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ON THE ITERATES OF DERIVATIONS OF PRIME RINGS

W. S. MARTINDALE, III AND C. ROBERT MIERS

In this paper we study properties of associative derivations whose iterates are related in rather special ways to the original derivation, or to the iterates of another derivation. An associative derivation $d: R \to R$ is an additive (or linear when appropriate) mapping on a ring R satisfying $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A derivation $d: R \to R$ is called *inner* if $d(x) = (\operatorname{ad} a)(x)$ for some $a \in R$ where $(\operatorname{ad} a)(x) = [a, x] = ax - xa$. In particular we ask when can the iterate of an inner derivation be an inner derivation? When can the iterates of two derivations commute? More precisely, we characterize elements $a, b \in R$, R a prime ring, for which $(\operatorname{ad} a)^n(x) = (\operatorname{ad} b)(x)$ for all $x \in R$, and we characterize derivations $d: R \to R$, $\delta: R \to R$ for which $[d^n(x), \delta^n(y)] = 0$ for all $x, y \in R$, R prime. Applications are made to C^*-algebras.

1. Introduction. In **[15]** it was shown that if d_1 and d_2 are derivations of a prime ring not of characteristic 2 with $d_1 \circ d_2$ a derivation, then either $d_1 = 0$ or $d_2 = 0$. Consequently if $d^2(x) = 0$ for all x, d a derivation on such a ring, then $d = 0$. In **[6]** it was shown that if $(\operatorname{ad} a)^n(x) = 0$ for all x in a simple ring there exists a scalar λ such that $(a - \lambda e)^{[(n+1)/2]} = 0$. And in **[14]** it was shown that if $(\operatorname{ad} a)^3(x) = (\operatorname{ad} a)(x)$ for a self-adjoint a and all x in a von Neumann algebra then $(a - z)^2 = a - z$ for some central element z. In [7] it was shown that if $[d(x), d(y)] = 0$ for all $x, y \in R$ where R is a prime ring and $\operatorname{char} R \neq 2$, then $d = 0$ or R is a commutative integral domain. Our results show that if d and δ are as above, and R is prime of characteristic 0 then R is commutative, or $d^{3n-1} = 0$, or $\delta^{3n-1} = 0$. If $d = \delta = \operatorname{ad} b$ there exists λ in the extended centroid of R such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.

In §§2 and 3 we prove results in the full generality of prime rings. Crucial use is made of the notions of extended centroid and central closure of a prime ring and of a key result on tensor products of closed prime algebras. We summarize these constructions by quoting from **[11]**. Let R be a prime ring and let T be the totality of all right R-homomorphisms $f: U_R \to R_R$, where U ranges over the non-zero ideals of R. An equivalence relation $\sim$ is defined on T as follows: f (acting on U) $\sim g$ (acting on V) if $f = g$ on W where W is non-zero ideal contained in $U \cap V$. The set $Q = \{\hat{f}\}$ of all equivalence classes forms a ring under the operations induced by addition and composition of representatives of the equivalence classes. $R \subseteq Q$ via the map $a \to \hat{a}_l$ where a_l is left multiplication by a acting on R. The center C of Q is a field containing the centroid of R and is called the *extended centroid* of R. The C-algebra $A = RC + C$ is again a prime ring and is called the *central closure* of R. In general we define a prime algebra S over a field F to be a *closed prime algebra* over F if S is its own central closure, i.e., the extended center of S is just F itself. We list some examples of closed prime algebras which are important for the purposes of this paper:

(1) The central closure of a prime ring ([**12**], p. 503).

(2) $A \otimes_C F$, where A is a closed prime algebra over C and F is an extension field of C ([**2**], Theorem 3.5).

(3) Any 2-fold transitive algebra of linear transformations on a complex vector space ([**4**], Theorem 2.1.3 and [**11**], Theorem 12).

Finally, if A is an algebra over F we denote by A_l the algebra of left multiplications a_l of A determined by the elements of A and A_r, the algebra of right multiplications a_r determined by the elements of A. A key result on tensor products is

THEOREM 1. *If S is a closed prime algebra over F and S^0 is the opposite algebra of S then*

$$S \underset{F}{\otimes} S^0 \cong S_l S_r \text{ via the map } u \otimes v \to u_l v_r.$$

In §4 the results of §§2 and 3 are applied to C^*-algebras. Although C^*-algebras, in general, are not prime they have a complete set of (algebraically) irreducible representations and, in our case, a phenomenon which occurs in each of these representations can be translated to a corresponding result for the original algebra. For a full account of C^*-algebras we refer the reader to [**1**].

