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THE PERTURBATION THEORY FOR LINEAR OPERATORS OF DISCRETE TYPE

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Using the theory of unconditional bases, we discuss the perturbation theory of linear operators of discrete type.

The principal abstract perturbation theorem about discrete spectral operators was introduced by J. T. Schwartz, and extended by H. P. Kramer to the general case ([**1**], XIX.2 Theorem 7). In this paper, we shall give a simple proof for Schwartz-Kramer's Theorem by using the theory of unconditional bases, and omit the condition of weak completeness in their theorem. In the proof of [**1**], XIX.2 Theorem 7, because of using [**1**], XVIII.2 Corollary 33, so that it needs the condition of weak completeness. On the other hand, all perturbant generalized eigenvectors consist of an unconditional basis, so we can prove the theorem without using the above corollary and omit the condition of weak completeness.

DEFINITION 1. A linear operator T in Banach space B is called discrete type ((D) type), if $\rho(T) \neq \varnothing$, and there exist an unconditional basis $\{x_n\}$ of B, a sequence of complex numbers $\{\lambda_n\}$ and a positive integer N, such that $\lim_n |\lambda_n| = +\infty$, $\lambda_n \neq \lambda_m$, $\forall\, n, m \in \mathbf{N}$, $m > N$ and $n \neq m$, $Tx_n = \lambda_n x_n$, $\forall\, n > N$, $T[x_1,\ldots,x_N] \subset [x_1,\ldots,x_N]$ and $\sigma(T\,|\,[x_1,\ldots,x_N]) = \{\lambda_1,\ldots,\lambda_N\}$.

PROPOSITION 2. *Let T be a linear operator of (D) type in Banach space B, $\{x_n\}$, $\{\lambda_n\}$ and N as in Definition* 1. *Then* $\sigma(T) = \{\lambda_n\}$,

$$\mathfrak{D}(T) = \left\{x \in B \,\middle|\, \text{if } x = \sum_n \alpha_n x_n, \text{ then } \sum_{n>N} \lambda_n \alpha_n x_n \in B\right\}$$

$$Tx = \sum_{n=1}^{N} \alpha_n T x_n + \sum_{n>N} \lambda_n \alpha_n x_n, \qquad \forall\, x = \sum_n \alpha_n x_n \in \mathfrak{D}(T).$$

However, for each $\lambda \notin \sigma(T)$, $R(\lambda, T) = (T - \lambda I)^{-1}$ is compact and

$$R(\lambda, T)x = \sum_{n=1}^{N} \alpha_n (T - \lambda I)^{-1} x_n + \sum_{n>N} \frac{\alpha_n}{\lambda_n - \lambda} x_n,$$
$$\forall\, x = \sum_n \alpha_n x_n \in B.$$

Proof. Define a linear operator T_0 in B as follows

$$\mathfrak{D}(T_0) = \left\{ x \in B \mid \text{if } x = \sum_n \alpha_n x_n, \text{ then } \sum_{n>N} \lambda_n \alpha_n x_n \in B \right\},$$

$$T_0 x = \sum_{n=1}^{N} \alpha_n T x_n + \sum_{n>N} \lambda_n \alpha_n x_n, \qquad \forall\, x = \sum_n \alpha_n x_n \in \mathfrak{D}(T_0).$$

Because T is closed, so $T \supset T_0$. Without loss of generality, we can assume $0 \in \rho(T)$. Then by $|\lambda_n| \to \infty$ and [3], Ch. II Lemma 16.1, let

$$y = \sum_{n=1}^{N} \alpha_n T^{-1} x_n + \sum_{n>N} \frac{\alpha_n}{\lambda_n} x_n \in \mathfrak{D}(T_0)$$

for $x = \Sigma_n \alpha_n x_n \in B$, and $T_0 y = x$, so that $T_0 \mathfrak{D}(T_0) = B$. Therefore $T = T_0$.

