PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No 1, 1983

ON THE BEST CONDITIONS ON THE GRADIENT OF PRESSURE FOR UNIQUENESS OF VISCOUS FLOWS IN THE WHOLE SPACE

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In this paper we prove a uniqueness theorem for the Cauchy problem of the Navier-Stokes equations under the assumption on the gradient of pressure ∇p that it either belongs to some L^q space for some $q \in (1, \infty)$ or tends to zero at large spatial distances. As shown by means of a counterexample, in the class where uniqueness is proven the above hypotheses cannot be relaxed to ∇p only bounded.

Introduction. Let Ω be an unbounded domain of the Euclidean 3-dimensional space E_3. During the last years, the problem of uniqueness (and, more generally, of continuous dependence) for solutions to non-steady Navier-Stokes equation in Ω, whose (global) kinetic energy need not be *a priori* a finite quantity, has drawn the attention of many writers [**1**–**10**]. As is well known, this kind of question traces back to early papers of J. Kampé de Fériet [**9**], D. Graffi [**6**] and J. Serrin [**12**]. The aim of these works is to try to recover uniqueness without giving *a priori* on the solutions decay or summability conditions which for some applications must be considered restrictive from the physical point of view (cf. e.g. [**9**], pp. 21–24, [**12**], pp. 63–64)[1].

The above results can be grouped, essentially, into two classes. Specifically, indicating by $\mathbf{v}$ and p the velocity and pressure fields, respectively, we have

(i) Uniqueness theorems without assumptions on p but with restrictive hypotheses (e.g. summability) on the behavior of $\mathbf{v}$ and, possibly, on its derivatives [**1**–**3**];

(ii) Uniqueness theorems with assumptions on p (e.g. summability) but with very mild hypotheses (e.g. boundedness or even "growth" at large spatial distances) on $\mathbf{v}$ and, possibly, on its first spatial derivatives [**3**–**10**].

In this paper we investigate uniqueness in a (ii)-type class, namely, we consider the class $\mathcal{C}$ of solutions where $\nabla \mathbf{v}$ is bounded and try to determine the "best assumptions" on ∇p for uniqueness to hold. Precisely, we prove that, for the Cauchy problem in the class $\mathcal{C}$, uniqueness

[1]We should notice that here we are not interested in uniqueness of solutions which are assumed to be square summable together with their first spatial derivatives. For this type of problem the reader is referred, e.g., to [**8**] and the references cited therein.

holds provided that either $\nabla p \in L^q(E_3 \times [0, T])$, $1 < q < \infty$, or $\nabla p = o(1)$ as $|x| \to \infty$. These assumptions on ∇p can be considered the best possible in the sense that uniqueness fails in the class $\mathcal{C}$ if ∇p is *only* bounded, as can be proven by means of a counterexample.

The paper is subdivided into two sections. In §1, after briefly recalling some preliminaries, we state our thoerem and give a counterexample showing that in the class where $\mathbf{v}$ and $\nabla \mathbf{v}$ are bounded uniqueness fails if ∇p is only bounded too. In §2, employing the weight function approach [**4**–**11**] and some estimates concerning elliptic equations, we show that a motion $(\mathbf{v}, p)$ in E_3 with $\nabla \mathbf{v}$ bounded is unique in the class of motions $(\mathbf{v} + \mathbf{u}, p + \tau)$ such that $\nabla \mathbf{u}$ is bounded and either $\nabla \tau \in L^q(E_3 \times [0, T])$, $1 < q < \infty$, or $\nabla \tau = o(1)$ as $|x| \to \infty$, which is just the theorem enunciated in §1.