2. Iterates of an inner derivation.

LEMMA 1. *Let R be a prime ring of characteristic $> n > 1$, and suppose $(\text{ad}\, a)^n = \text{ad}\, b$ for some $a, b \in R$. Let C be the extended centroid of R, $A = RC + C$ the central closure of R, F the algebraic closure of C, and $S = A \otimes_C F$. Then a is algebraic over F of degree $\leq n$, and if $p(x) = (x - \lambda)^l g(x)$ is the minimum polynomial of a over F where $\lambda \in F$ is a root of multiplicity $l \geq 1$, then $b - \beta = c^n$, $\beta \in F$, and*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n \tag{1}$$

holds in $S \otimes_F S^0$ with $c = a - \lambda$.

Proof. The condition $(\text{ad } a)^n = \text{ad } b$ clearly lifts to S, i.e., $(a_l - a_r)^n = b_l - b_r$ holds in the algebra S_lS_r. By Theorem 1, $S \otimes_F S^0 \cong S_lS_r$ via $u \otimes v \to u_lu_r$ since S is a closed prime algebra over F, and the condition further translates to

$$(2) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} \otimes a^k = b \otimes 1 - 1 \otimes b \text{ in } S \underset{F}{\otimes} S^0.$$

It is clear that $b \in \text{span}\{1, a, \ldots, a^n\}$. Writing $b = \sum_{i=0}^{n} \beta_i a^i$, $\beta_i \in F$, and substituting this into (2) we see that

$$(3) \quad (a^n - b + \beta_0) \otimes 1 + (\beta_1 - na^{n-1}) \otimes a + \cdots$$
$$+ (\beta_{n-1} + (-1)^{n-1} na) \otimes a^{n-1} + (\beta_n + (-1)^n) \otimes a^n = 0.$$

If $1, a, \ldots, a^n$ are independent then, in particular, $\beta_{n-1} + (-1)^{n-1} na = 0$, whence $a = \pm \beta_{n-1}/n \in F$ a contradiction. Therefore $\{1, a, \ldots, a^n\}$ is a dependent set and we have established that a is algebraic over F of degree $\leq n$. Hence we may set $p(x) = (x - \lambda)^l g(x)$ as indicated in the statement of the lemma. Setting $c = a - \lambda$ we then see that $m(x) = p(x + \lambda) = x^l q(x)$ is the minimum polynomial of c over F with $\deg m(x) = \deg p(x) \leq n$ and $q(0) \neq 0$. Since $\text{ad } a = \text{ad } c$ we may rewrite (2) as

$$(4) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = b \otimes 1 - 1 \otimes b.$$

We set $v = c^{l-1} q(c) \neq 0$, note that $cv = 0$, and multiply (4) on the right by $1 \otimes v$ to obtain

$$(5) \qquad c^n \otimes v = b \otimes v - 1 \otimes \beta v,$$

whence $(c^n - b + \beta) \otimes v = 0$. Thus $b = c^n + \beta$ and (4) becomes

$$(6) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

which completes the proof of the lemma. □

In the following Theorem, C, $A = RC + C$ and $S = A \otimes_C F$ have the same meaning as in Lemma 1.

THEOREM 2. *Let R be a prime ring of char. $> n > 1$, and suppose $(\text{ad } a)^n = \text{ad } b$ for some $a, b \in R$. If the minimum polynomial $p(x)$ of a over F (which necessarily exists in view of Lemma 1) contains a root $\lambda \in F$ of multiplicity $l > 1$, then $\lambda \in C$ and $(a - \lambda)^{[(n+1)/2]} = 0$.*

Proof. By Lemma 1 we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n \tag{1}$$

holding in $S \otimes_F S^0$, where $c = a - \lambda$ and the minimum polynomial of c over F is $m(x) = x^l q(x)$, $l > 1$, $q(0) \neq 0$. We remark that $q(c)$, $cq(c), \cdots, c^{l-1}q(C)$ are F-independent. Indeed, one would have a dependency $a_{i_1} c^{i_1} q(c) + \cdots + a_{i_r} c^{i_r} q(c) = 0$, $\alpha_{i_j} \neq 0$, $i_1 < i_2 < \cdots < i_r < l$, with "length" $r > 1$ minimal. Multiplication of the dependency by c^{l-i_r} would result in a dependency of shorter "length", a contradiction.

