PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 1, 1983

THE FINE SPECTRA FOR WEIGHTED MEAN OPERATORS

B. E. RHOADES

In a recent paper [5] the fine spectra of integer powers of the Cesàro matrix were computed. In this paper the fine spectra of weighted mean methods are determined. In most cases investigated, the interior points belong to III_1, the boundary points, except 1, belong to II_2, and 1 and any isolated points belong to III_3, where III_1, II_2, and III_3 are portions of the state space as described in [3].

From Goldberg [3], if $T \in B(X)$, X a Banach space, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$,

(II) $\overline{R(T)} = X$, but $R(T) \neq X$, and

(III) $\overline{R(T)} \neq X$,

and three possibilities for T^{-1}

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

A weighted mean matrix A is a lower triangular matrix with entries $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geq 0$ for $n > 0$, and $P_n = \sum_{k=0}^{n} p_k$. The necessary and sufficient condition for the regularity of A is that $\lim P_n = \infty$.

In [2] it was shown that, for any regular weighted mean matrix A, the spectrum, $\sigma(A)$, contains the set $\{\lambda \mid |\lambda - (2-\delta)^{-1}| \leq (1-\delta)/(2-\delta)\} \cup S$, and is contained in the set $\{\lambda \mid |\lambda - (2-\gamma)^{-1}| \leq (1-\gamma)/(2-\gamma)\} \cup \overline{S}$, where $\delta = \limsup p_n/P_n$, $\gamma = \liminf p_n/P_n$, and $S = \{p_n/P_n \mid n \geq 0\}$

We shall first consider those regular weighted mean methods for which $\delta = \gamma$, i.e., for which the main diagonal entries converge.

THEOREM 1. *Let A be a regular weighted mean method such that $\delta = \lim p_n/p_n$ exists. If λ satisfies $|\lambda - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)$ and $\lambda \notin S$, then $\lambda \in \mathrm{III}_1\, \sigma(A)$; i.e., λ is a point of $\sigma(A)$ for which $\overline{R(T)} \neq X$ and T^{-1} exists and is continuous.*

Proof. First of all $\lambda I - A$ is a triangle, hence 1-1. Therefore $\lambda I - A \in 1 \cup 2$.

Consider the adjoint matrix $T^* = \lambda I - A^*$. Since A is regular, $A^* \in B(l)$ with entries $a^*_{00} = \chi(A) = \lim_n Ae - \Sigma_k \lim_n a_{nk} = 1$, $a^*_{n0} = a^*_{0n} = 0$ for $n > 0$ and $a^*_{nk} = a_{k-1,n-1}$ for $n, k > 0$, where $e = \{1, 1, 1, \ldots\}$.

Suppose $T^*x = 0$. Then

$$(\lambda - 1)x_0 = 0,$$

and

$$(1) \qquad \left(\lambda - \frac{p_{n-1}}{P_{n-1}}\right)x_n - \sum_{k=n+1}^{\infty} a^*_{nk}x_k = 0 \quad \text{for } n > 0.$$

Thus $x_0 = 0$ and, from (1), x_1 is arbitrary and, with $c_n = p_n/P_n$,

$$(2) \qquad x_n = \frac{p_{n-1}x_1}{p_0\lambda^{n-1}} \prod_{j=0}^{n-2} (\lambda - c_j) = \frac{p_{n-1}}{p_0} x_1 \prod_{j=0}^{n-2}\left(1 - \frac{c_j}{\lambda}\right)$$

$$= \left(1 - \frac{1}{\lambda}\right)\frac{p_{n-1}x_1}{P_{n-2}} \prod_{j=1}^{n-2}\left(1 + \left(1 - \frac{1}{\lambda}\right)\frac{p_j}{P_{j-1}}\right).$$

Now $|1 + (1 - \frac{1}{\lambda})p_j/P_{j-1}| < 1$ for all j sufficiently large if and only if

$$\left(1 + (1 + \alpha)\frac{p_j}{P_{j-1}}\right)^2 + \left(\beta\frac{p_j}{P_{j-1}}\right)^2 < 1, \quad \text{where } -\frac{1}{\lambda} = \alpha + i\beta.$$

Case I. Assume at most a finite number of the p_k are zero. Then the above inequality is equivalent to

$$2(1 + \alpha) + \left((1 + \alpha)^2 + \beta^2\right)\left(p_j/P_{j-1}\right) < 0$$

for all j sufficiently large. The above inequality will be true for all j sufficiently large if $2(1 + \alpha) + ((1 + \alpha)^2 + \beta^2)\delta/(1 - \delta) < 0$, which is equivalent to $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$.

