}^{\infty} (\lambda - i\eta)^{-1}\, d\mu(\eta) = (2\pi)^{-1}\int_{-\infty}^{\infty} (\bar{\lambda} + i\eta)^{-1}\, d\mu(\eta) - G(\bar{\lambda}).$$

Since the left-hand side of this equality is analytic in λ and since the right-hand side is analytic in $\bar{\lambda}$, it follows that each side is constant. Since $\sup\{\omega\,|\,F(\omega)\,|: \omega > 0\}$ is finite, we conclude that this constant is zero.

REMARK. Let h be as in Lemma 1.2. Fix $\omega' > 0$. By (a) the harmonic function

$$(\omega, \xi) \mapsto h(\omega + \omega', \xi) - \frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + (\xi - \eta)^2} h(\omega', \eta)\, d\eta, \qquad \omega > 0, \xi \in \mathbf{R},$$

is bounded. Since it has boundary value 0, it vanishes identically. So

$$h(\omega, \xi) = \lim_{\omega' \downarrow 0} h(\omega + \omega', \xi) = \lim_{\omega' \downarrow 0} \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + (\xi - \eta)^2} h(\omega', \eta)\, d\eta.$$

An application of the Riesz representation theorem for $C_0(\mathbf{R})$ (e.g. see Hewitt and Ross [**4**, Theorem 14.4, p. 168]) and of the fact that the dual unit ball of $C_0(\mathbf{R})$ is weak* compact, yields the existence of a measure μ for which (b) holds.

2. Operators which are similar to a unitary one. In this section $\mathbf{H}$ is a complex Hilbert space and T: $\mathbf{H} \to \mathbf{H}$ is a continuous linear operator with a bounded everywhere defined inverse S. The theorem we want to prove reads as follows. Notice that the equivalence of (i) and (iii) yields a positive answer to Question 1 in Stampfli [**5**, p. 149].

2.4. THEOREM. *The following assertions are equivalent*:
(i) *T is similar to a unitary operator*;
(ii) *T and S are power bounded*;
(iii) *T is power bounded, $(I - \lambda S)^{-1}$ exists for $|\lambda| < 1$ and*

$$\sup\{(1 - |\lambda|)\|(I - \lambda S)^{-1}\| : |\lambda| < 1\}$$

is finite;
(iv) *For each x in $\mathbf{H}$ the expressions*

$$\sup\left\{(n+1)^{-1} \sum_{k=0}^{n} \|T^k x\|^2 : n \in \mathbf{N}\right\}$$

and

$$\sup\left\{(n+1)^{-1} \sum_{k=0}^{n} \|(T^*)^k x\|^2 : n \in \mathbf{N}\right\}$$

are finite, $(I - \lambda S)^{-1}$ exists for $|\lambda| < 1$ and

$$\sup\{(1 - |\lambda|)\|(I - \lambda S)^{-1}\| : |\lambda| < 1\}$$

is finite;
(v) *For every x in $\mathbf{H}$ the expressions*

$$\sup\left\{(n+1)^{-1} \sum_{k=0}^{n} \|(T^*)^k x\|^2 : n \in \mathbf{N}\right\}$$

and

$$\sup\left\{(n+1)^{-1}\sum_{k=0}^{n}\|S^k x\|^2 : n \in \mathbf{N}\right\}$$

are finite;

(vi) *For* $|\lambda| < 1$ *the inverses* $(I-\lambda T)^{-1}$ *and* $(I-\lambda S)^{-1}$ *exist and for every* x *and* y *in* $\mathbf{H}$ *the expression*

$$\sup\left\{(1-r^2)\int_{-\pi}^{+\pi}\left|\left((I-re^{-i\theta}T)^{-1}(I-re^{i\theta}S)^{-1}x,\, y\right)\right|d\theta : 0 \le r < 1\right\}$$

is finite.

The proof of the equivalency of (ii) and (iii) is contained in Van Casteren [**8**, Theorem 1].

Proof. The proof of the equivalency of (i) and (ii) appears in Sz.-Nagy [7]. The implication (ii) ⇒ (iii) is easy. The implication (iii) ⇒ (iv) is trivial.

