PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 1, 1983

A CHARACTERIZATION OF A NEUBERGER TYPE ITERATION PROCEDURE THAT LEADS TO SOLUTIONS OF CLASSICAL BOUNDARY VALUE PROBLEMS

THOMAS H. PATE

Recent papers of Neuberger and Pate have been concerned with iteration procedures for solving non-linear partial differential equations. Up to this time it has not been clear how to apply these methods to the solution of boundary value problems. In this paper we characterize the solutions obtained by applying one of these methods and then use this characterization to derive solutions to the classical boundary value problems.

Introduction. This paper is concerned with the Neuberger type iteration procedure for solving Partial Differential Equations that is analysed by Pate in [**5**] and [**6**]. In these papers as well as the papers [3], [7], and [**10**] of Neuberger's, general existence and convergence theorems were considered while little mention was made of the possible application of the Neuberger type iteration procedures to the solution of the classical boundary value problems.

In this paper we give alternate characterizations for some of the solutions that Neuberger obtains in [3]. These characterizations are then used along with the iteration procedure described by Pate in [7] and [**8**] to obtain analytic solutions to the classical boundary value problems that arise from the heat equation, the wave equation and Laplace's equation.

In [9] Neuberger comments that one might begin a study of boundary value problems by considering those characteristics of the initial estimate that are held invariant by the iteration procedure. The author has found this suggestion to be of value at least in the cases examined so far. In the cases considered here we begin our iteration with an initial estimate that satisfies the boundary conditions and show that these conditions are held fixed throughout the iteration, thus forcing our limit function to satisfy the boundary conditions. It seems that this approach offers a viable alternative to the usual methods used to solve linear problems.

If m and n are positive integers we let $S(m, n)$ denote the collection of all symmetric real-valued n-linear functions on E_m. If $A \in S(m, n)$ then we let

$$\|A\| = \left(\sum_{p_1=1}^{m} \sum_{p_2=1}^{m} \cdots \sum_{p_n=1}^{m} \left(A\left(e_{p_1}, e_{p_2}, \ldots, e_{p_n}\right)\right)^2 \right)^{1/2}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal basis for E_m (see [3], [4], and [5]). This norm is basis independent and generates an inner product on $S(m, n)$. If $A, B \in S(m, n)$ then we let $\langle A, B \rangle$ or $Ad\mathrm{B}$ denote the inner product of A and B. If U is a real-valued function from some open set $D \in E_m$ then $U^{(p)}(x)$ denotes the pth Fréchet derivative of U at x. As long as U is p times continuously differentiable $U^{(p)}$ is a continuous function from D to $S(m, p)$. We let $H_r(0)$ denote the collection of all infinitely differentiable functions f such that

$$f(x) = \sum_{p=0}^{\infty} \frac{1}{p!} f^{(p)}(0)x^p \quad \text{for } \|x\| < r, \tag{1}$$

$$\sum_{p=0}^{\infty} \frac{1}{p!} \|f^{(p)}(0)\| s^p < \infty \quad \text{for } 0 \leq s < r. \tag{2}$$

Here $f^{(p)}(0)x^p$ denotes $f^{(p)}(0)(\overbrace{x, x, \ldots, x}^{p})$ or $\langle f^{(p)}(0), x^p \rangle$ where x^p is the member of $S(m, p)$ such that $x^p(y_1, y_2, \ldots, y_p) = \prod_{i=1}^{p} \langle x, y_i \rangle$. The summation (2) is denoted by $\|f\|_s$. Clearly $H_r(0)$ is a vector space over R. It is within this space of analytic functions or within $H_r(\alpha)$ for some $\alpha \in E_m$, $\alpha \neq 0$ that the iterations were carried out in [3], [7], [8], and [9].

If $A \in S(m, n)$ and $B \in S(m, p)$ then $A \cdot B$ denotes the symmetric product of A and B—i.e., $A \cdot B$ is the member of $S(m, p + n)$ that is closest to $A \otimes B$, the tensor product of A and B (see [3], [4] or [5]). If $A \in S(m, n)$ while $B \in S(m, p)$ where $p > n$ then AB denotes the member of $S(m, p - n)$ such that

$$AB(y_1, y_2, \ldots, y_{p-n}) = \langle A, B(y_1, y_2, \ldots, y_{p-n}) \rangle$$

where $B(y_1, y_2, \ldots, y_{p-n})$ denotes the member of $S(m, n)$ that is obtained by filling the first $p - n$ places of B with the y_i's.