Let $z_n = \Pi_{j=1}^{n-2}(1 + (1 - 1/\lambda)p_j/P_{j-1})$. Then

$$|z_{n+1}/z_n| = |1 + (1 - 1/\lambda)p_{n-1}/P_{n-2}|.$$

From the hypothesis on λ, and the discussion in the preceding paragraph,

$$(3) \qquad \left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{p_{n-1}}{P_{n-2}}\right| \leq \xi < 1.$$

for all n sufficiently large, and $\Sigma|z_n|$ is convergent.

Since $|(1 - 1/\lambda)p_{n-1}x_1/P_{n-2}|$ is bounded, it follows that $\Sigma|x_n|$ is convergent, so that $(\lambda I - A^*)x = 0$ has nonzero solutions.

By [**3**, Theorem II 3.7], $\lambda I - A$ does not have dense range. Therefore $\lambda I - A \in \text{III}$ and hence $\lambda I - A \in \text{III}_1 \cup \text{III}_2$.

To verify that $\lambda I - A \in \mathrm{III}_1$ it is sufficient, from [3, Theorem II 3.11], to show that $\lambda I - A^*$ is onto.

Suppose $y = (\lambda I - A^*)x$, $x, y \in l$. Then $(\lambda - 1)x_0 = y_0$ and

$$(4) \qquad (\lambda - c_{n-1})x_n - \sum_{k=n+1}^{\infty} p_{n-1}x_k/P_{k-1} = y_n, \qquad n > 0.$$

Choose $x_1 = 0$ and solve for x in terms of y to get

$$(5) \qquad -p_0 \sum_{k=2}^{\infty} x_k/P_{k-1} = y_1$$

$$(6) \qquad (\lambda - c_{n-1})x_n = y_n + p_{n-1} \sum_{k=n+1}^{\infty} x_k/P_{k-1}.$$

For example, substituting (5) into (6), with $n = 2$, yields

$$(\lambda - c_1)x_2 = y_2 + p_1 \sum_{k=3}^{\infty} x_k/P_{k-1} = y_2 + p_1\left(\sum_{k=2}^{\infty} x_k/P_{k-1} - x_2/P_1 \right),$$

so that $x_2 = (y_2 - p_1y_1/p_0)/\lambda$.

Continuing this process, if B is a lower triangular matrix defined by $By = x$, then B has entries

$$b_{00} = \frac{1}{\lambda - 1}, \quad b_{nn} = \frac{1}{\lambda}, \qquad n > 1,$$

$$b_{21} = \frac{-p_1}{\lambda p_0}, \quad b_{n,n-1} = -\frac{p_{n-1}}{P_{n-2}\lambda^2}, \qquad n > 2,$$

$$b_{n1} = -\frac{p_{n-1}}{\lambda p_0} \prod_{j=1}^{n-2} \left(1 - \frac{c_j}{\lambda}\right), \qquad n > 2,$$

$$b_{nk} = -\frac{p_{n-1}}{\lambda^2 P_{k-1}} \prod_{j=k}^{n-2} \left(1 - \frac{c_j}{\lambda}\right), \qquad 1 < k < n - 1,$$

and $b_{nk} = 0$ otherwise.

To show that $B \in B(l)$ it is sufficient to show that $\Sigma_n | b_{nk} |$ is finite, independent of k.

$\Sigma_n | b_{n0} = 1/| \lambda - 1 |$. We may write $1 - c_j/\lambda = 1 - p_j/\lambda P_j = (P_{j-1}/P_j)(1 + (1 - 1/\lambda)p_j/P_{j-1})$. Also, $\sup_{n>1} | p_{n-1}/P_{n-2} | \leq M < \infty$. Therefore

$$\sum_n |b_{n1}| \leq \frac{1}{|\lambda|}\left(M + M \sum_{n=3}^{\infty} \prod_{j=1}^{n-2} \left| 1 + \left(1 - \frac{1}{\lambda}\right) \frac{p_j}{P_{j-1}} \right| \right),$$

and, for $k > 1$,

$$\sum_n |b_{nk}| \le \frac{1}{|\lambda|} + \frac{M}{|\lambda|2} + \frac{M}{|\lambda|2} \sum_{n=k+2}^{\infty} \prod_{j=k}^{n-2} \left|1 + \left(1 - \frac{1}{\lambda}\right) \frac{p_j}{P_{j-1}}\right|.$$

Since $k > 1$, the series in the second inequality is dominated by the series in the first inequality which, from (3), is absolutely convergent. Therefore $\|B\|_1 < \infty$.