1. Preliminaries and statement of the theorem. As is known, the motion of a viscous, homogeneous Newtonian fluid occurring in the domain $\Omega \subseteq E_3$, is governed by the following Navier-Stokes system (we assume, for the sake of simplicity, the kinetic viscosity to be equal to 1)

$$
(1) \qquad \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Delta_2 \mathbf{v} + \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}(x, t) = \mathbf{v}^*(x, t), \qquad (x, t) \in \partial\Omega \times [0, T], \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x), \qquad x \in \Omega, \end{cases}
$$

where $\mathbf{v}$ is the velocity, p is the pressure, $\mathbf{f}$ is the external force, and finally, $\mathbf{v}^*$ and $\mathbf{v}_0$ are ascribed functions. From (1) it comes out that the uniqueness of a given motion $(\mathbf{v}, p)$ in the class of motions $(\mathbf{v} + \mathbf{u}, p + \tau)$ is then reduced to investigate the uniqueness of the null solution of the following initial boundary value problem

$$
(2) \qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} + \mathbf{u}) \cdot \nabla \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{v} - \nabla \tau + \Delta_2 \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(x, t) = 0, \qquad (x, t) \in \partial\Omega \times [0, T], \\ \mathbf{u}(x, 0) = 0, \qquad x \in \Omega. \end{cases}
$$

The aim of this paper is to prove the following theorem.

THEOREM. *Let $\Omega \equiv E_3$ and $\nabla \mathbf{v}$, $\nabla \mathbf{u}$ uniformly bounded in $E_3 \times [0, T]$. Then if either*

$$
\nabla \tau \in L^q(E_3 \times [0, T]) \qquad (q \in (1, \infty))
$$

or

$$\nabla\tau = o(1)$$

the solution $\mathbf{u}(x, t)$ *to problem* (2) *is identically zero in* $E_3 \times [0, T]$.

As we mentioned in the Introduction, the above assumptions on $\nabla\tau$ *cannot* be relaxed to $\nabla\tau$ *only* bounded. Actually, we have the following counterexample to uniqueness. Let us take in $(1)_1$ $\mathbf{f} = 0$ and $\Omega = E_3$. It can be readily seen that $\mathbf{v} = 0$, $p = \text{const}$ and $\mathbf{v}' = (\sin t, 0, 0)$, $p' = -x_1 \cos t$ are two solutions to problem (1)[2] assuming the *same* initial data and, moreover, $\mathbf{v}, \mathbf{v}', \nabla\mathbf{v}, \nabla\mathbf{v}', \nabla(p - p')$ are *only* uniformly bounded in $E_3 \times [0, T]$.

2. Proof of the uniqueness theorem. The proof of the theorem will be given in several steps. First of all we propose a quite general lemma concerning an *a priori* estimate for regular solutions to problem (2) in unbounded domains. We have

LEMMA 1. *Let* Ω *be any unbounded domain in* E_3 *whose* (*2-dimensional*) *boundary, if any, is sufficiently smooth to allow the application of the divergence theorem. Moreover, assume that with respect to a fixed spherical coordinates system* (r, γ) *with the origin at a given point* $O \in E_3$ *it holds*

$$\begin{cases} v_r, u_r = O(r), & \mathbf{u} = O(r^m) \qquad (m \geq 0), \\ \nabla\mathbf{v} = O(1), & \nabla\mathbf{u} = O(r^k) \qquad (k \geq 0) \end{cases}$$

where v_r *and* u_r *are the radial components of* $\mathbf{v}$ *and* $\mathbf{u}$ *respectively. Then for* $q \in (1, \infty)$

$$\nabla\tau \in L^q(\Omega \times [0, T]) \Rightarrow \mathbf{u}(t) \in L^q(\Omega) \quad \forall\, t \in [0, T]$$

and the following estimate holds ($u = |\mathbf{u}|$)

$$\int_\Omega u^q(x, t)\, dx \leq A \int_0^t \int_\Omega |\nabla\tau|^q\, dx\, ds \quad \forall\, t \in [0, T] \tag{3}$$

where A *is a positive constant independent of* t.

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[2]We would remark that these solutions are a particular case of the following general class of potential-like solutions to (1) in E_3 with $\mathbf{f} = 0$:

$$\begin{cases} \mathbf{v}(x, t) = a(t)\nabla\psi \\ p(x, t) = -\dfrac{da}{dt}\psi + (a^2/2)(\nabla\psi)^2 + c(t) \end{cases}$$

where $a \in C^1(0, T)$, $\triangle_2\psi = 0$ and $c(t)$ is an arbitrary function of $t \in [0, T]$.