For a fixed $A \in S(m, n)$ we define the operators M_p: $S(m, p) \to S(m, p)$ and G_p: $S(m, p) \to S(m, p)$ as follows: $M_p(B) = A(A \cdot B)$ and $G_p(B) = A \cdot (AB)$. We define G_p only in case $p \geq n$. These operators are crucial to the iteration procedure as presented in [3], [7], [8], and [9] and are the reason for the papers [4], [5], and [6] which deal primarily with estimates for $\|M_p^{-1}\|$.

We describe briefly the iteration procedure. We begin with a partial differential equation of the form $AU^{(n)}(x) = G(x)$ where $U, G \in H_r(0)$. Suppose we take $Q \in H_r(0)$ as our initial estimate. We have

$$Q(x) = \sum_{p=0}^{n-1} \frac{1}{p!} Q^{(p)}(0)x^p + \int_0^1 \frac{(1-j)^{n-1}}{(n-1)!} Q^{(n)}(jx)x^n \, dj$$

for $\|x\| < r$ by the Taylor formula. To find a member of $H_r(0)$ that is nearer to being a solution we replace $Q^{(n)}(jx)$ in the above formula with

$w_j(x)$ where $w_j(x)$ is the member of $S(m, n)$ that is closest to $Q^{(n)}(jx)$ among those members B of $S(m, n)$ with the property $\langle A, B\rangle = G(jx)$. A calculation reveals that $w_j(x) = Q^{(n)}(jx) - (\langle A, Q^{(n)}(jx)\rangle - G(jx))A$. We define the operator T_G on $H_r(0)$ so that

$$T_G(Q)(x) = \sum_{p-0}^{n-1} \frac{1}{p!} Q^{(p)}(0)x^p + \int_0^1 \frac{(1-j)^{n-1}}{(n-1)!} w_j(x)x^n\, dj.$$

In [3] Neuberger shows that T_G is a linear transformation from $H_r(0)$ to $H_r(0)$. Furthermore, he shows that $AU^{(n)}(x) = G(x)$ for each x with $\|x\| < r$ if and only if $T_G(U) = U$. Thus as would be expected the solutions for our P.D.E. are merely the fixed points of T_G. Also, in [3], the series representation

$$T_G(U)(x) = \sum_{p=0}^{n-1} \frac{1}{p!} U^{(p)}(0)x^p + \sum_{p=n}^{\infty} \frac{1}{p!}(I - G_p)\big(U^{(p)}(0)\big)x^p$$

is derived and it is proven furthermore that for each $U \in H_r(u)$ the sequence $\{T_G^k(U)\}_{k=1}^{\infty}$ converges $\|\cdot\|_s$ for s with $0 \le s < r$ to a member F of $H_r(0)$ such that $T_G(F) \equiv F$. We denote this solution F by $\Phi(U, G)$. The series representation for $\Phi(U, G)$ is

$$\sum_{p=n}^{n-1} \frac{1}{p!} U^{(p)}(0)x^p + \sum_{p=n}^{\infty} \frac{1}{p!} L_p\big(U^{(p)}(0)\big)x^p + \sum_{p=n}^{\infty} \frac{1}{p!} A \cdot M_{p-n}^{-1}\big(G^{(p-n)}(0)\big)x^p$$

where L_p denotes the orthogonal projection of $S(m, p)$ onto the subspace for $S(m, p)$ that consists of those functions that are orthogonal to A. A formula for $L_p(B)$ is $B - A \cdot M_{p-n}^{-1}(AB)$. The first two summations above define a function which we denote by H_U. This H_U is the solution to the homogeneous equation $AU^{(n)}(x) = 0$ that is generated by our procedure. The last term we denote by $\mathcal{G}(G)$. The function $\mathcal{G}(G)$ is a particular solution to $AU^{(n)}(x) = G(x)$, more particularly it is the solution one obtains by beginning the iteration with the zero function. Henceforth $\mathcal{G}(G)$ will be called the primary solution. Of course H_U and $\mathcal{G}(G)$ are linear in U and G and produce members of $H_r(0)$.

We are now in a position to describe the procedure for the general constant coefficient linear partial differential equation of order n. Such an equation may be written

$$AU^{(n)}(x) = \sum_{i=0}^{n} B_i U^{(n-i)}(x) + F(x)$$

where $A \in S(m, n)$ and $B_i \in S(m, n-i)$ for $0 \leq i \leq n$ and $F \in H_r(0)$. Here we allow some terms of order n on the right-hand side whereas in the previous papers (see [7], [8], and [9]) only lower order derivatives were allowed there. We define the differential operator D: $H_r(0) \to H_r(0)$ such that if $U \in H_r(0)$ then $D(U)(x) = \sum_{i=1}^{n} B_i U^{(n-i)}(x)$. Our equation now takes the simpler form $AU^{(n)}(x) = D(U)(x) + F(x)$. To begin the iteration we choose an initial estimate $Q \in H_r(0)$. Now using the procedure described we solve $AU^{(n)}(x) = D(Q)(x) + F(x)$ using Q as initial estimate. We obtain $U(x) = H_Q(x) + (\mathcal{G} \circ D)(Q)(x) + \mathcal{G}(F)(x)$. We note that $H_U = 0$ (see [7], [8], or [9]) as can be seen by direct substitution into the series for H_U. Denoting U by U_1 we now repeat the procedure. We solve $AU^{(n)}(x) = D(U_1)(x) + F(x)$ obtaining