Since $(\lambda I - A)^{-1}$ is bounded, it is continuous, and $\lambda \in \mathrm{III}_1\sigma(A)$.

Case II. Suppose an infinite number of the p_k are zero. Since $\lim_n p_n = \infty$, there are an infinite number of nonzero p_k. Denote these by $\{p_{n_k}\}$.

From (2) for $n \neq 1 + n_k$, $x_n = 0$. Otherwise,

$$x_{1+n_k} = (1 - 1/\lambda) \frac{p_{n_k}}{P_{n_k - 1}} \prod_{j=1}^{r} \left(1 - (1 - 1/\lambda) \frac{p_{n_j}}{P_{n_j - 1}}\right).$$

Now apply the same analysis as in Case I to verify that $\Sigma |z_n|$ converges, and hence $\lambda I - A^*$ has nonzero solutions.

To show that $\lambda I - A^*$ is onto, the presence of an infinite number of $p_k = 0$ merely introduces more zero entries in B. For the non-zero entries, the same argument as Case I applies.

THEOREM 2. *Let A be a regular weighted mean method such that $\delta = \lim p_n/P_n$ exists, and $\delta < 1$. Suppose no diagonal entry of A occurs an infinite number of times. If $\lambda = \delta$, or $\lambda = a_{nn}$, $n = 1,2,3,\ldots$ and $\delta/(2-\delta) < \lambda < 1$, then $\lambda \in \mathrm{III}_1\sigma(A)$.*

Proof. First assume that A has distinct diagonal entries, and fix $j \ge 1$. Then the system $(a_{jj}I - A)x = 0$ implies $x_k = 0$ for $k = 0, 1, \ldots, j-1$, and, for $n \ge j$,

$$(a_{jj} - a_{nn})x_n - \sum_{k=0}^{n-1} a_{nk}x_k = 0.$$

The above system yields the recursion relation

$$x_{n+1} = p_j P_n x_n / P_j P_{n+1}(c_j - c_{n+1}),$$

which can be solved for x_n to yield

$$x_{j+m} = \frac{P_j x_j c_j^m}{P_{j+m} \prod_{i=1}^{m} (c_j - c_{j+i})} = x_j \prod_{i=1}^{m} \left(\frac{1 - c_{j+i}}{1 - c_{j+i}/c_j}\right). \tag{7}$$

Since $(1 - c_{j+i})/(1 - c_{j+i}/c_j) = (P_{j+i} - p_{j+i})/(P_{j+i} - p_{j+i}/c_j) = P_{j+i-1}/(P_{j+i-1} + (1 - 1/c_j)p_{j+i}) = (1 + (1 - 1/c_j)p_{j+i}/P_{j+i-1})^{-1}$,

$$x_{j+m} = x_j \Big/ \prod_{i=1}^{m} \left(1 + \left(1 - 1/c_j\right)p_{j+i}/P_{j+i-1}\right).$$

Since $0 < c_j < 1$, the argument of Theorem 1 implies that

$$\left| 1 + \left(1 - 1/c_j\right)p_{j+i}/P_{j+i-1} \right| \leq \xi < 1$$

for all i sufficiently large. Therefore $x \in c$ implies $x = 0$ and $a_{jj}I - A$ is 1-1, so that $c_jI - A \in 1 \cup 2$.

Clearly $c_jI - A \in \text{III}$. It remains to show that $c_jI - A^*$ is onto.

Suppose $(c_jI - A^*)x = y$, $x, y \in l$. By choosing $x_{j+1} = 0$ we can solve for $x_0, \cdots, x_j$ in terms of $y_0, \cdots, y_{j+1}$. As in Theorem 1, the remaining equations can be written in the form $x = By$, where the nonzero entries of B are

$$b_{j+m,j+m} = 1/c_j, \qquad m > 1; \tag{8}$$

$$b_{j+2,j+1} = -p_{j+1}/c_jp_j;\ b_{j+m,j+m-1} = -p_{j+m-1}/c_j^2P_{j+m-2}, \qquad m > 2;$$

$$b_{j+m,j+k} = -\frac{p_{j+m-1}}{c_j^2P_{j+k-1}} \prod_{i=j+k}^{j+m-2} \left(1 - \frac{c_i}{c_j}\right),\ 1 < k < m - 1, \qquad m > 2;$$

$$b_{j+m,j+1} = -\frac{p_{j+m-1}}{c_jp_j} \prod_{i=j+1}^{j+m-2} \left(1 - \frac{c_i}{c_j}\right), \qquad m > 2.$$