(iv) ⇒ (vi) Put

$$M = \sup\left\{\left\|\left\{(1-|\lambda|^2)(I-\lambda S)^{-1} - I\right\}T\right\||\lambda|^{-1} : 0 < |\lambda| < 1\right\}.$$

From (iv) it follows that M is finite. Next fix x and y in $\mathbf{H}$ and put

$$M_1(x)^2 = \sup\left\{(1-r^2)\sum_{k=0}^{\infty} r^{2k}\|T^k x\|^2 : 0 \le r < 1\right\}$$

and

$$M_2(y)^2 = \sup\left\{(1-r^2)\sum_{k=0}^{\infty} r^{2k}\|(T^*)^k y\|^2 : 0 \le r < 1\right\}.$$

From Lemma 1.1 and assertion (iv) it follows that $M_1(x)$ and $M_2(y)$ are finite. For $0 < |\lambda| < 1$ we have

$$\begin{aligned}&\left((I-\bar{\lambda}T)^{-1}(I-\lambda S)^{-1}x,\, y\right)\\ &\quad = \left(\lambda^{-1}\left\{(1-|\lambda|^2)(I-\lambda S)^{-1} - I\right\}T(I-\bar{\lambda}T)^{-1}x,\, (I-\lambda T^*)^{-1}y\right).\end{aligned}$$

Hence

$$\left|\left((I-\bar{\lambda}T)^{-1}(I-\lambda S)^{-1}x,\, y\right)\right| \le M\left\|(I-\bar{\lambda}T)^{-1}x\right\|\left\|(I-\lambda T^*)^{-1}y\right\|.$$

So

$$\int_{-\pi}^{+\pi}\left|\left((I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1}x, y\right)\right| d\theta$$

$$\leq M\int_{-\pi}^{+\pi}\left\|(I - re^{-i\theta}T)^{-1}x\right\|\left\|(I - re^{i\theta}T^*)^{-1}y\right\| d\theta$$

$$\leq M\left(\int_{-\pi}^{+\pi}\left\|(I - re^{-i\theta}T)^{-1}x\right\|^2 d\theta\right)^{1/2}\left(\int_{-\pi}^{+\pi}\left\|(I - re^{i\theta}T^*)^{-1}y\right\|^2 d\theta\right)^{1/2}$$

$$= 2\pi M\left(\sum_{k=0}^{\infty} r^{2k}\|T^k x\|^2\right)^{1/2}\left(\sum_{k=0}^{\infty} r^{2k}\left\|(T^*)^k y\right\|^2\right)^{1/2}$$

$$\leq 2\pi M M_1(x)M_2(y)(1 - r^2)^{-1}.$$

Hence (vi) follows.

(vi) $\Rightarrow$ (ii) Fix x and y in $\mathbf{H}$. Since $T = S^{-1}$ it follows that

$$(1 - r^2)(I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1} = \sum_{k=-\infty}^{\infty} r^{|k|}e^{ik\theta}S^k, \qquad 0 \leq r < 1,\ -\pi \leq \theta \leq +\pi.$$

So

$$r^{|n|}(T^n x, y) = \frac{1 - r^2}{2\pi}\int_{-\pi}^{+\pi} e^{in\theta}\left((I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1}x, y\right) d\theta, \qquad n \in \mathbf{Z}.$$

With $r = |n|(|n| + 1)^{-1}$ we infer

$$|(T^n x, y)| \leq e \sup_{0 \leq r < 1} \frac{1 - r^2}{2\pi}\int_{-\pi}^{+\pi}\left|\left((I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1}x, y\right)\right| d\theta.$$

So $\sup\{|(T^n x, y)| : n \in \mathbf{Z}\}$ is finite for each x and y in $\mathbf{H}$. By a Banach Steinhaus argument it follows that $\sup\{\|T^n\|: n \in \mathbf{Z}\}$ is finite. Hence (ii) follows.

(ii) $\Rightarrow$ (v) Trivial.