Now we multiply (1) on the right by $1 \otimes q(c)$ to obtain

$$\sum_{k=0}^{l-1} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k q(c) = 0. \tag{7}$$

By the preceding remark $\{c^k q(c) \mid k = 1, \cdots, l-1\}$ is an independent set so in particular $c^{n-(l-1)} = 0$. Thus c is nilpotent, with $m(x) = x^l$ where $l \leq n - (l-1)$, i.e., $l \leq (n+1)/2$, and so $c^{[(n+1)/2]} = 0$. In terms of a this says that $(a - \lambda)^{[(n+1)/2]} = 0$ and $p(x) = (x - \lambda)^l$.

It remains to prove that $\lambda \in C$. Since $c = a \otimes 1 - 1 \otimes \lambda \in A \otimes_C F$ (when written more precisely we see from $p(x) = (x-\lambda)^l$ that $0 = (a \otimes 1 - 1 \otimes \lambda)^l = \Sigma_{k=0}^{l} (-1)^k \binom{l}{k} a^{l-k} \otimes \lambda^k$ holds in $A \otimes_C F$. It follows that a is algebraic over C of degree $\leq l$. Let $h(x)$ be the minimum polynomial of a over C. On the one hand we must have $\deg h(x) \leq \deg p(x) = l$, and so $h(x) = p(x)$. Therefore the coefficients of $p(x)$ lie in C and so from $p(x) = (x-\lambda)^l = x^l - l\lambda x^{l-1} + \cdots$ we have $l\lambda = \alpha \in C$, whence $\lambda = \alpha / l \in C$. □

COROLLARY 1. *Let R be a prime ring of char. $> n > 1$, and suppose $(\mathrm{ad}\, a)^n = \mathrm{ad}\, b$ for some $a, b \in R$. Then $(a - \lambda)^{[(n+1)/2]} = 0$ for some λ in the extended centroid C if either of the following conditions hold*:

(a) n is *even*;

(b) $b = 0$.

Proof. By Lemma 1 we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n \tag{8}$$

where $c = a - \lambda$ has minimum polynomial $m(x) = x^l q(x)$, $q(0) \neq 0$. We show that if either (a) or (b) holds then c is nilpotent whence the conclusion follows from Theorem 2, since then $m(x) = x^l$ and $l > 1$. If we are given (a) then (8) becomes

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = -2 \otimes c^n. \tag{9}$$

Multiplication of (9) by $c^{l-1}q(c) \otimes 1$ on the left yields $0 = -2c^{l-1}q(c) \otimes c^n$ whence $c^n = 0$.

If we are given (b) then (8) reads

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = 0. \tag{10}$$

Multiplication of (10) by $1 \otimes c^{l-1}q(c)$ yields $c^n \otimes c^{l-1}q(c) = 0$ so again we obtain $c^n = 0$. □

REMARK. Corollary 1(b) was first proved for simple rings by Herstein [**6**] and conjectured for prime rings by Kovacs [**10**] and Herstein and the proof announced in [**13**].

The conclusion of Corollary 1 is false for prime rings in general. As an example, let $n = 5$, $R = M_2(\mathbf{C})$, the ring of 2×2 matrices over $\mathbf{C}$, and $a = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+\sqrt{-3}}{2} \end{bmatrix}$. One may verify that $(\operatorname{ad} a)^5 = \operatorname{ad}(a^5)$, but there is no $\lambda \in \mathbf{C}$ such that $(a - \lambda)$ is nilpotent. A more complicated example of the same nature can be constructed as follows:

Let $n = 5$, $R_j = M_2(\mathbf{C})$ for $j = 1, 2, \cdots, R = \oplus_{j=1}^{\infty} R_j$, $\{\theta_j\}_{j=1}^{\infty}$ a collection of distinct real numbers such that $0 < \theta_j < \pi/6$, $z_j = e^{i\theta_j}$ where $i^2 = -1$, $a_j = \begin{pmatrix} z_j & 0 \\ 0 & z_j\frac{1+\sqrt{-3}}{2} \end{pmatrix}$, and $a = \oplus a_j$. Then $(\operatorname{ad} a)^5 = \operatorname{ad}(a^5)$ and a has infinitely many distinct eigenvalues.

COROLLARY 2. *Let R be a 2-fold transitive algebra of linear transformations on a complex vector space H and suppose* $(\operatorname{ad} a)^n = \operatorname{ad} b$, $n > 1$, *for some $a, b \in R$. Then a is algebraic, and if the minimum polynomial $p(x)$ of a contains a repeated root λ (in particular if either n is even or $b = 0$) then* $(a - \lambda)^{[(n+1)/2]} = 0$.