If $\lambda \neq \lambda_n$, $\forall\, n$, because of $|\lambda - \lambda_n| \to \infty$ and above Lemma 16.1, it is easy to see $(T - \lambda I)\mathfrak{D}(T) = B$. So that $\sigma(T) = \{\lambda_n\}$, and we have the formula about $R(\lambda, T)$.

We can assume $\|x_n\| = 1$, $\forall\, n$. Let $f_m \in B^*$, such that $f_m(x_n) = \delta_{n,m}$, $\forall\, n, m$. Then there exists a positive constant M_1, such that $\|f_m\| \leq M_1$, $\forall\, m$.

For each n, let P_n, Q_n be the projections, such that $P_n + Q_n = I$, and $P_n B = [x_1, \ldots, x_n]$, $Q_n B = [x_{n+1}, \ldots, x_m, \ldots]$. By [3], Ch. II Th. 17.1, there exists a positive constant M_2, such that $\|Q_n\| \leq M_2$, $\forall\, n$.

Again by above Th. 17.1, there exists a positive constant M_3, such that

$$\left\| \sum_n \beta_n \alpha_n x_n \right\| \leq M_3 \|x\|, \qquad \forall\, x = \sum_n \alpha_n x_n \in B \text{ and } |\beta_n| \leq 1 \ (\forall\, n).$$

Let $\lambda \in \rho(T)$ and $\{y_n\}$ be a bounded sequence of B, i.e. $\|y_n\| \leq M_4$, $\forall\, n$. Because $|f_m(y_n)| \leq M_1 M_4$, we can assume that

$$f_m(y_n) =: \alpha_m^{(n)} \to \alpha_m, \quad \forall\, m$$

(replacing a subsequence of $\{y_n\}$, if necessary). For $\varepsilon > 0$, there exists N_1 $(> N)$ such that $|1/(\lambda_n - \lambda)| < \varepsilon$, $\forall\, n \geq N_1$. Then for sufficiently large n, m

$$\|R(\lambda, T) Q_N (y_n - y_m)\| \leq \sum_{k=N+1}^{N_1} \left| \frac{\alpha_k^{(n)} - \alpha_k^{(m)}}{\lambda_k - \lambda} \right| + \left\| \sum_{k>N_1} \frac{\alpha_k^{(n)} - \alpha_k^{(m)}}{\lambda_k - \lambda} x_k \right\|$$

$$\leq \varepsilon + \varepsilon M_3 \|Q_{N_1} y_n - Q_{N_1} y_m\| \leq (1 + 2M_2 M_3 M_4)\varepsilon$$

Therefore $R(\lambda, T) Q_N$ and $R(\lambda, T)$ are compact. □

LEMMA 3. *Let $\{x_n\}$ be an unconditional basis of Banach space B, J be a subset of* $\mathbf{N}$. *Then $\{x_n \mid n \in J\}$ is an unconditional basis of $[x_n \mid n \in J]$, where $[x_n \mid n \in J]$ is the closed subspace generated by $\{x_n \mid n \in J\}$, and*

$$B = [x_n \mid n \in J] \dotplus [x_n \mid n \notin J].$$

However, let $P(J)$ be the projection from B onto $[x_n \mid n \in J]$ such that $(I - P(J))B = [x_n \mid n \notin J]$, then $J \to P(J)$ is countably additive in the strong operator topology from the σ-field of all subsets of $\mathbf{N}$ *into the projections in B, and $P(\mathbf{N}) = I$, $P(\varnothing) = 0$.*