$$\begin{aligned} U(x) &= H_{u_1}(x) + (\mathcal{G} \circ D)(U_1)(x) + \mathcal{G}(F)(x) \\ &= H_Q(x) + \mathcal{G}(F)(x) + (\mathcal{G} \circ D)(H_2 + \mathcal{G}(F))(x) \\ &\quad + (\mathcal{G} \circ D)^2(Q)(x). \end{aligned}$$

We denote this function by $U_2(x)$. Continuing in this manner we generate a sequence $\{U_k\}_{k=1}^{\infty}$ of members of $H_r(0)$ where

$$U_u(x) = \sum_{p=0}^{k-1} (\mathcal{G} \circ D)^p (H_Q + \mathcal{G}(F))(x) + (\mathcal{G} \circ D)^k (Q)(x).$$

Allowing $p \to \infty$ we obtain ideally $\sum_{p=0}^{\infty} (\mathcal{G} \circ D)^p (H_Q + \mathcal{G}(F))(x)$ which is easily seen to be a formal solution to our problem. This series will be used to develop solutions to the above mentioned boundary value problems.

Our first theorem presents a much needed formula for $A \cdot M_p^{-1}(B)$ for $B \in S(m, p)$ where A is of special form — i.e., $A = y^n$ for some $y \in E_m$. Theorem 1 is used to derive Theorem 2 wherein we present a characterization of the primary solution one obtains if one applies the Neuberger procedure to the equation

$$\frac{\partial^n U}{\partial t^n}(x_1, x_2, \ldots, x_n, t) = G(x_1, \ldots, x_m, t).$$

This characterization is then used along with the series above to develop solutions to the three basic boundary value problems described previously.

THEOREM 1. *If $y \in E_m$ and we let $A = y^n$ then the operator M_p is invertible and*

$$M_p^{-1}(B) = \sum_{j=0}^{p} (-1)^j \cdot \binom{n+p}{n+j} \cdot \binom{n+j-1}{n-1} \cdot y^j \cdot B(y^j).$$

We note that M_p is invertible for arbitrary A so long as $A \neq 0$. To prove that M_p^{-1} has the form above in the special case $A = y^n$ we need the following binomial identity: if $1 \leq k \leq n$ then

$$\sum_{j=0}^{k} (-1)^j \cdot \binom{n+j-1}{j} \cdot \binom{n}{k-j} = 0,$$

and if $n + 1 \leq k$ then

$$\sum_{j=k-n}^{k} (-1)^j \binom{n+j-1}{j} \cdot \binom{n}{k-j} = 0.$$

In case $k = 0$ the above summations reduce to 1. Proofs for these identities are not included but are easily constructed.

Proof of Theorem 1. Let $A = y^n$, then

$$\begin{aligned}
&y^n\left(y^n \cdot \left\{\sum_{j=0}^{p} (-1)^j \binom{n+p}{n+j} \cdot \binom{n+j-1}{n-1} y^j \cdot B(y^j)\right\}\right) \\
&\quad = \sum_{j=0}^{p} (-1)^j \cdot \binom{n+p}{n+j} \cdot \binom{n+j-1}{n-1} \cdot y^n(y^{n+j} \cdot B(y^j)) \\
&\quad = \sum_{j=0}^{p} (-1)\binom{n+p}{n+j}\binom{n+j-1}{n-1} \\
&\qquad \times\left\{\sum_{i=0}^{\inf(n,p-j)} \binom{n+j}{n-i}\binom{p-j}{i}\binom{n+p}{n}^{-1} y^{i+j} \cdot B(y^{i+j})\right\} \\
&\qquad + \sum_{j=0}^{p-n} \sum_{i=0}^{n} (-1)^j \binom{n+p}{n}^{-1} \binom{n+p}{n+j}\binom{n+j-1}{n-1}\binom{n+j}{n-1}\binom{p-j}{i} \\
&\qquad \times y^{i+j} \cdot B(y^{i+j}) \\
&\quad = \sum_{j=p-n}^{p} \sum_{i=0}^{p-j} (-1)^j \binom{n+p}{n}^{-1} \binom{n+p}{n+j}\binom{n+j-1}{n-1}\binom{n+j}{n-1}\binom{p-j}{i} \\
&\qquad \times y^{i+j} \cdot B(y^{i+j}).
\end{aligned}$$