The following corollary shows that the example following Corollary 1 is typical.

COROLLARY 3. *Let R be a 2-fold transitive algebra of linear transformations on a complex vector space H, and suppose* $(\operatorname{ad} a)^n = \operatorname{ad} b$ *for some $a, b \in R$ with n odd, $n > 1$. If the minimum polynomial $p(x)$ of a has distinct roots $\lambda_1, \cdots, \lambda_k$, $k \leq n$, there exist idempotents $p_1, \cdots, p_k$ with $p_i p_j = 0$ for $i \neq j$, $\sum_{i=1}^{k} p_i = 1$ and $a = \sum_{i=1}^{k} \lambda_i p_i$. If $k > 2$ and* 0, λ, μ *are distinct eigenvalues of a, then $\lambda^n - \mu^n = (\lambda - \mu)^n$. Conversely, if R is an algebra over a field F of characteristic* 0, *with idempotents $p_1, \cdots, p_k$ such* *that* $1 = \Sigma_{i=1}^k p_i$, $p_i p_j = 0$ *and* $\lambda_1, \cdots, \lambda_k$ *distinct elements of* F *such that* $\lambda_i^n - \lambda_j^n = (\lambda_i - \lambda_j)^n$, $1 \le i, j \le n$, *and* n *odd, then* $(\text{ad } a)^n = \text{ad}(a^n)$ *and* $a = \Sigma_{i=1}^k \lambda_i p_i$.

Proof. Standard linear algebra gives the existence of the p_i with the desired properties. Equation (1) of Lemma 1 becomes $\Sigma_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} x c^k = [c^n, x]$ for all $x \in R$ where $c = a - \lambda$. Since n is odd we have

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} x c^k \quad \text{for all } x \in R.$$

Let $v \neq 0$ be such that $av = \mu v$. Then $cv = (a - \lambda)v = (\mu - \lambda)v \neq 0$. Hence

$$\begin{aligned} 0 &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} x c^k v \\ &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} (\mu - \lambda)^k x v \quad \text{for all } x \in R. \end{aligned}$$

By transitivity,

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} (\mu - \lambda)^k = (c - (\mu - \lambda))^n - c^n + (\mu - \lambda)^n.$$

In terms of a this says $0 = (a - \mu)^n - (a - \lambda)^n + (\mu - \lambda)^n$. Therefore, the minimum polynomial $p(x)$ must divide $(x - \mu)^n - (x - \lambda)^n + (\mu - \lambda)^n$. Since 0 is an eigenvalue of a we have $(-\mu)^n - (-\lambda)^n + (\mu - \lambda)^n = 0$. The other part of the corollary is a straightforward calculation. □

3. **Commuting iterates of derivations.**

THEOREM 3. *Let* R *be a prime ring and let* d *and* δ *be derivations on* R *such that* $[d^n(x), \delta^n(y)] = 0$ *for all* $x, y \in R$. *Then either* R *is commutative, or* $d^{3n-1} = 0$, *or* $\delta^{3n-1} = 0$. *Furthermore, if* $n = 1$, *and the characteristic of* $R \neq 2$, *then either* R *is commutative, or* $d = 0$, *or* $\delta = 0$.

Proof. Let W be the subring generated by $\{d^n(x) \mid x \in R\}$ and note that $d(W) \subseteq W$. By the Leibniz formula we have

$$(1) \qquad d^n(xy) = \sum_{k=0}^{n} \binom{n}{k} d^k(x) d^{n-k}(y) \in W \quad \text{for all } x, y \in R.$$

For $l = 1, 2, \cdots, n$ we substitute $d^{l-1}(x)$ for x and $d^{2n-l}(y)$ for y in (1) to obtain