From (8),

$$\sum_{n=j+1}^{\infty} |b_{n,j+1}| = \frac{p_{j+1}}{c_jp_j} + \frac{1}{c_jp_j} \sum_{n=j+3}^{\infty} p_{n-1} \prod_{i=j+1}^{n-2} \left|1 - \frac{c_i}{c_j}\right|. \tag{9}$$

For $m > 1$,

$$\sum_{n=m+j}^{\infty} |b_{n,m+j}| = \frac{1}{c_j} + \frac{p_{j+m}}{c_j^2P_{j+m-1}} + \frac{1}{c_j 2} \sum_{n=m+j+2}^{\infty} \frac{p_{n-1}}{P_{j+m-1}} \prod_{i=j+m}^{n-2} \left|1 - \frac{c_i}{c_j}\right|.$$

Since p_{j+m}/P_{j+m-1} is bounded, and $p_j/P_{j+m-1} \leq 1$ for each $m > 1$, to show that $\|B\|_1$ is finite, it is sufficient to show that the series in (9) converges. We may write

$$\frac{p_{n-1}}{p_j} = \frac{p_{n-1}P_{n-2}P_{n-3}\cdots P_j}{P_{n-2}P_{n-3}\cdots P_jp_j} = \frac{p_{n-1}}{P_{n-2}} \cdot \frac{1}{\left(1 - c_{n-2}\right)\cdots\left(1 - c_{j+1}\right)c_j}.$$

Substituting in (9) the series then becomes

$$\frac{1}{c_j^2}\sum_{n=j+3}^{\infty}\frac{p_{n-1}}{P_{n-2}}\prod_{i=j+1}^{n-2}\left|\frac{1-c_i/c_j}{1-c_i}\right|.$$

Note that

$$(10)\qquad \frac{1-c_i/c_j}{1-c_i}=\frac{P_i-p_i/c_j}{P_i-p_i}=\frac{p_i}{P_{i-1}}+1-\frac{p_i}{c_jP_{i-1}}$$

$$=1+\left(1-\frac{1}{c_j}\right)\frac{p_i}{P_{i-1}}.$$

From the hypothesis on λ, (3) is satisfied for all i sufficiently large, and the series in (9) is absolutely convergent.

Suppose A does not have distinct diagonal entries. The restriction on λ guarantees that no zero diagonal entries are being considered. Let $c_j \neq 0$ be any diagonal entry which occurs more than once, and let k, r denote, respectively, the smallest and largest integers for which $c_j = c_k = c_r$. From (7) it follows that $x_n = 0$ for $n \geq r$. Also, $x_n = 0$ for $0 \leq n < k$. Therefore the system $(c_jI - A)x = 0$ becomes

$$(11)\qquad (c_j-c_n)x_n-\sum_{i=j}^{n-1}a_{ni}x_i=0,\qquad k<n\leq r.$$

Case I. $r = k + 1$. Then (11) reduces to the single equation

$$(c_j-c_{k+1})x_{j+1}-a_{k+1,k}x_k=0,$$

which implies $x_k = 0$, since $c_j = c_r = c_{k+1}$, and $p_j \neq 0$. Therefore $x = 0$.

Case II. $r > k + 1$. From (11) one obtains the recursion formula $x_n = P_{n+1}(c_j - c_{n+1})x_{n+1}/c_jP_n$, $k < n < r$. Since $x_d\text{r} = 0$ it then follows that $x_n = 0$ for $k < n < r$. Using (11) with $n = k + 1$ yields $x_k = 0$ and so again $x = 0$.

To show that $c_jI - A^*$ is onto, suppose $(c_jI - A^*)x = y$, x, $y \in l$. By choosing $x_{j+1} = 0$ we can solve for $x_0, x_1, \ldots, x_j$ in terms of $y_0, y_1, \ldots, y_{j+1}$. As in Theorem 1 the remaining equations can be written in the form $x = By$, where the entries of B are as in (8), with the other entries of B clearly zero.