(v) $\Rightarrow$ (vi) Fix x and y in $\mathbf{H}$. Put

$$M_1(x)^2 = \sup\left\{(1 - r^2)\sum_{k=0}^{\infty} r^{2k}\|S^k x\|^2 : 0 \leq r < 1\right\}$$

and put

$$M_2(y)^2 = \sup\left\{(1 - r^2)\sum_{k=0}^{\infty} r^{2k}\left\|(T^*)^k y\right\|^2 : 0 \leq r < 1\right\}.$$

From Lemma 1.1 and (v) it follows that $M_1(x)$ and $M_2(y)$ are finite. For $0 \leq r < 1$ we have

$$
\begin{aligned}
(1-r^2)&\int_{-\pi}^{+\pi}\left|\left((I-re^{-i\theta}T)^{-1}(-re^{i\theta}S)^{-1}x,\, y\right)\right|d\theta \\
&= (1-r^2)\int_{-\pi}^{+\pi}\left|\left((I-re^{i\theta}S)^{-1}x,\,(I-re^{i\theta}T^*)^{-1}y\right)\right|d\theta \\
&\leq (1-r^2)\left(\int_{-\pi}^{+\pi}\left\|(I-re^{i\theta}S)^{-1}x\right\|^2 d\theta\right)^{1/2} \\
&\quad\times\left(\int_{-\pi}^{+\pi}\left\|(I-re^{i\theta}T^*)^{-1}y\right\|^2 d\theta\right)^{1/2} \leq 2\pi M_1(x)M_2(y).
\end{aligned}
$$

Hence (vi) follows.

Next we shall give an example of an operator $T\colon l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ which is not similar to a unitary operator but for which, for each x in $l^2(\mathbf{Z})$, the expression

$$\sup\left\{(2n+1)^{-1}\sum_{j=-n}^{+n}\|T^jx\|^2 : n \in \mathbf{N}\right\}$$

is finite.

EXAMPLE 1. Fix $0 < 2\gamma < 1$ and put $\alpha_k = (1+|k|)^{-\gamma}$, $k \in \mathbf{Z}$. Let $(e_k\colon k \in \mathbf{Z})$ be the standard basis in $l^2(\mathbf{Z})$ and define the operator $T\colon l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ by

$$Te_k = \frac{\alpha_{k+1}}{\alpha_k}e_{k+1}, \qquad k \in \mathbf{Z}.$$

Then

$$T^je_k = \frac{\alpha_{k+j}}{\alpha_k}e_{k+j}, \qquad k, j \in \mathbf{Z}.$$

Fix x in $l^2(\mathbf{Z})$. Then

$$\sum_{j=-n}^{+n}\|T^jx\|^2 \leq \|x\|^2 . \sup_{k\in\mathbf{Z}}\alpha_k^{-2}\sum_{j=-n}^{+n}\alpha_{k+j}^2$$

Since $0 < 2\gamma < 1$, it follows that

$$k \in \sup_{\mathbf{Z},n\in\mathbf{N}}\frac{1}{(2n+1)\alpha_k^2}\sum_{j=-n}^{+n}\alpha_{k+j}^2$$

is finite. Consequently

$$\sup\left\{(2n+1)^{-1}\sum_{j=-n}^{+n}\|T^j x\|^2 : n \in \mathbf{N}\right\}$$

is finite. Since $\|T^j\| = (1+|j|)^\gamma$, it follows that T cannot be similar to a unitary operator.

3. Operators which are similar to a selfadjoint one. In this section A denotes a linear operator with domain and range in a Hilbert space $\mathbf{H}$. The operator iA is said to generate a strongly continuous semigroup $\{P_t : t \geq 0\}$ if

$$iA = \operatorname{s-lim}_{t\downarrow 0}\frac{P_t - I}{t}.$$

It generates a strongly continuous group $\{P_t : t \in \mathbf{R}\}$ if

$$iA = \operatorname{s-lim}_{t\to 0}\frac{P_t - I}{t}.$$

For more details on semigroups see Yosida [**10**, Chapter IX]. For the proof of Theorem 3.1. we need Stone's representation theorem; see Yosida [**10**, Corollary 2, p. 253]. Furthermore we shall use Plancherel's theorem in $L^2(\mathbf{R}, \mathbf{H})$; e.g. see Edwards and Gaudry [**2**, §3.4, p. 53] or Stein [**6**, Chapter II, §5, pp. 45–47].