Proof. We shall employ the so called "generalized weighted energy equality" which we already introduced in **[11]**. This inequality is obtained by multiplying both sides of $(2)_1$ by $gu^{q-2}\mathbf{u}$ $(q > 1)$, where g is a rapidly decreasing (smooth) function (weight function) and integrating by parts over Ω. Thus we have (cf. **[11]**)

$$(4)\quad \begin{cases} \dfrac{dE^{(q)}}{dt} = \displaystyle\int_\Omega \Big\{ \frac{u^{(q)}}{q}\Big[\frac{\partial g}{\partial t} + (\mathbf{u}+\mathbf{v})\cdot\nabla g\Big] \\ -gu^{q-2}\big[\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{u} + \nabla\mathbf{u}\colon\nabla\mathbf{u} + (q-2)u^{-2}(\nabla\mathbf{u}\cdot\mathbf{u})^2 + \nabla\tau\cdot\mathbf{u}\big] \\ \qquad\qquad + (1/q)\Delta_2 g u^q \Big\}\, dx \end{cases}$$

with

$$E^{(q)} = (1/q)\int_\Omega gu^q\, dx.$$

Let us choose in (4) $g(x,t) = \exp[-\alpha(t+t_0)^\beta r]$ $(\alpha, \beta, t_0 > 0)$. We easily have the following inequalities

(i) $$-gu^{q-2}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{u} \le Mgu^q \quad (M = \text{const} > 0);$$

(ii) $$-gu^{q-2}\nabla\tau\cdot\mathbf{u} \le g(1/q)|\nabla\tau|^q + g\frac{(q-1)}{q}u^q;$$

(iii) $$(1/q)\Delta_2 g u^q \le \alpha^2(T+t_0)^{2\beta}(1/q)gu^q;$$

(iv) $$-u^{q-2}\big[\nabla\mathbf{u}\colon\nabla\mathbf{u} + (q-2)u^{-2}(\nabla\mathbf{u}\cdot\mathbf{u})^2\big] \le 0, \qquad q\in(1,\infty).$$

Moreover, by assumption there is a constant M' such that $|u_r + v_r| \le M'(r+1)$. Thus,

$$(5)\quad \begin{cases} \dfrac{\partial g}{\partial t} + (\mathbf{u}+\mathbf{v})\cdot\nabla g \le gM'\alpha(T+t_0)^\beta \\ \qquad + \alpha(t+t_0)^\beta gr\Big[-\dfrac{\beta}{(t+t_0)} + M'\Big]. \end{cases}$$

Choosing

$$\beta \ge M'(T+t_0)$$

from (5) we then deduce

(v) $$\frac{\partial g}{\partial t} + (\mathbf{u}+\mathbf{v})\cdot\nabla g \le gM'\alpha(T+t_0).$$

Therefore, collecting (i)–(v), from (4) we obtain

$$\frac{dE^{(q)}}{dt} \le c_1\left[E^{(q)} + \int_\Omega |\nabla\tau|^q\, dx\right] \tag{6}$$

where c_1 is a suitable positive constant (with respect to t). Consequently, integrating (6) and letting $\alpha \to 0$ prove (3).

LEMMA 2. *Let φ be a classical solution to the problem (in E_3)*

$$\Delta_2\varphi = \nabla\cdot\mathbf{f} \tag{7}$$

with $\mathbf{f}$ *belonging to $L^q(E_3)$, for some $q \in (1,\infty)$. Assume that either*

$$\nabla\varphi \in L^q(E_3) \tag{8}$$

or

$$\nabla\varphi = o(1) \quad \text{as } |x| \to \infty, \tag{9}$$

then it follows that in both cases (8), (9) $\nabla\varphi \in L^q(E_3)$ *and the following estimate holds*

$$\int_{E_3} |\nabla\varphi|^q\, dx \le C\int_{E_3} |\mathbf{f}|^q\, dx \qquad (C = \text{const} > 0). \tag{10}$$