$$\sum_{k=0}^{n-l} \binom{n}{k} d^{k+l-1}(x) d^{3n-k-l}(y) \in W, \tag{2}$$

$$l = 1, 2, \cdots, n \text{ for all } x, y \in R.$$

For $l = n$, (2) reads

$$d^{n-1}(x) d^{2n}(y) \in W, \tag{3}$$

and for $l = n - 1$, (2) reads

$$d^{n-2}(x) d^{2n+1}(y) + \binom{n}{1} d^{n-1}(x) d^{2n}(y) \in W. \tag{4}$$

Together (3) and (4) imply that $d^{n-2}(x)d^{2n+1}(y) \in W$. Continuing in this fashion by comparing successive decreasing values of l from $l = n$ to $l = 1$ we have that $xd^{3n-1}(y) \in W$ for $x, y \in R$. Therefore, $Rd^{3n-1}(y) \subseteq W$ for all $y \in R$. Similarly for all x, $R\delta^{3n-1}(x) \subseteq V$, the subring generated by $\{\delta^n(t) \mid t \in R\}$. Since $[W, V] = 0$ by assumption, the left ideals $Rd^{3n-1}(y)$ and $R\delta^{3n-1}(x)$ commute for all $x, y \in R$. Without loss of generality we may assume that $d^{3n-1}(y) \neq 0$ for some y and that $\delta^{3n-1}(x) \neq 0$ for some x. Since we have two commuting non-zero left ideals $Rd^{3n-1}(y)$ and $R\delta^{3n-1}(x)$ in the prime ring R, R must be commutative.

For $n = 1$, if R is not commutative, we may assume $d^2 = 0$. From $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$ we have $d(x)d(y) = 0$. In particular, $0 = d(xy)d(x) = [d(x)y + xd(y)]d(x)$ whence $d(x)yd(x) = 0$ for all $x, y \in R$. Since R is prime it follows that $d(x) = 0$ for all $x \in R$. □

THEOREM 4. *Let R be a prime ring of characteristic $\geq 3n$ and let $d = \operatorname{ad} b$ be an inner derivation of R satisfying $[d^n(x), d^n(y)] = 0$ for all $x, y \in R$. Then there exists an element λ in the extended centroid of R such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.*

Proof. We can assume, by [7], that $n > 1$. The condition on d clearly extends to the central closure $A = RC + C$ of R. By Theorem 3, $d^{3n-1} = 0$ and hence by Corollary 1(b) there exists $\lambda \in C$ such that $a = b - \lambda \in A$ satisfies $a^{[3n/2]} = 0$. If l is the degree of nilpotency of a we have

$$l \leq \frac{3n}{2} \tag{5}$$

and, assuming the theorem to be false, we may also suppose that

$$l > \frac{2n+3}{3}, \tag{6}$$

in other words, $3l - 2n - 4 \geq 0$. We then set

$$p = q = \frac{3l - 2n - 4}{2} \quad \text{if } l \text{ is even} \tag{7}$$

and

$$p = \frac{3l - 2n - 5}{2}, q = \frac{3l - 2n - 3}{2} \quad \text{if } l \text{ is odd.} \tag{8}$$

In either case $p + q = 3l - 2n - 4$, which we wish to view in the form $2n + p + q = 3(l - 1) - 1$.

Expansion of $[d^n(x), d^n(y)] = 0$ by the Leibniz formulas, followed by replacement of x by ax, yields

$$0 = a^p[d^n(ax), d^n(y)]a^q = g - h$$

where

$$g = \sum_{j,k=0}^{n} (-1)^{n-j+k}\binom{n}{j}\binom{n}{k} a^{p+j+1}xa^{2n-j-k}ya^{q+k}$$

and

$$h = \sum_{j,k=0}^{n} (-1)^{n-j+k}\binom{n}{j}\binom{n}{k} a^{p+j}ya^{2n+1-j-k}xa^{q+k}.$$

Since $2n + 1 + p + q = 3(l - 1)$ the only possible surviving summand of g occurs when $j + p + 1 = q + k = l - 1$; for h it occurs when $j + p = q + k = l - 1$. To see that these terms actually occur it is necessary to show that the j and k thus determined are indeed within the range $0 \leq j$, $k \leq n$. This means verifying

$$0 \leq l - p - 2 \leq n, \quad 0 \leq l - p - 1 \leq n, \quad 0 \leq l - q - 1 \leq n. \tag{9}$$

We leave it to the reader to check that the various substitutions of (7) and (8) in (9) lead to the following inequalities

$$0 \leq 2n - l + i \leq 2n, \qquad i = 0, 1, 2, 3 \tag{10}$$

where $i = 0, 2$ when l is even and $i = 1, 3$ when l is odd. But the inequalities (10) follow readily from (5) and (6), so that we have established

$$g = (-1)^{n+p-q+1}\binom{n}{l-p-2}\binom{n}{l-q-1} a^{l-1}xa^{l-1}ya^{l-1} \tag{11}$$

and

$$h = (-1)^{n+p-q}\binom{n}{l-p-1}\binom{n}{l-q-1} a^{l-1}ya^{l-1}xa^{l-1}. \tag{12}$$

Setting $x = y$ in (11) and (12), noting that the coefficients of $a^{l-1}xa^{l-1}xa^{l-1}$ in (11) and (12) are of opposite parity, and knowing $g = h$, we may conclude that $a^{l-1}xa^{l-1}xa^{l-1} = 0$ for all $x \in A$. This means that the non-zero right ideal $a^{l-1}A$ is nil of bounded degree which in view of [5], Lemma 1.1 provides a contradiction since A is a prime ring.