We want to prove the following theorem

3.1. THEOREM. *Let A be a linear operator with domain and range in a Hilbert space $\mathbf{H}$. The following assertions are equivalent.*

(i) *The operator A is similar to a selfadjoint operator*;

(ii) *The operator iA generates a bounded strongly continuous group*;

(iii) *The operator $-iA$ generates a bounded strongly continuous semi-group, $(\lambda I - iA)^{-1}$ exists for* $\operatorname{Re}\lambda > 0$ *and*

$$\sup\left\{\operatorname{Re}\lambda\|(\lambda I - iA)^{-1}\| : \operatorname{Re}\lambda > 0\right\}$$

is finite;

(iv) *The operator $-iA$ generates a strongly continuous semi-group $\{P_t : t \geq 0\}$ for which the expressions*

$$\sup\left\{t^{-1}\int_0^t \|P_s x\|^2\,ds : t > 0\right\} \quad \textit{and} \quad \sup\left\{t^{-1}\int_0^t \|P_s^* x\|^2\,ds : t > 0\right\}$$

are finite for each x in **H**, *for which* $(\lambda I - iA)^{-1}$ *exists for* Re $\lambda > 0$ *and for which*

$$\sup\left\{\operatorname{Re}\lambda\left\|(\lambda I - iA)^{-1}\right\| : \operatorname{Re}\lambda > 0\right\}$$

is finite;

(v) *The operator* $-iA$ *generates a strongly continuous group* $\{P_t : t \in \mathbf{R}\}$ *for which the expressions*

$$\sup\left\{t^{-1}\int_0^t \|P_s^* x\|^2\, ds : t > 0\right\} \quad \textit{and} \quad \sup\left\{t^{-1}\int_0^t \|P_{-s} x\|^2\, ds : t > 0\right\}$$

are finite for each x in **H**;

(vi) *The operator A is closed*, $(\lambda I - iA)^{-1}$ *exist for* Re $\lambda \neq 0$ *and for each* x, y *in* **H** *the expression*

$$\sup\left\{2\omega\int_{-\infty}^{\infty}\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1} x, y\right)\right| d\xi : \omega > 0\right\}$$

is finite and

$$\lim_{\lambda\to\infty} \lambda\left((\lambda I + iA)^{-1} x, y\right) = (x, y).$$

Proof. The equivalency of (i) and (ii) follows from Stone's Theorem (see Yosida l.c.) and from Sz.-Nagy [7].

(ii) ⇒ (iii) Let $-iA$ be the generator of the bounded group $\{P_t : t \in \mathbf{R}\}$. For Re $\lambda > 0$ we have

$$(\lambda I - iA)^{-1} x = \int_0^{\infty} e^{-\lambda s} P_{-s} x\, ds, \qquad x \in \mathbf{H}.$$

Put $M = \sup\{\|P_{-s}\| : s \geq 0\}$. Then

$$\operatorname{Re}\lambda\left\|(\lambda I - iA)^{-1}\right\| \leq M, \qquad \operatorname{Re}\lambda > 0.$$

(iii) ⇒ (iv) Trivial.

(iv) ⇒ (vi) Let $-iA$ and $\{P_t : t \geq 0\}$ be as in (iv). Put

$$M_1(x)^2 = \sup\left\{t^{-1}\int_0^t \|P_s x\|^2\, ds : t > 0\right\}, \qquad x \in \mathbf{H}, \tag{1}$$

put

$$M_2(x)^2 = \sup\left\{t^{-1}\int_0^t \|P_s^* x\|^2\, ds : t > 0\right\}, \qquad x \in \mathbf{H}, \tag{2}$$

and put

$$M = \sup\left\{\left\|2\operatorname{Re}\lambda(\lambda I - iA)^{-1} - I\right\| : \operatorname{Re}\lambda > 0\right\}. \tag{3}$$

Let $\lambda = \omega + i\xi$ in $\mathbf{C}$ be such that $\omega = \operatorname{Re}\lambda > 0$ and let x and y be in $\mathbf{H}$. Then

$$\begin{aligned}&\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\\ &\qquad = \left(\left\{2\omega(\lambda I - iA)^{-1} - I\right\}(\bar{\lambda} I + iA)^{-1}x, (\lambda I - iA^*)^{-1}y\right).\end{aligned}$$

From (3), the definition of M, it follows that

$$\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\right| \le M\left\|(\bar{\lambda} I + iA)^{-1}x\right\|\left\|(\lambda I - iA^*)^{-1}y\right\|.$$

So by Schwarz' inequality we infer

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\right| d\xi\right)^2\\ &\qquad \le M^2\int_{-\infty}^{\infty}\left\|((\omega - i\xi)I + iA)^{-1}x\right\|^2 d\xi\\ &\qquad\quad \cdot\int_{-\infty}^{\infty}\left\|((\omega + i\xi)I - iA^*)^{-1}y\right\|^2 d\xi.\end{aligned} \tag{4}$$