We make the change of variables $i + j = k$ and eliminate i thus obtaining

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{n+p}{n}^{-1} \binom{n+p}{n+j} \binom{n+j-1}{n-1} \times \binom{n+j}{n-k+j} \binom{p-j}{k-j} y^k \cdot B(y^k)$$
$$+ \sum_{k=n+1}^{p} \sum_{j=k-n}^{k} (-1)^j \binom{n+p}{n}^{-1} \binom{n+p}{n+j} \binom{n+j-1}{n-1} \times \binom{n+j}{n-k+j} \binom{p-j}{k-j} y^k \cdot B(y^k)$$
$$= \sum_{k=0}^{n} \binom{p}{k} y^k \cdot B(y^k) \left\{ \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} \binom{n+j-1}{j} \right\}$$
$$+ \sum_{k=n+1}^{p} \binom{p}{k} y^k \cdot B(y^k) \left\{ \sum_{j=k-n}^{k} (-1)^j \binom{n}{k-j} \binom{n+j-1}{j} \right\}.$$

But the terms in the braces are zero unless $k = 0$ in which case we get only B. This proves Theorem 1.

This formula for M_p^{-1} is important for reasons other than the proof of the following theorem: it has been useful in various attempts to prove that the Cauchy-Kowaleska theorem is obtainable if one "correctly" applies the iteration procedure presented in this paper.

LEMMA. *Suppose $F \in H_r(0)$. The primary solution to the equation*

$$\frac{\partial^n U}{\partial x_i^n}(x_1, \ldots, x_i, \ldots, x_m) = F(x_1, \ldots, x_i, \ldots, x_m)$$

is the function Q where

$$Q(x_1, x_2, \ldots, x_i, \ldots, x_m) = \int_0^1 \frac{(1-j)^{n-1}}{(n-1)!} F(x_1, \ldots, jx_i, \ldots, x_m) x_i^n \, d_j.$$

Again we need a binomial identity. This time it is

$$\binom{p+n}{i} = \sum_{j=0}^{p-i} (-1)^j \binom{p+n}{n+j} \binom{n+j-1}{j} \binom{p-j}{i}.$$

An induction proof is easily constructed (induction on n).

Proof of Lemma. We consider only the case $m = 2$ and denote an arbitrary vector by $\binom{x}{y} = xe_1 + ye_2$ where $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. We let the x_i in the theorem by y and $A = y^n$, so that the equation is of the form $AU^{(n)}(x, y) = F(x, y)$. The cases for $m > 2$ are identical except for the amount of writing required. We know from Theorem 1 that

$$A \cdot M_p^{-1}(B) = \sum_{j=0}^{p} (-1)\binom{n+p}{n+j}\binom{n+j-1}{j} e_2^j \cdot B(e_2^j)$$

our primary solution to $AU^{(n)} \equiv F$ is

$$\sum_{p=0}^{\infty} ((p+n)!)^{-1} A \cdot M_p^{-1}(F^{(p)}(0))\binom{x}{y}^p.$$

Hence by substitution of the formula for $A \cdot M_p^{-1}(F^{(p)}(0))$ we have

$$\sum_{p=0}^{\infty} ((p+n)!)^{-1}\left\{\sum_{j=0}^{p} (-1)^j \binom{n+p}{n+j}\binom{n+j-1}{j}\right.$$
$$\left.\times y^{j+n} F^{(p)}(0)\left(e_2^j, \binom{x}{y}^{p-j}\right)\right\}$$
$$= \sum_{p=0}^{\infty} ((p+n)!)\left\{\sum_{j=0}^{p}\sum_{i=0}^{p-j} (-1)^j \binom{n+p}{n+j}\binom{n+j-1}{j}\binom{p-j}{i}\right.$$
$$\left.\times y^{p+n-i} x^i F^{(p)}(0)(e_1^i, e_2^{p-i})\right\}$$
$$= \sum_{p=0}^{\infty} ((p+n)!)^{-1}\left\{\sum_{i=0}^{p} x^i y^{p+n-i} F^{(p)}(0)(e_1^i, e_2^{p-i})\right.$$
$$\left.\times\left[\sum_{j=0}^{p-i} (-1)^j \binom{n+p}{n+j}\binom{n+j-1}{j}\binom{p-j}{i}\right]\right\}$$
$$= \sum_{p=0}^{\infty} ((p+n)!)^{-1}\left\{\sum_{i=0}^{p} \binom{p+n}{i} x^i y^{p+n-i} F^{(p)}(0)(e_1^i, e_2^{p-i})\right\}$$

where we have used our binomial identity.