Since $k \leq j \leq r$, there are two cases to consider.

Case I. $j = r$. Then the proof proceeds exactly as the argument following (8).

Case II. $j < r$. Then, from (8), $b_{j+m,j+k} = b_{j+m,j+1} = 0$ at least for $m \geq r - j + 2$. If there are other values of $n, j < n < r$ for which $c_n = c_j$, then additional entries of B will be zero. These zero entries do not affect the validity of the argument showing that (9) converges.

If $\delta = 0$, then 0 does not lie inside the disc, and so it is not considered in this theorem.

Let $\lambda = \delta > 0$. If $a_{nn} \neq \delta$ for each $n \geq 1$, all i sufficiently large, then the argument of Theorem 1 applies and $\delta I - A \in \mathrm{III}_1$. If $a_{nn} = \delta$ for some n, then the proof of Theorem 2 applies, with c_j replaced by δ, and, again, $\delta I - A \in \mathrm{III}_1$.

Therefore, in all cases, $c_j I - A \in 1 \cup 2$.

THEOREM 3. *Let A be a regular weighted mean method such that $\delta = \lim p_n/P_n$ exists and $p_n/P_n \geq \delta$ for all n sufficiently large. If λ satisfies $|\lambda - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)$, $\lambda \neq 1$, $\delta/(2-\delta)$, then $\lambda \in \mathrm{II}_2\sigma(A)$.*

Proof. Fix $\lambda \neq 1$, $\delta/(2-\delta)$, and satisfying $|\lambda - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)$. Since $\lambda I - A$ is a triangle, it is 1-1 and $\lambda I - A \in 1 \cup 2$.

Now consider $(\lambda I - A^*)x = 0$. As in Theorem 1, $x_0 = 0$, x_1 is arbitrary and $\{x_n\}$ satisfies (2) for all $n > 0$. From the hypothesis there exists a positive integer N such that $n \geq N$ implies $c_n \geq \delta$. This fact, together with the condition on λ, implies that $|1 + (1 - 1/\lambda)p_n/P_{n-1}| \geq 1$ for $n \geq N$. Thus $|x_n| \geq c p_{n-1}/P_{n-2}$ for $n \geq N$, where c is a constant independent of n. We may write

$$\begin{aligned} p_{n-1}/P_{n-2} &= (p_{n-1}/P_{n-1})(p_{n-1}/P_{n-2}) \\ &= (p_{n-1}/P_{n-1})(1 + p_{n-1}/P_{n-2}) \geq p_{n-1}/P_{n-1}. \end{aligned}$$

From [4, p. 290], $\Sigma p_n/P_n$ diverges, so $\{x_n\} \in l$ implies $x_1 = 0$, hence $x = 0$ and $\lambda I - A^* \in 1 \cup 2$. Since $\lambda \in \sigma(A)$, and $\lambda \neq 1$, $\delta/(2-\delta)$, $\lambda \in \mathrm{II}_2\sigma(A)$.

THEOREM 4. *Let A be a regular weighted mean method. Then $1 \in \mathrm{III}_3\sigma(A)$.*

Proof. Since $(I - A)e = 0$, $I - A$ is not 1-1 and hence $I - A \in 3$. It remains to show that $\overline{R(I - A)} \neq c$. Let $z \in c$ such that $z_0 \neq 0$. Then $\|(I - A)x - z\| \geq |z_0| > |z_0|/2$ for all $x \in c$. Therefore $z \notin \overline{R(I - A)}$.

THEOREM 5. *Let A be a regular weighted mean method with $\gamma = \liminf p_n/P_n$. If there exists values of n such that $0 \leq c_n \leq \gamma/(2-\gamma)$, then $\lambda = c_n$ implies $\lambda \in \mathrm{III}_3\sigma(A)$.*

Proof. Let c_k be any diagonal entry satisfying $0 < c_k \le \gamma/(2-\gamma)$. Let j be the smallest integer such that $c_j = c_k$. Since $c_0 = 1, j > 0$. By setting $x_n = 0$ for $n > j+1$, $x_0 = 0$, the system $(c_j I - A^*)x = 0$ reduces to a homogeneous linear system of j equations in $j+1$ unknowns, so that nontrivial solutions exist. Therefore $c_j I - A \in \text{III}$.

If $c_j = \gamma/(2-\gamma)$ then clearly $c_j I - A \in 3$. Assume $0 < c_j < \gamma/(2-\gamma)$ and let r denote the largest integer such that $c_r = c_k$. Solving $(c_r I - A)x = 0$ leads to (7) with $j = r$.