Since $-iA$ generates the semigroup $\{P_t : t \ge 0\}$, the operator $+iA^*$ generates the semigroup $\{P_t^* : t \ge 0\}$; see Yosida [**10**, Chapter VII, Theorem 3, p. 196 and Chapter IX, §13]. Since

$$((\omega - i\xi)I + iA)^{-1}x = \int_0^{\infty} e^{-(\omega - i\xi)s}P_s x\, ds$$

and since

$$((\omega + i\xi)I - iA^*)^{-1}x = \int_0^{\infty} e^{-(\omega + i\xi)s}P_s^* x\, ds$$

it follows form (4) and from Plancherel's theorem in Hilbert space that

$$\begin{aligned}&\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\right| d\xi\right)^2\\ &\qquad \le M^2\int_0^{\infty} e^{-2\omega s}\left\|P_s x\right\|^2 ds\int_0^{\infty} e^{-2\omega s}\left\|P_s^* y\right\|^2 ds.\end{aligned}$$

From Lemma 1.1(b) and by (1) and (2) it follows that

$$2\omega\int_{-\infty}^{\infty}\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\right| d\xi \le 2\pi M\, M_1(x)\, M_2(y).$$

Since M, $M_1(x)$ and $M_2(y)$ are finite, assertion (vi) follows.

(vi) $\Rightarrow$ (ii) Fix x and y in $\mathbf{H}$. The function

$$(\omega, \xi) \mapsto 2\omega\Big(\big(\omega^2 I + (\xi I - A)^2\big)^{-1} x, y\Big), \qquad \omega > 0, \xi \in \mathbf{R},$$

is harmonic and by (vi)

$$\sup\left\{2\omega \int_{-\infty}^{\infty} \left|\Big(\big(\omega^2 I + (\xi I - A)^2\big)^{-1} x, y\Big)\right| d\xi \colon \omega > 0\right\}$$

is finite. So by Lemma 1.2(a) there exists a complex Borel measure $\mu_{x,y}$, which is of bounded total variation, such that

$$\begin{aligned} &2\omega\Big(\big(\omega^2 I + (\xi I - A)^2\big)^{-1} x, y\Big) \\ &\qquad = \frac{\omega}{\pi} \int_{-\infty}^{\infty} \big(\omega^2 + (\xi - \eta)^2\big)^{-1} d\mu_{x,y}(\eta), \qquad \omega > 0, \xi \in \mathbf{R}. \end{aligned}$$

Since $\lim_{\lambda \to \infty} \lambda((\lambda I + iA)^{-1} x, y) = (x, y)$ it follows from Lemma 1.2(c) that

$$\big((\lambda I - iA)^{-1} x, y\big) = (2\pi)^{-1} \int_{-\infty}^{\infty} (\lambda - i\eta)^{-1} d\mu_{x,y}(\eta), \qquad \operatorname{Re} \lambda > 0. \tag{5}$$

It also follows that

$$\big((\lambda I + iA)^{-1} x, y\big) = (2\pi)^{-1} \int_{-\infty}^{\infty} (\lambda + i\eta)^{-1} d\mu_{x,y}(\eta), \qquad \operatorname{Re} \lambda > 0. \tag{6}$$

We also conclude that

$$2\pi(x, y) = \mu_{x,y}(\mathbf{R}). \tag{7}$$

Next consider, for $\omega > 0$ and t in $\mathbf{R}$, the identities

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{\omega|t| + i\xi t} \cdot 2\omega\Big(\big(\omega^2 I + (\xi I - A)^2\big)^{-1} x, y\Big)\, d\xi \\ &\quad = \int_{-\infty}^{\infty} e^{\omega|t| + i\xi t}\left(\frac{\omega}{\pi} \int_{-\infty}^{\infty} \big(\omega^2 + (\xi - \eta)^2\big)^{-1} d\mu_{x,y}(\eta)\right) d\xi \\ &\quad = \int_{-\infty}^{\infty} e^{i\eta t} e^{\omega|t|}\left(\frac{\omega}{\pi} \int_{-\infty}^{\infty} \big(\omega^2 + (\xi - \eta)^2\big)^{-1} e^{i(\xi - \eta)t}\, d\xi\right) d\mu_{x,y}(\eta) \\ &\quad = \int_{-\infty}^{\infty} e^{i\eta} e^{\omega|t|}\left(\frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega t \xi}{1 + \xi^2}\, d\xi\right) d\mu_{x,y}(\eta) \\ &\quad = \int_{-\infty}^{\infty} e^{i\eta t}\, d\mu_{x,y}(\eta). \end{aligned}$$