We now begin reductions on the other side of our identity. We claim the primary solution to $AU^{(n)} = F$ is simply $\int_0^1((n-1)!)^{b11}F(x, jy)y^n\, dj$.

Since $F(x, y) + \sum_{p=0}^{\infty}(p!)^{-1}F^{(p)}(0)\binom{x}{y}^p$ we have

$$F(x, jy) = \sum_{p=0}^{\infty} (p!)^{-1}F^{(p)}(0)(xe_1 + jye_2)^p$$

$$= \sum_{p=0}^{\infty} (p!)^{-1}F^{(p)}(0)\left\{\sum_{i=0}^{p}\binom{p}{i}\binom{x}{0}^i \cdot \binom{0}{y}^{p-i} j^{p-i}\right\}$$

$$= \sum_{p=0}^{\infty}\sum_{i=0}^{p}(p!)^{-1}\binom{p}{i}x^i y^{p-i} j^{p-i} F^{(p)}(0)\left(e_1^i, e_2^{p-i}\right).$$

Now, multiplying by $(1-j)^{n-1}((n-1)!)^{-1}y^n$ and integrating we obtain

$$\sum_{p=0}^{\infty}\sum_{i=0}^{p}(p!)^{-1}\binom{p}{i}x^i y^{p-i} F^{(p)}(0)\left(e_1^i, e_2^{p-i}\right)$$

$$\times \int_0^1 ((n-1)!)^{-1}(1-j)^{n-1} j^{p-i} y^n \, dj.$$

Now, recognizing that

$$\int_0^1 (1-j)^{n-1} j^{p-i}\, dy = (n-1)!\,(p-i)!\,((p+n-i)!)^{-1}.$$

If we substitute this in our last summation above and rearrange the factorial terms we obtain

$$\sum_{p=0}^{\infty}((p+n)!)^{-1}\left\{\sum_{i=0}^{p}\binom{p+n}{i}x^i y^{p+n-i}F^{(p)}(0)\left(e_1^i, e_2^{p-i}\right)\right\}$$

which is identical to the expression we obtained by reducing the Neuberger solution. This completes the proof of the lemma.

THEOREM 2. *Suppose $F \in H_r(0)$. If one uses the Neuberger iteration procedure as presented in* [3] *to solve the equation*

$$\frac{\partial^n U}{\partial x_i^n}(x_1,\ldots,x_m) = F(x_1,\ldots,x_m)$$

beginning with initial estimate $Q \in H_r(0)$ then the solution obtained is

$$\sum_{k=0}^{n-1}(h!)^{-1}x_i^k Q_k(x_1,\ldots,x_m) + \int_0^1 \frac{(1-j)^{n-1}}{(n-1)!}F(x_1,\ldots,jx_i,\ldots,x_m)x_i^n\, dj$$

where Q_k denotes $\partial^k Q/\partial x_i^k$ with a zero placed in the ith position.

Proof. The second term above is (F) by the lemma, hence we merely need to show that the first term H_Q. Again we restrict our attention to the case $m = 2$ is $\partial^n U/\partial y^n = F(x, y)$. If $B \in S(2, p)$ where $p \geq n$ then $B = \sum_{j=0}^{p} \binom{p}{j} e_2^j \cdot e_1^{p-j} B(e_2^j, e_1^{p-j})$ hence it is easily seen that $L_p(B) = \sum_{j=0}^{n-1} \binom{p}{j} B(e_2^j, e_1^{p-j}) e_2^j \cdot e_1^{p-j}$. In case $m > 2$ the formulae are more complicated but the principles are the same (see [**6**]). Now,

$$\frac{\partial^k Q}{\partial y^k}(x, y) = \sum_{p=k}^{\infty} ((p-k)!)^{-1} Q^{(p)}(0)\left(e_2^k, \binom{x}{y}^{p-k}\right)$$

so that

$$\begin{aligned}\sum_{k=0}^{n-1} (k!)^{-1} y^k Q_k(x) &= \sum_{k=0}^{n-1} \sum_{p=k}^{\infty} (k!\,(p-k)!)^{-1} Q^{(p)}(0)(e_2^k, e_1^{p-k}) y^k x^{p-k} \\ &= \sum_{p=0}^{n-1} (p!)^{-1} \left\{ \sum_{k=0}^{p} \binom{p}{k} Q^{(p)}(0)(e_2^k, e_1^{p-k}) e_2^k \cdot e_1^{p-k} \right\} \binom{x}{y}^p \\ &\quad + \sum_{p=n}^{\infty} (p!)^{-1} \left\{ \sum_{h=0}^{n-1} \binom{p}{k} Q^{(p)}(0)(e_2^k, e_1^{p-k}) e_2^k \cdot e_1^{p-k} \right\} \binom{x}{y}^p \\ &= \sum_{p=0}^{n-1} (p!)^{-1} Q^{(p)}(0) \binom{x}{y}^p + \sum_{p=n}^{\infty} (p!)^{-1} L_p(Q^{(p)}(0)) \binom{x}{y}^p.\end{aligned}$$