Proof. Let P_n be the projection from B to $[x_n]$ such that $(I - P_n)B = [x_m \mid m \neq n]$, $\forall\, n$. If $x = \sum_n \alpha_n x_n \in [x_n \mid n \in J]$, by $P_m x = 0$, $\forall\, m \notin J$, so that $\alpha_m = 0$, $\forall m \notin J$ and $x = \sum_{n \in J} \alpha_n x_n$. This series is also unconditionally convergent, therefore $\{x_n \mid n \in J\}$ is an unconditional basis of $[x_n \mid n \in J]$. Similarly, $\{x_n \mid n \notin J\}$ is an unconditional basis of $[x_n \mid n \notin J]$, so that

$$B = [x_n \mid n \in J] \dotplus [x_n \mid n \notin J].$$

We notice the following fact: if $x = \sum_n \alpha_n x_n \in B$ and $\varepsilon > 0$, then there exists a positive integer N, such that

$$\Big\| \sum_{n \in \Lambda} \alpha_n x_n \Big\| < \varepsilon$$

where Λ is an arbitrary subset of $\mathbf{N}$ and $\Lambda \cap \{1, \ldots, N\} = \varnothing$. In fact, we have N such that

$$\Big\| \sum_{n > N} \alpha_n x_n \Big\| < \frac{\varepsilon}{M}$$

where M is the constant such that $\|\sum_n \varepsilon_n \beta_n x_n\| \leq M \|y\|$, $\forall\, y = \sum_n \beta_n x_n \in B$ and $|\varepsilon_n| \leq 1$, $\forall\, n \in \mathbf{N}$ ([3], Ch. II, Th. 17.1). Let

$$\varepsilon_n = \begin{cases} 1 & n \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

then

$$\Big\| \sum_{n \in \Lambda} \alpha_n x_n \Big\| = \Big\| \sum_{n > N} \varepsilon_n \alpha_n x_n \Big\| \leq M \Big\| \sum_{n > N} \alpha_n x_n \Big\| < \varepsilon.$$

Now let $J_i \subset \mathbf{N}$, $J_i \cap J_j = \varnothing$, $\forall\, i \neq j$, $J = \bigcup_{i=1}^{\infty} J_i$, and $x = \sum_n \alpha_n x_n \in B$, $\varepsilon > 0$. Take above N and a positive integer K such that

$$\bigcup_{i=1}^{K} J_i \supset J \cap \{1, \ldots, N\}$$

then

$$\left\| \left(\sum_{i=1}^{K} P(J_i) - P(J) \right) x \right\| = \left\| \sum_{n \in \Lambda} \alpha_n x_n \right\| < \varepsilon$$

where $\Lambda = J \setminus \bigcup_{i=1}^{K} J_i$ and $\Lambda \cap \{1,\ldots,N\} = \varnothing$. So that

$$P(J) = \operatorname*{s-lim}_{K} \sum_{i=1}^{K} P(J_i)$$

and $J \to P(J)$ is countably additive in the strong operator topology. □

Now we recall that a linear operator T in Banach space B is spectral as in [**1**], XVIII. 2, Definition 1, and T is discrete as in [**1**], XIX. 2, Definition 1, i.e. every resolvent $R(\lambda, T)$ of T is compact. We also say that a discrete spectral operator T satisfies condition (F), if for all but a finite number of spectral points λ, the space of generalized eigenvectors of T corresponding to λ is one-dimensional.

PROPOSITION 4. *Let T be a linear operator in Banach space B. Then T is (D) type, if and only if, T is a discrete spectral operator which satisfies condition (F).*

Proof. Let T be (D) type, $\{x_n\}$, $\{\lambda_n\}$ and N as in Definition 1. We assume that $\{\lambda_1,\ldots,\lambda_N\} = \{\lambda_1,\ldots,\lambda_k\}$, where $k \leq N$ and $\lambda_i \neq \lambda_j$, $\forall\, 1 \leq i \neq j \leq k$. If B_i is the space of generalized eigenvectors of T corresponding to λ_i, $1 \leq i \leq k$, then $N = \sum_{i=1}^{k} \dim B_i$. We can also assume that there is a partition $\{1,\ldots,N\} = \bigcup_{i=1}^{k} \Lambda_i$, such that $B_i = [x_n \mid n \in \Lambda_i]$, $1 \leq i \leq k$. Let $B_n = [x_n]$, $\forall\, n > N$ and $\mathfrak{B}$ be all Borel subsets of complex plane $\mathbf{C}$, and

$$P(\Delta) = \dot{+}\, \{B_n \mid \lambda_n \in \Delta\} \quad \forall \Delta \in \mathfrak{B}$$

then by Lemma 3, $\Delta \to P(\Delta)$ is countably additive in the strong operator topology.