From these identities we conclude that the expression

$$\int_{-\infty}^{\infty} e^{\omega|t|+i\xi t}\cdot 2\omega\Big(\big(\omega^2 I+(\xi I-A)^2\big)^{-1}x,y\Big)\,d\xi$$

does not depend on the choice of ω. So for each t in $\mathbf{R}$ there exists a continuous linear map P_t: $\mathbf{H}\to\mathbf{H}$ such that

$$\begin{aligned}(8)\quad (P_t x,y) &= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{\omega|t|+i\xi t}\cdot 2\omega\Big(\big(\omega^2 I+(\xi I-A)^2\big)^{-1}x,y\Big)\,d\xi\\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\eta t}\,d\mu_{x,y}(\eta).\end{aligned}$$

Next pick λ in $\mathbf{C}$, $\operatorname{Re}\lambda>0$. From (8) and (5) it follows that

$$\begin{aligned}(9)\qquad \int_0^{\infty} e^{-\lambda t}(P_t x,\,y)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\lambda-i\eta)^{-1}\,d\mu_{x,y}(\eta)\\ &= \big((\lambda I-iA)^{-1}x,y\big).\end{aligned}$$

Since $\sup\{|(P_t x,\ y)|:\ t\in\mathbf{R}\}$ is finite and since $\lim_{t\to 0}(P_t x,\ y)=(P_0 x,y)$, it follows from (9) and (7) that

$$(P_0 x,y)=\lim_{\lambda\to\infty}\lambda\int_0^{\infty}e^{-\lambda t}(P_t x,y)\,dt=(2\pi)^{-1}\mu_{x,y}(\mathbf{R})=(x,y).$$

Hence $P_0=I$. From (9) it also follows that

$$(10)\qquad \int_0^{\infty}e^{-\lambda t}P_t x\,dt=(\lambda I-iA)^{-1}x.$$

Since the map $\lambda\mapsto(\lambda I-iA)^{-1}$, $\operatorname{Re}\lambda>0$, satisfies the resolvent equation, to wit

$$\begin{aligned}&(\lambda I-iA)^{-1}-(\mu I-iA)^{-1}\\ &\qquad=(\mu-\lambda)(\lambda I-iA)^{-1}(\mu I-iA)^{-1},\qquad \operatorname{Re}\lambda,\operatorname{Re}\mu>0,\end{aligned}$$

it follows that $\{P_s\colon s\geq 0\}$ is a semigroup. This means

$$P_{s+t}=P_s\circ P_t,\qquad s,t\geq 0.$$

This semigroup is weakly continuous. By a standard result on semigroups it is strongly continuous. By (10) its generator is given by iA. Similarly the family $\{P_{-s}\colon s\geq 0\}$ is a strongly continuous semigroup with generator $-iA$. Consequently the collection $\{P_s\colon s\in\mathbf{R}\}$ is a strongly continuous group for which $\sup\{|(P_s x,\ y)|:\ s\in\mathbf{R}\}$ is finite for each x, y in $\mathbf{H}$. It follows that $\sup\{\|P_s\|\colon s\in\mathbf{R}\}$ is finite. This proves (ii).

(ii) $\Rightarrow$ (v) Trivial.

(v) $\Rightarrow$ (vi) Let $-iA$ be the generator of a group $\{P_s\colon s \in \mathbf{R}\}$ for which (v) is satisfied. Then A is closed, $\lim_{\lambda \to \infty} \lambda((\lambda I + iA)^{-1}x, y) = (x, y)$, x, y in $\mathbf{H}$ and $(\lambda I - iA)^{-1}$ exists for $\operatorname{Re}\lambda \neq 0$. By Plancherel's theorem in $L^2(\mathbf{R}, \mathbf{H})$ we conclude