Proof. We notice that, given $\mathbf{f} \in L^q(E_3)$, it is possible to prove the existence of a weak solution to (7) satisfying (10), i.e., of a function $\bar\varphi$ with $\nabla\bar\varphi \in L^q(E_3)$ such that

$$\begin{cases} \displaystyle\int_{E_3} \nabla\bar\varphi\cdot\nabla\psi\, dx = \int_{E_3} \mathbf{f}\cdot\nabla\psi\, dx \\ \displaystyle\int_{E_3} |\nabla\bar\varphi|^q\, dx \le C\int_{E_3} |\mathbf{f}|^q\, dx, \end{cases} \tag{11}$$

where ψ is an arbitrary (measurable) function with $\nabla\psi \in L^{q'}(E_3)$ ($q' = q/(q-1)$) and C is a positive constant[3]. On the other hand, since φ is a classical solution to (7), from (7) and $(11)_1$ we must have

$$\int_{E_3} \nabla w\cdot\nabla\psi^*\, dx = 0$$

[3]In fact, if $\mathbf{f} \in C_0^\infty(E_3)$ we have $\bar\varphi(x) = (1/4\pi)\int_{E_3}\nabla_x(1/|x-y|)\cdot\mathbf{f}\, dy$. If $\mathbf{f} \in L^q(E_3)$ we can approximate it with smooth functions of compact support and then apply the well-known Calderon-Zygmund theorem on the boundedness in L^q of the singular integrals.

where $w = \varphi - \overline{\varphi}$ and ψ^* is arbitrary from $C_0^\infty(E_3)$, so that w is a weak solution to $\triangle_2 w = 0$ in E_3. From well-known results on the regularity of weak solutions to elliptic equations we then deduce that w is in fact a classical solution to $\triangle_2 w = 0$ in E_3. Now, fix any point $x_0 \in E_3$ and choose x_0 as the origin of a spherical coordinates system (R, θ). By using the Hölder inequality, it can be readily seen that from either (8) or (9) and from $(11)_2$ it turns out the existence of a sequence of radii $\{R_n\}_{n \in N}$ starting from x_0 with $\lim_{n\to\infty} R_n = \infty$ such that

$$\lim_{n\to\infty} \int_{S_1} \nabla w(R_n, \theta)\, d\theta = 0, \tag{12}$$

where S_1 is the unit sphere centered at x_0. Therefore, by the mean value theorem for harmonic functions applied to $\nabla w(x_0)$ we deduce $\nabla w(x_0) = 0\ \forall\, x_0 \in E_3$. This fact, in virtue of $(11)_2$, completely proves the lemma.

We are now in a position *to prove the theorem*. From $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0$, one obtains the obvious identity

$$\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{u}) = \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{v}).$$

As a consequence, taking the divergence operator of both sides of $(2)_1$ we have that at each $t \in [0, T]$ τ satisfies

$$\begin{cases} \triangle_2 \tau = \nabla \cdot \boldsymbol{\Psi} \\ \boldsymbol{\Psi} = -\mathbf{u} \cdot (\nabla \mathbf{u} + 2\nabla \mathbf{v}). \end{cases}$$

From the assumptions of the theorem and from Lemma 2 we then deduce the existence of a constant B such that

$$\int_0^t \int_{E_3} |\nabla \tau|^q\, dx\, ds \leq B \int_0^t \int_{E_3} u^q\, dx\, ds \quad \forall\, t \in [0, T].$$

On the other hand, given $w \in C^1(E_3 \times [0, T])$ with $\sup_{E_3 \times [0,T]} |\nabla w| = N < \infty$, it is

$$|w(x, t)| \leq Nr + |w(0, T)| \leq Nr + N'$$

where $N' = \max_{t \in [0,T]} |w(0, t)|$. Consequently, by the assumptions of the theorem we have

$$\sup_{E_3 \times [0, T]} |\mathbf{u}/(1 + r)|, \qquad \sup_{E_3 \times [0, T]} |\mathbf{v}/(1 + r)| < \infty. \tag{14}$$

Thus, employing (14), Lemma 1 and (13) we finally obtain $\forall\, t \in [0, T]$

$$\int_{E_3} u^q(x, t)\, dx \leq D \int_0^t \int_{E_3} u^q(x, t)\, dx\, ds \qquad (D = \text{const} > 0)$$

which in turn implies $\mathbf{u}(x, t) \equiv 0$ in $E_3 \times [0, T]$.

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Received July 9, 1980. The research described in this paper was made under the auspices of G.N.F.M. of the Italian C.N.R.

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