COROLLARY 4. *Let R be a 2-fold transitive ring of linear transformations on a complex vector space H and let $d = \operatorname{ad} b$ be an inner derivation of R satisfying $[d^n(x), d^n(y)] = 0$ for all $x, y \in R$. Then there exists a complex scalar λ such that $a = b - \lambda$ satisfies $a^{[(2n+3)/3]} = 0$.* □

4. Applications. If A is an algebraically irreducible algebra of operators on a complex Banach space H then A is 1-fold transitive (since it is irreducible) and hence by [3] it is m-fold transitive if H is infinite dimensional or is of dimension at least m. In particular, $\mathcal{L}(H)$, the algebra of all bounded linear operators on H is 2-fold transitive so that the previous results apply.

Let A be a C^*-algebra of operators, containing the identity operator 1, acting on a complex Hilbert space H. Let $R = A''$ be the ultraweak closure of A and let M be the universal enveloping von Neumann algebra of R. If ϕ is any $*$-representation of R and π the natural injection of R into M, there exists a normal $*$-representation $\tilde{\phi}$ of M such that $\phi(x) = \tilde{\phi}(\pi(x))$. We have that $\tilde{\phi}(M) = \phi(R)''$. If ϕ is irreducible, $\tilde{\phi}(M) = \phi(R)'' = \mathcal{L}(H_\phi)$, the ring of all bounded linear operators on H_ϕ, where H_ϕ is the representation Hilbert space. If $\tilde{\phi}$ is a normal homomorphism of M onto a von Neumann algebra N, there exists a central projection $c \in M$ and a $*$-isomorphism $\tilde{\psi}$ of M_c onto N such that $\tilde{\phi}(x) = \tilde{\psi}(xc)$ for all x in M.

THEOREM 5. *Let A be a C^*-algebra of operators acting on a complex Hilbert space H, and assume A contains the identity operator* 1. *Let $R = A''$, the ultraweak closure of A, and suppose $(\operatorname{ad} a)^n(x) = (\operatorname{ad} b)(x)$ for some $a, b \in A$ and all $x \in A$. If n is odd there exists a central projection $c \in R$ and a central element z in R such that $((a - z)c)^{(n+1)/2} = 0$, $1 - c = \sum_{\beta \in B'} d_\beta$, and $\operatorname{ad}_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta r_j^\beta$ where the λ_i^β are distinct complex numbers, the r_j^β are (not necessarily self-adjoint) orthogonal idempotents, and the d_β are orthogonal central projections. If n is even there exists a central element $z \in R$ such that $(a - z)^{n/2} = 0$.*