This is precisely the expression for H_Q as given in the introduction. This completes the proof of Theorem 2.

We note at this point that the solution to the equation $\partial^n U/\partial x_i^n = F(x_1, \ldots, x_m)$ obtained by applying the Neuberger iteration procedure as presented in [**3**] agrees with the initial estimate for $x_i = 0$. Also, each of the first $n - 1$ partial derivatives of our solution with respect to x_i is the same as the corresponding partial derivative of the initial estimate when $x_i = 0$.

We illustrate the usefulness of Theorem 2 by showing that if the initial estimates are chosen properly then the Neuberger type iteration procedure generates solutions to three of the classical boundary value problems.

EXAMPLE 1. The wave equation. We solve the following boundary value problem

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x^2}, \quad 0 < x < \pi, t > 0$$

(1) $$F(x, 0) = g(x)$$

(2) $$F_t(x, 0) = h(x)$$

(3) $$F(0, t) = F(\pi, t) = 0.$$

We assume that g and h are entire periodic functions of period 2π, and as is the custom we assume that g and h are odd. The differential operator $D = \partial^2/\partial x^2$. The primary solution of $\partial^2 U/\partial t^2 = G(x, t)$ is $U(x, t) = \int_0^t (t - s)G(x, s)\, ds$. We know from Theorem 2 that the iteration procedure for the equation $U_{tt} = G$ leaves the values of the initial estimate as well as its first partial with respect to t fixed along the x-axis. Thus we simply choose $Q(x, t) = g(x) + th(x)$ as our initial estimate. If we solve $\partial^2 F/\partial t^2 = \partial^2 Q/\partial x^2$ using Q as our initial estimate we obtain $F(x, t) = Q(x, t) + \int_0^t (t - s)Q_{xx}(x, s)\, ds$. We let $F_1 = F$ and repeat the procedure: We solve $\partial^2 F/\partial t^2 = \partial^2 F_1/\partial x^2$ using F_1 as our initial estimate. We obtain

$$F_2(x, t) = Q(x, t) + \int_0^t (t - s)\frac{\partial^2 Q}{\partial x^2}(x, s)\, ds + \int_0^t (3!)^{-1}(t - s)^3 \frac{\partial^4 Q}{\partial x^4}(x, s)\, ds.$$

Continuing in this manner we obtain a sequence of analytic functions $F_1, F_2, F_3, \cdots$ each of which satisfies the boundary conditions. Furthermore, we have $F_p(x, t) = \sum_{q=0}^{p} \int_0^t ((2q - 1)!)^{-1}(\partial^{2q} Q/\partial x^{2q})(x, s)\, ds$ and since $Q(x, t) = q(x) + th(x)$ we have

$$\begin{aligned} F_p(x, t) &= \sum_{q=0}^{p} \int_0^t ((2q - 1)!)^{-1}(t - s)^{2q-1}\{q^{(2q)}(x) + sh^{(2q)}(x)\}\, ds \\ &= \sum_{q=0}^{p} ((2q)!)^{-1} t^{2q} g^{(2q)}(x) + \sum_{q=0}^{p} ((2q + 1)!)^{-1} t^{2q+1} h^{(2q)}(x). \end{aligned}$$

Observe that since g and h are odd periodic functions of period 2π we must have $g^{(2q)}(n\pi) = h^{(2q)}(n\pi) = 0$ for n an integer. Thus $F_p(0, t) = F_p(\pi, t) = 0$ and each of the functions F_p satisfies all of the boundary conditions. Now, letting $p \to \infty$ we obtain the series

$$\sum_{q=0}^{\infty} \{((2q)!)^{-1} t^{2q} g^{(2q)}(x) + ((2q + 1)!)^{-1} t^{2q+1} h^{(2q)}(x)\}$$

which we denote by $F(x, t)$. The function is at least a formal solution to the wave equation as is easily verified by term by term differentiations. The boundary conditions are also trivially satisfied as long as one assumes that the series is convergent. Since g and h are entire we know that for each $r > 0$ there is $M_r > 0$ such that