$$
\begin{aligned}
2\omega\int_{-\infty}^{\infty}&\left|\left(\left(\omega^2 I + (\xi - A)^2\right)^{-1}x, y\right)\right|d\xi \\
&= 2\omega\int_{-\infty}^{\infty}\left|((\omega - i\xi)I + iA)^{-1}((\omega + i\xi)I - iA)^{-1}x, y\right|d\xi \\
&= 2\omega\int_{-\infty}^{\infty}\left|((\omega + i\xi)I - iA)^{-1}x, ((\omega + i\xi)I - iA^*)^{-1}y\right|d\xi \\
&\leq 2\omega\int_{-\infty}^{\infty}\left\|((\omega + i\xi)I - iA)^{-1}x\right\|\left\|((\omega + i\xi)I - iA^*)^{-1}y\right\|d\xi \\
&\leq 2\omega\left(\int_{-\infty}^{\infty}\left\|((\omega + i\xi)I - iA)^{-1}x\right\|^2 d\xi\right)^{1/2} \\
&\quad\times\left(\int_{-\infty}^{\infty}\left\|((\omega + i\xi)I - iA^*)^{-1}y\right\|^2 d\xi\right)^{1/2} \\
&= 2\omega\left(\int_{-\infty}^{\infty}\left\|\int_0^{\infty} e^{-(\omega+i\xi)s}P_{-s}x\, ds\right\|^2 d\xi\right)^{1/2} \\
&\quad\times\left(\int_{-\infty}^{\infty}\left\|\int_0^{\infty} e^{-(\omega+i\xi)s}P_s^* y\, ds\right\|^2 d\xi\right)^{1/2} \\
&= 2\omega\cdot 2\pi\left(\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2\, ds\right)^{1/2}\left(\int_0^{\infty} e^{-2\omega s}\|P_s^* y\|^2\, ds\right)^{1/2} \\
&\leq 2\pi\cdot\left(\sup_{\omega>0} 2\omega\int_0^{\infty} e^{-2\omega s}\|P_{-s}x\|^2\, ds\right)^{1/2} \\
&\quad\times\left(\sup_{\omega>0} 2\omega\int_0^{\infty} e^{-2\omega s}\|P_s^* y\|^2\, ds\right)^{1/2}.
\end{aligned}
$$

From (v) and from Lemma 1.1 it follows that the latter expression is finite. Hence

$$
\sup\left\{2\omega\int_{-\infty}^{\infty}\left|\left(\left(\omega^2 I + (\xi I - A)^2\right)^{-1}x, y\right)\right|d\xi\colon \omega > 0\right\}
$$

is finite. Whence (vi) follows.

REMARK. Under suitable modifications the implication (vi) $\Rightarrow$ (ii) is valid in Banach spaces too.

We conclude with an example of a group $(P_s\colon s \in \mathbf{R})$ of linear operators acting on $L^2(\mathbf{R})$ which is not bounded but for which

$$\sup\left\{(2t)^{-1}\int_{-t}^{+t}\|P_s f\|^2\,ds\colon t > 0\right\}$$

is finite for each f in $L^2(\mathbf{R})$.

EXAMPLE 2. Fix $0 < 2\gamma < 1$ and put $\varphi(x) = (1 + |x|)^\gamma$, $x \in \mathbf{R}$. Define for each s in $\mathbf{R}$ the operator $P_s\colon L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$P_s f(x) = \frac{\varphi(x+s)}{\varphi(x)} f(x+s), \qquad x \in R, f \in L^2(\mathbf{R}).$$

Then the family $\{P_s\colon s \in \mathbf{R}\}$ is a strongly continuous group for which $\|P_s\| = \varphi(s)$ and for which

$$\sup\left\{(2t)^{-1}\int_{-t}^{+t}\|P_s f\|^2\,ds\colon t > 0\right\}$$

is finite for each f in $L^2(\mathbf{R})$. Define the operator A as follows. Its domain $D(A)$ is given by

$$D(A) = \{f \in L^2(\mathbf{R})\colon f' \in L^2(\mathbf{R})\}$$

and $Af, f \in D(A)$, is given by

$$Af(x) = if'(x) + \frac{i\gamma x}{|x|}\cdot\frac{1}{1+|x|}\cdot f(x), \qquad x \in \mathbf{R}.$$

Then $-iA$ generates the group $\{P_s\colon s \in \mathbf{R}\}$. Since $\sup\{\|P_s\|\colon s \in \mathbf{R}\} = \infty$, the operator A is not similar to a self-adjoint one.

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