Pick $\varepsilon = \min\{\gamma(1-\gamma)/(2-\gamma)2,\ \gamma/2 - 1/(1+1/c_j)\}$. (Since $c_j < \gamma/(2-\gamma)$, $\gamma/2 - 1/(1+1/c_j) > 0$.) Choose N large enough so that, for $m \ge N$, $c_{m+j+1} > \gamma - \varepsilon$. From $c_j < \gamma/(2-\gamma)$ it follows that

$$c_{j+m+1}/c_j - 1 > ((2-\gamma)/\gamma)c_{j+m+1} - 1 > ((2-\gamma)/\gamma)(\gamma-\varepsilon) - 1 > 0$$

since $\varepsilon < \gamma(1-\gamma)/(2-\gamma)$.

For $m \ge N$, from (7),

$$\frac{|x_{j+m+1}|}{|x_{j+m}|} = \left|\frac{1-c_{j+m+1}}{1-\dfrac{c_{j+m+1}}{c_j}}\right| = \frac{1-c_{j+m+1}}{\dfrac{c_{j+m+1}}{c_j} - 1} < \frac{1-\gamma+\varepsilon}{\dfrac{\gamma-\varepsilon}{c_j} - 1} < 1,$$

since $\varepsilon < \gamma - 2/(1+1/c_j)$. Consequently $\{x_n\} \in l$, hence $\{x_n\} \in c$, and $c_j I - A$ is not 1-1.

Suppose A has a zero on the main diagonal and $\gamma > 0$. Let j denote the smallest positive integer for which $c_j = 0$. Let e^j denote the coordinate sequence with a 1 in the jth positive and all other entries zero. Then $Ae^j = 0$, and $c_j I - A = -A$ is not 1-1. By setting $x_0 = 0$, $x_n = 0$ for $n > j+1$, the system $(c_j I - A^*)x = 0$ reduces to a homogeneous linear system of j equations in $j+1$ unknowns.

When the diagonal entries of A do not converge, it was shown in **[2]** that the spectrum need no longer be a disc. This fact was illustrated by considering weighted mean methods with diagonal entries $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, $n > 0$, where $1 < p < q$. Under these conditions, $\sigma(A) = \{\lambda \mid (p-1)(q-1)|\lambda|^2 \ge |1-p\lambda||1-q\lambda|\}$. The boundary of the spectrum is either an oval, two ovals tangent at a point on the x-axis between 0 and 1, or two disjoint ovals, depending on the relative sizes of p and q. It will now be shown that the fine spectra of these methods behave exactly as the fine spectra for the weighted mean methods considered in Theorems 1–5.

THEOREM 6. *Let A be a regular weighted mean method defined by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, $n > 0$, where $1 < p < q$. If $\lambda \ne 1/p, 1/q, 1$ and satisfies $(p-1)(q-1)|\lambda|^2 > |1-p\lambda||1-q\lambda|$, then $\lambda \in \text{III}_1\sigma(A)$.*

Proof. Since $\lambda \neq 1/p, 1/q$, or 1, $\lambda I - A$ is a triangle, so it is 1-1, and $\lambda I - A \in 1 \cup 2$.

Suppose $(\lambda I - A^*)x = 0$. Then, as in the proof of Theorem 1, $x_0 = 0$, x_1 is arbitrary and, from (2),

$$x_{2n} = \frac{p_{2n-1}}{p_0} x_1 \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{p\lambda}\right)^{n-1}\left(1 - \frac{1}{q\lambda}\right)^{n-1},$$

$$x_{2n+1} = \frac{p_{2n}}{p_0} x_1 \left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{p\lambda}\right)^{n-1}\left(1 - \frac{1}{q\lambda}\right)^{n}.$$

From the hypotheses on A it follows that

$$p_{2n} = \frac{p_0 p^{n-1} q^n}{(p-1)^n (q-1)^n}, \quad p_{2n-1} = \frac{p_0 (pq)^{n-1}}{(p-1)^{n-1}(q-1)^n},$$

so that

$$\begin{aligned} \frac{|x_{2n+2}|}{|x_{2n}|} = \frac{|x_{2n+1}|}{|x_{2n-1}|} &= \frac{pq}{(p-1)(q-1)}\left|1 - \frac{1}{p\lambda}\right|\left|1 - \frac{1}{q\lambda}\right| \\ &= \frac{pq}{(p-1)(q-1)} \frac{|1 - p\lambda||1 - q\lambda|}{pq|\lambda|^2} \\ &< \frac{pq}{(p-1)(q-1)} \frac{|\lambda|^2 (p-1)(q-1)}{pq|\lambda|^2} = 1. \end{aligned}$$

Consequently there exist nonzero sequences $\{x_n\} \in l$ such that $(\lambda I - A^*)x = 0$ and $\lambda I - A^*$ is not 1-1. Therefore $\lambda I - A \in \mathrm{III}$.