$$| g^{(m)}(x) | \leq M_r \cdot r \cdot m! | (r - x) |^{-(m+1)}$$

and

$$| h^{(m)}(x) | \leq M_r \cdot r \cdot m! | (r - x) |^{-(m+1)}$$

for each x such that $|x| < r$. These estimates guarantee the uniform convergence on compact subsets of R^2 of the series as well as all series obtained from it by term by term differentiations. Thus $F(x, t)$ is a solution of our boundary value problem. If one expands $g^{(2g)}(t)$ and $h^{(2q)}(t)$ in power series and interchanges the order of summation one obtains $F(x, t) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$. This is the D'Alembert solution of the wave equation.

EXAMPLE 2. The heat equation. We solve boundary value problem

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial F}{\partial t}, \qquad t > 0, 0 < x < \pi$$

$$F(0, t) = F(\pi, t) = 0 \tag{1}$$

$$F(x, 0) = g(x). \tag{2}$$

We assume that g is odd, entire, and has period 2π. The iteration procedure does not lend itself to this problem directly. We solve first the Cauchy type problem

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial F}{\partial t}, \qquad t > 0, 0 < x < \pi$$

$$F(0, t) = \delta(t) \tag{$(1)^1$}$$

$$F_x(0, t) = \gamma(t) \tag{$(2)^1$}$$

the series

$$Q(x, t) + \sum_{p=1}^{\infty} \int_0^x ((2p-1)!)^{-1}(x-s)^{2p-1} \frac{\partial^p Q}{\partial t^p}(s, t)\, ds$$
$$= \sum_{p=0}^{\infty} \left\{((2p)!)^{-1} x^{2p} \delta^{(p)}(t) + ((2p+1)!)^{-1} x^{2p+1} \gamma^{(p)}(t)\right\}.$$

Term by term differentiations show that F is a formal solution of the heat equation. Clearly, $F(0, t) = \delta(t)$ and $F_x(0, t) = \gamma(t)$ as long as the appropriate series are assumed to be convergent. For $t_0 > 0$ there is a constant M such that $|\delta^{(p)}(t)| \leq M^p \cdot p!$ and $|\gamma^{(p)}(t)| \leq M^p p!$ for $|t| \leq t_0$. These estimates guarantee the uniform convergence on compact subsets of R^2 of the series as well as all series obtained from it by term by term differentiations. Thus F is a legitimate solution of our Cauchy type problem.

We return to the original problem. Since $F(0, t) = \delta(t)$ we must have $\delta = 0$. Since we require $F(x, 0) = \sum_{p=0}^{\infty}((2p+1)!)^{-1} x^{2p+1} \gamma^{(p)}(0) = g(x)$

it is clear that we must take $\gamma^{(p)}(0) = g^{(2p+1)}(0)$ so that

$$\gamma(t) = \sum_{p=0}^{\infty} (p!)^{-1} g^{(2p+1)}(0) t^p.$$

To insure the convergence of this series we assume that there is for each $t_0 > 0$ a constant $M > 0$ such that $|g^{(2p+1)}(0)| \leq M \cdot p!(t_0)^{-2p-1}$. This assumption is not really very restrictive since any function with a finite Fourier series is of this type and all functions that are periodic analytic are uniform limits of their Fourier series.

With δ and γ thus defined we have a solution to the heat equation that also satisfied (2) and is zero when $x = 0$. Now,

$$F(\pi, t) = \sum_{q=0}^{\infty} ((2q+1)!)^{-1} \pi^{q+1} \gamma^{(q)}(t).$$

If $t = 0$ then $F(\pi, t) = g(\pi) = 0$. Let $\phi(t) = F(\pi, t)$. Then

$$\phi^{(k)}(t) = \sum_{q=0}^{\infty} ((2q+1)!)^{-1} \pi^{2q+1} \gamma^{(q+k)}(t)$$

and

$$\phi^{(k)}(0) = \sum_{q=0}^{\infty} ((2q+1)!)^{-1} \pi^{2q+1} g^{(2q+2k+1)}(0) = g^{(2k)}(\pi) = 0.$$

Hence, since ϕ is analytic we must have $F(\pi, t) = 0$ for each t. Thus F is a legitimate solution to our boundary value problem. The same procedure is successful in solving the boundary value problem that arises by replacing (1) by $(\partial F/\partial x)(0, t) = (\partial F/\partial x)(\pi, t) = 0$.