Proof. Let $\{\phi_\beta\}_{\beta \in B}$ be a complete set of irreducible representations of R. Then $\phi_\beta(x) = \tilde{\phi}_\beta(\pi(x))$ where $\tilde{\phi}_\beta$ is a normal $*$-homomorphism of M on $\mathcal{L}(H_{\phi_\beta})$. As above, for each β, there exists a central projection c_β in M and a $*$-isomorphism $\tilde{\psi}_\beta$ of M_{c_β} on $\mathcal{L}(H_{\phi_\beta})$ such that $\tilde{\phi}_\beta(x) = \tilde{\psi}_\beta(xc_\beta)$ for all $x \in M$. Now $(\text{ad } a)^n(x) = (\text{ad } b)(x)$ for all $x \in A$ implies $(\text{ad } \pi(a))^n(x) = (\text{ad } \pi(b))(x)$ for all $x \in M$ so that $(\text{ad } \tilde{\phi}_\beta(\pi(a)))^n(x) = (\text{ad } \tilde{\phi}_\beta(\pi(b)))(x)$. Since $\tilde{\phi}_\beta$ is irreducible, the results of Theorem 2 and Corollaries 2 and 3 apply. If n is odd then either (i) there exists $\lambda_\beta \in \mathbf{C}$ such that $(\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = 0$ (in the case that the minimum polynomial of $\tilde{\phi}_\beta(\pi(a))$ has a repeated root or $\tilde{\phi}_\beta(\pi(a))$ is central) or (ii) $\tilde{\phi}_\beta(\pi(a)) = \sum_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta$ where the p_i^β are mutually orthogonal idemotents and the λ_i^β are distinct. Since $\{\phi_\beta\}_{\beta \in B}$ is complete we have LUB $c_\beta = 1$. Choose mutually orthogonal central projections $d'_\beta \in M$ such that $d'_\beta \leq c_\beta$ and $\sum_{p \in B} d'_\beta = 1$. Let $c' = \sum_{\beta \in B_1} d'_\beta$ where $B_1 = \{\beta \mid \text{(i) holds}\}$. Then $1 - c' = \sum_{\beta \in B \setminus B_1 = B'} d'_\beta$. If $\beta \in B_1$ then $0 = (\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\phi_\beta(a) - \lambda_\beta)^{(n+1)/2}$ so that $|\lambda_\beta| \leq \|\phi_\beta(a)\| \leq \|a\|$. Hence $z' = \sum_{\beta \in B_1} \lambda_\beta d'_\beta \in Z_M$. Moreover, $\beta \in B_1$ implies $0 = (\tilde{\phi}(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\tilde{\psi}_\beta((\pi(a) - \lambda_\beta)c_\beta))^{(n+1)/2}$ so that $0 = (\pi(a) - \lambda_\beta)^{(n+1)/2} c_\beta$ since $\tilde{\phi}_\beta$ is an isomorphism. $(\pi(a) - z')^{(n+1)/2} c' = ((\pi(a) - z')c')^{(n+1)/2} c_\beta d'_\beta = 0$ for each $\beta \in B_1$. Similarly if $\beta \in B \setminus B_1$, $\tilde{\phi}_\beta(\pi(a)) = \sum_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta = \tilde{\psi}_\beta(\pi(a)c_\beta)$ so that $\pi(a)c_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta$ and $\pi(a)d'_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta d'_\beta$.

Let i: $R \to R$ be the identity map, $\tilde{i}$ the normal homomorphism of M on R for which $\tilde{i}(\pi(x)) = i(x)$ for all $x \in R$. Let c'' be a central projection in M and $\tilde{j}$ an isomorphism of $M_{c''}$ on R such that $\tilde{i}(\pi(x)) = \tilde{j}(\pi(x)c'')$. $\tilde{j}$ induces an isomorphism between $Z_{M_{c''}}$ the center of $M_{c''}$ and Z_R. Hence $c'c'' + (1 - c')c''$ which is the identity of $M_{c''}$ is sent by $\tilde{j}$ to 1, the identity of R. Let $c = \tilde{j}(c', c'')$ and $z = \tilde{j}(z', c'')$, $d_\beta = \tilde{j}(d'_\beta c'')$, and $r_j^\beta = \tilde{j}(q_i^\beta d'_\beta c'')$. □

THEOREM 6. *Let A be a C^*-algebra, d a derivation on A such that $[d^n(x), d^n(y)] = 0$ for all $x, y \in A$. There exists $s \in R$, $z \in Z_R$ the centre of R, such that $d(x) = [s, x]$ for all $x \in A$ and $(s - z)^{[(2n+3)/3]} = 0$.*

Proof. By [**16**: 4.1.7] there exists such an s. Moreover d extends in this way to a derivation on R. The result follows from Theorem 4 and an argument as in Theorem 5. □

We finish with a result which does not fit the title of the paper but which contains the same methods in its proof.

LEMMA [*See* 8: *Theorem*]. *If R is a prime ring not of characteristic 2 and d: $R \to R$ a derivation, then then either $d = 0$ or $\{x \in R \mid [x, d(r)] = 0$ for all $r \in R\} \subseteq Z_R$, the centre of R.* □

Proof. Let $b \in \{x \in R \mid [x, d(r)] = 0$ for all $r \in R\}$. Then $0 = [b, d(r)] = ((\text{ad } b) \circ d)(r)$ for all $r \in R$. By [**15**: Theorem 1] either $d = 0$ or ad $b = 0$. If ad $b = 0$ then $[b, r] = 0$ for all r so $b \in Z_R$. □

THEOREM 7. *Let A be a C^*-algebra with identity e, $R = A''$, and $d: A \to A$ a derivation. There exists a central projection $c \in Z_R$ such that $dc(a) = 0$ for all $a \in A$, and $\{a(e-c) \mid [a, d(b)](e-c) = 0$ for all $b \in A\} \subseteq Z_A(e-c)$.*

Proof. d extends to a derivation (denoted by d) from R to R. Let ϕ be an irreducible representation of R and consider $d_\phi: \phi(R) \to \phi(R)$ given by $d_\phi(\phi(r)) = \phi(d(r))$. Then d_ϕ is a derivation on the irreducible algebra $\phi(R)$. By the lemma either $d_\phi = 0$ or $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0$ for all $s \in R\} \subseteq Z_{\phi(R)}$.