It will now be shown that $\lambda I - A^*$ is onto. Suppose $y = (\lambda I - A^*)x$, $x, y \in l$. Then $(\lambda - 1)x_0 = y_0$ and (4) holds. Solving (4) for x in terms of y yields the matrix B, from $x = By$, with entries as described in the paragraph following (6). $\Sigma_n |b_{n0}| = 1/|\lambda - 1| < \infty$.

$$\sum_{n=1}^{\infty} |b_{n1}| = \frac{p_1}{p_0|\lambda|} + \sum_{n=3}^{\infty} \frac{p_{n-1}}{p_0|\lambda|} \prod_{j=1}^{n-2}\left|1 - \frac{c_j}{\lambda}\right|. \tag{12}$$

The series on the right can be written in the form $\Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{p_{2n}}{p_0|\lambda|} \prod_{j=1}^{2n-1} \left|1 - \frac{c_j}{\lambda}\right|,$$

$$\Sigma_2 = \sum_{n=2}^{\infty} \frac{p_{2n-1}}{p_0|\lambda|} \prod_{j=1}^{2n-2} \left|1 - \frac{c_j}{\lambda}\right|.$$

$$\Sigma_1 = \frac{1}{p_0|\lambda|} \sum_{n=1}^{\infty} \frac{p_0 p^{n-1} q^n}{(p-1)^n (q-1)^n} \left|1 - \frac{1}{p\lambda}\right|^{n-1} \left|1 - \frac{1}{q\lambda}\right|^n$$

$$= \frac{|q\lambda - 1|}{(p-1)(q-1)|\lambda|^2} \sum_{n=1}^{\infty} R^{n-1},$$

where $R = |p\lambda - 1||q\lambda - 1|/(p-1)(q-1)|\lambda|^2$.

Similarly,

$$\Sigma_2 = \frac{1}{(q-1)|\lambda|} \sum_{n=2}^{\infty} R^{n-1}.$$

From the hypothesis on λ, both series are convergent geometric series. For $k > 1$,

$$\sum_{n=k}^{\infty} |b_{nk}| = \frac{1}{|\lambda|}\left(1 + \frac{p_k}{|\lambda| P_{k-1}}\right) + \sum_{n=k+2}^{\infty} \frac{p_{n-1}}{|\lambda| P_{k-1}} \prod_{j=k}^{n-2} \left|1 - \frac{c_j}{\lambda}\right|.$$

Since the above series is dominated by the series in (12), $\|B\|_1 < \infty$.

THEOREM 7. *Let A be as in Theorem 6. If $\lambda = 1/p, 1/q$ then $\lambda \in$ $\mathrm{III}_1\sigma(A)$.*

Proof. Suppose $\lambda = 1/p$. Then $\lambda I - A$ maps x into $\{(1/p - 1)x_0$, $-a_{10}x_0 + (1/p - 1/q)x_1$, $-a_{20}x_0 - a_{21}x_1, -a_{30}x_0 - a_{31}x_1 - a_{32}x_2 + (1/p - 1/q)x_3, \cdots\}$, so that $(\lambda I - A)x = 0$ implies $x_0 = x_1 = 0$. By induction, one solves successively a pair of equations of the form

$$-a_{2n+1,2n}x_{2n} + (1/p - 1/q)x_{2n+1} = 0,$$
$$-a_{2n+2,2n}x_{2n} - a_{2n+2,2n+1}x_{2n+1} = 0,$$

whose only solution is $x_{2n} = x_{2n+1} = 0$, since the determinant of the coefficients is equal to $p_{2n}/pP_{2n+2} \neq 0$.