EXAMPLE 3. Laplace's Equation. We solve the boundary value problem

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0, \qquad 0 < x < \pi, 0 < y < p$$

(1) $$F(x, 0) = g(x)$$

(2) $$F(0, y) = F(\pi, y) = 0$$

(3) $$F(x, p) = 0$$

where g is an odd entire function of period 2π. We proceed as in Example 2. Let δ and γ denote entire functions of the variable y. We solve the alternative problem

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0, \qquad -\infty < x < \infty, -\infty < y < \infty$$

$$F(0, y) = \delta(y)$$

$$F_x(0, y) = \gamma(y)$$

and then show that δ and γ may be chosen so that the solution to our alternative problem is also a solution to our original problem. The primary solution for $\partial^2 U/\partial x^2 = G$ is $U(x, y) = \int_0^x (x - s)G(s, y)\, ds$ and the operator $D = -\partial^2/\partial y^2$. As before we let $Q(x, y) = \delta(y) + x\gamma(y)$. Applying the procedure as in the previous examples we obtain the series

$$Q(x, y) + \sum_{q=1}^{\infty} (-1)^q \int_0^x ((2q-1)!)^{-1}(x-s)^{2q-1} \frac{\partial^{2q} Q}{\partial y^{2q}}(s, y)\, ds$$

$$= \sum_{q=0}^{\infty} (-1)^q \left\{((2q)!)^{-1} x^{2q} \delta^{(2q)}(y) + ((2q+1)!)^{-1} x^{2q+1} \gamma^{(2q)}(y)\right\}.$$

We denote this last series by $F(x, y)$. Again our assumptions on δ and γ guarantee the uniform convergence of the series on compact subsets of R^2 and it is again permissable to differentiate term by The boundary conditions are easily seen to be satisfied. This completes the solution of the alternative problem.

Now, we want $F(0, y) = \delta(y) = 0$ so δ must the zero function. Also, we want $F(x, 0) = g(x) = \Sigma_{q=0}^{\infty}(-1)^q((2q+1)!)^{-1}x^{2q+1}\gamma^{(2q)}(0)$. This can be accomplished by choosing $\gamma^{(2q)}(0) = (-1)^q g^{(2q+1)}(0)$. We still must choose the odd order derivatives of γ at 0. We hope to be able to do this so that $f(x, p) = 0$. This is easily accomplished by choosing $\gamma^{(2q+1)}(0) = (-1)^q p^{-1} g^{(2q+1)}(0)(2q+1)$ so that

$$\gamma(y) = \sum_{q=0}^{\infty} (-1)^q g^{(2q+1)}(0)((2q)!)^{-1} y^{2q}$$

$$+ \sum_{q=0}^{\infty} (-1)^{q+1} p^{-1}((2q)!)^{-1} g^{(2q+1)}(0) y^{2q+1}$$

$$= (1 - yp^{-1}) \sum_{q=0}^{\infty} (-1)^q ((2q)!)^{-1} g^{(2q+1)}(0) y^{2q}.$$

With γ defined in this way we have $\gamma^{(2q)}(p) = 0$ for each q. We are left with the condition $F(\pi, y) = 0$. This can be accomplished by showing that if $\phi(y) = F(\pi, y)$ then $\phi^{(k)}(0) = 0$ for each k. This is accomplished as in Example 2.

References

1. R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw Hill Inc., New York, 1969.
2. L. Hörmander, *Linear Partial Differential Operators*, Die Grundlehren der Math. Wiss., Springer-Verlag, Berlin, 1963.
3. J. W. Neuberger, *Tensor products and successive approximations for partial differential equations*, Israel J. Math., **6** (1968), 121–132.
4. _____, *Norm of symmetric product compared with norm of tensor product*, J. Linear and Multilinear Algebra, **2** (1974), 115–121.
5. T. H. Pate, *Lower bounds for the norm of the symmetric product*, Linear Algebra and Its Applications, **14** (3), (1976), 115–121.
6. _____, *Invariant subspaces for certain linear transformations on a space of tensors*, J. Linear and Multilinear Algebra, **5** (1), (1977).
7. _____, *Some improvements in the Neuberger iteration procedure for solving partial differential equations*, to appear in J. Differential Equations.
8. _____, *An application of the Neuberger iteration procedure to the constant co-efficient linear partial differential equation*, J. Mathematical Analysis and Applications, to appear.
9. _____, *An iterative method for solving non-linear partial differential equations*, Advances Math., **19** (1976), 245–265.
10. _____, *A resolvent for an iteration procedure for non-linear partial differential equations*, Trans. Amer. Math. Soc., **226** (1977), 321–343.
11. Francois Treves, *Linear Partial Differential Equations with Constant Coefficients*, Gordon and Beach, New York, 1966.

Received April 11, 1979.

AUBURN UNIVERSITY
AUBURN, AL 36830