If $\tilde{\psi}$, π, and c are as in the beginning of this section,

$$\begin{aligned}\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0 \text{ for all } s \in R\}\\ = \{\tilde{\psi}(\pi(r)c) \mid [\tilde{\psi}(\pi(r)c), \tilde{\psi}(\pi(d(s))c)] = 0 \text{ for all } s \in R\}\\ = \{\tilde{\psi}(\pi(r)c) \mid [\pi(r)c, \pi(d(s))c] = 0 \text{ for all } s \in R\}.\end{aligned}$$

Since $\tilde{\psi}$ is a $*$-isomorphism from M_c onto $\mathfrak{L}(H_\phi)$ it carries centers to centers so that if $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0$ for all $s \in R\} \subseteq Z_{\phi(R)}$ we must have $\{\pi(r)c \mid [\pi(r)c, \pi(d(s))c] = 0$ for all $s \in R\} \subseteq Z_{M_c}$.

Let $\{\phi_\beta\}$ be a complete set of irreducible $*$-representations of R and $d\phi_\beta$ as above. If $d\phi_\beta = 0$ there exists a central projection c_β in M such that $0 = d\phi_\beta(\phi_\beta(x)) = \phi_\beta(d(x)) = \tilde{\phi}_\beta(\pi(d(x))) = \tilde{\psi}_\beta(\pi(d(x))c_\beta)$ so that $\pi(d(x))c_\beta = 0$ for all $x \in R$. If $d\phi_\beta \neq 0$ there exists c_β in M such that $\{\pi(r)c_\beta \mid [\pi(r)c_\beta, \pi(d(s))c_\beta] = 0$ for all $s \in R\} \subseteq Z_{M_{c_\beta}}$.

Since $\{\phi_\beta\}$ is complete, LUB $c_\beta = e$, choose mutually orthogonal central projections c'_β in M such that $c'_\beta \leq c_\beta$ and $\Sigma c'_\beta = e$. Let $c_0 = \Sigma c'_\beta$ where the sum is over all β such that $\pi(d(x))c_\beta = 0$ for all $x \in R$.

Let i, $\tilde{i}$ and $\tilde{j}$ be as above with c_1 a central projection in M such that $\tilde{j}$ is an isomorphism of M_{c_1} on R. There exists $c \in Z_R$ such that $\tilde{j}(c_0c_1) = c$. We have $\tilde{i}(c_0) = c$. Now $0 = \pi(d(r))c_0$ for all $r \in R$ so that $0 = \tilde{i}(\pi(d(r))c_0) = d(r)c$ for all $r \in R$. Moreover,

$$\begin{aligned}\{\pi(r)(e-c_0) \mid\mid \pi(r), \pi(d(s)) \mid (e-c_0) = 0 \text{ for all } s \in R\}\\ = \{\pi(r)(e-c_0) \mid \tilde{i}([\pi(r), \pi(d(s))](e-c_0)) = 0 \text{ for all } s \in R\}\\ = \{\pi(r)(e-c_0) \mid [r, d(s)](e-c) = 0 \text{ for all } s \in R\}.\end{aligned}$$

Hence $\{\pi(r)(e-c_0) \mid [\pi(r), \pi(d(s))](e-c_0) = 0$ for all $s \in R\} \subseteq Z_{M_{e-c_0}}$ implies $\{r(e-e) \mid [r, d(s)](e-c) = 0$ for all $s \in R\} \subseteq \tilde{i}(Z_{M_{e-c_0}}) = Z_{R_{e-c}}$.

Finally,

$$\{a(e-c) \mid [a, d(b)](e-c) = 0 \text{ for all } b \in A\}$$
$$\subseteq \{r(e-c) \mid [r, d(s)](e-c) = 0 \text{ for all } s \in R\}$$

by the ultra weak continuity of d. Therefore

$$\{a(e-c) \mid [a, d(b)](e-c) = 0 \text{ for all } b \in A\}$$
$$\subseteq A(e-c) \cap Z_R(e-c) = Z(e-c). \qquad \square$$

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UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA, CANADA V8W 2Y2