If $\lambda = 1/q$, then $\lambda I - A$ maps x into $\{(1/q - 1)x_0, -a_{10}x_0, -a_{20}x_0 - a_{21}x_1 + (1/q - 1/p)x_2, \cdots\}$, and $(\lambda I - A)x = 0$ forces $x_0 = 0$. Again one solves successively pairs of equations, this time of the form

$$-a_{2n+2,2n+1}x_{2n+1} + (1/q - 1/p)x_{2n+2} = 0,$$
$$-a_{2n+3,2n+1}x_{2n+1} - a_{2n+3,2n+2}x_{2n+2} = 0.$$

Since the determinant of the coefficients is equal to $p_{2n+1}/qP_{2n+3} \neq 0$, $x = 0$, and $\lambda I - A$ is 1-1. Clearly $\lambda I - A \in \mathrm{III}$, and it remains to show that $\lambda I - A^*$ is onto.

Suppose $(\lambda I - A^*)x = y$, x, $y \in l$. As in the proof of Theorem 2, choosing $x_{j+1} = 0$ we can solve for $x_0, \cdots, x_j$ in terms of $y_0, \cdots, y_{j+1}$, and the remaining values of x are determined from $x = By$, where B is as defined in (8). Since $c_{j+2} = c_j$ for each $j > 0$, it is clear from (8) that $b_{j+m,j+m-k} = b_{j+m,j+1} = 0$ for $k \geq 3$, $m \geq 4$. Also, $b_{j+m,j+m-2} = 0$ for m even. Consequently B has at most three nonzero diagonals, with bounded elements, and $B \in B(l)$.

THEOREM 8. *If A is defined as in Theorem 6, and λ satisfies*

$$(p-1)(q-1)|\lambda|^2 = |1 - p\lambda||1 - q\lambda|, \qquad \lambda \neq 1,$$

then $\lambda \in \mathrm{II}_2\sigma(A)$.

Proof. Since $\lambda I - A$ is a triangle, it is 1-1, so that $\lambda I - A \in 1 \cup 2$. Now consider $(\lambda I - A^*)x = 0$. Then, as in the proof of Theorem 1, $x_0 = 0$ and x_n satisfies (2) for $n > 0$. It then follows that

$$|x_{2n}| = \frac{|x_1|}{q-1}\left|1 - \frac{1}{\lambda}\right|$$

and

$$|x_{2n+1}| = \frac{|x_1||\lambda - 1|}{|p\lambda - 1|}.$$

Therefore $\{x_n\} \in l$ implies $x = 0$ and $\lambda I - A^* \in 1 \cup 2$. It then follows that $\lambda \in \mathrm{II}_2\sigma(A)$.

From Theorem 4 it follows that $1 \in \mathrm{III}_3\sigma(A)$.

Cartlidge [1] demonstrated that certain weighted mean methods belong to $B(l^p)$, $p \geq 1$, and computed their spectra. For example, he showed that, if $\delta = \lim p_n/P_n > 0$, then $A \in B(l^p)$ and

$$\sigma(A) = \left\{\lambda \,||\, \lambda - (2-\delta)^{-1}| \leq (1-\delta)/(2-\delta)\right\} \cup S.$$

It can be shown that the results of Theorems 1–5 are true for each such A.

Based on the results established in this paper, the following is a reasonable conjecture.

Let A be a weighted mean method, $A \in B(l^p)$, for some p satisfying $1 \leq p \leq \infty$. Then all interior points of $\sigma(A)$ belong to III_1, all boundary points, except 1, and possibly $\gamma/(2-\gamma)$, belong to II_2, and 1 and all isolated points belong to III_3. If $\gamma/(2-\gamma)$ is a diagonal element of A, then $\gamma/(2-\gamma) \in \mathrm{III}_3$. Otherwise $\gamma/(2-\gamma) \in \mathrm{II}_2$.

References

1. J. P. Cartlidge, *Weighted Mean Matrices as Operators on l^p*, Ph.D. Dissertation, Indiana University, 1978.
2. Frank P. Cass and B. E. Rhoades, *Mercerian theorems via spectral theory*, Pacific J. Math., **73** (1977), 63–71.
3. S. Goldberg, *Unbounded Linear Operators*, McGraw Hill, New York, 1966.
4. K. Knopp, *Theory and Application of Infinite Series*, Hafner Publishing Co., New York, 1971.
5. Robert B. Wenger, *The fine spectra of the Hölder summability operators*, Indian J. Pure and Appl. Math., **6** (1975), 695–712.

Received May 1, 1979 and in revised form January 29, 1982.

INDIANA UNIVERSITY
BLOOMINGTON, IN 47405