Let $\Delta \in \mathfrak{B}$ and $x \in \mathfrak{D}(T) \cap P(\Delta)B$, we can write

$$x = \sum_{\substack{n \in \Lambda_i \\ \text{and } \lambda_i \in \Delta}} \alpha_n x_n + \sum_{\substack{n > N \\ \text{and } \lambda_i \in \Delta}} \alpha_n x_n$$

by Proposition 2,

$$Tx = \sum_{\substack{n \in \Lambda_i \\ \text{and } \lambda_i \in \Delta}} \alpha_n T x_n + \sum_{\substack{n > N \\ \text{and } \lambda_n \in \Delta}} \lambda_n \alpha_n x_n \in P(\Delta)B.$$

However, $T \mid P(\Delta)B$ also satisfies the assumptions of Proposition 2, so that $\sigma(T \mid P(\Delta)B) = \{\lambda_i \mid \lambda_i \in \Delta\} \subset \bar{\Delta}$. Therefore T is a discrete spectral operator which satisfies condition (F).

Conversely, let T be a discrete spectral operator which satisfies condition (F). Let $P(\cdot)$ be the resolution of the identity for T and assume that the different eigenvalues of T are $\lambda_1, \ldots, \lambda_k, \lambda_{N+1}, \ldots, \lambda_n, \ldots$ such that

$$N = \dim \bigoplus_{i=1}^{k} P(\{\lambda_i\})B, \qquad \dim P(\{\lambda_n\})B = 1, \forall\, n > N.$$

Let

$$[x_1, \ldots, x_N] = \bigoplus_{i=1}^{k} P(\{\lambda_i\})B, \qquad [x_n] = P(\{\lambda_n\})B, \forall\, n > N$$

because

$$P(\sigma(T)) = I, \qquad P\big(\{\lambda_n \mid n > N\}\big) = \operatorname*{s-lim}_n \bigoplus_{i=N+1}^{n} P\big(\{\lambda_{\sigma(i)}\}\big)B$$

for every permutation σ of $\{N+1, \ldots, n, \ldots\}$, so that $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis of B. Therefore T is (D) type. □

LEMMA 5. *Let $\{x_n\}$ be an unconditional basis of Banach space B, $\|x_n\| = 1$, $\forall\, n$, $\{y_n\}$ be a ω-linearly independent sequence of B,* i.e., *if $\Sigma_n \alpha_n y_n = 0$, then $\alpha_n = 0$, $\forall\, n$.*

(1) *If $\Sigma_n \|x_n - y_n\| < +\infty$, then $\{y_n\}$ is also an unconditional basis of B;*

(2) *If B is a Hilbert space, and $\Sigma_n \|x_n - y_n\|^2 < +\infty$, then $\{y_n\}$ is also an unconditional basis of B.*

Proof. (1) It follows by **[3]**, Ch. I, Th. 10.2, (a) $2^0. \Leftrightarrow 4^0.$, and **[3]**, Ch. II, Th. 17.1, $1^0. \Leftrightarrow 2^0.$;

(2) By **[3]**, Ch. II, Th. 18.1, we can assume that $\{x_n\}$ is an orthogonal normalized basis of B. Let N such that

$$\sum_{n>N} \|x_n - y_n\|^2 = \lambda^2 < 1, \qquad 0 \le \lambda < 1$$

and

$$z_n = \begin{cases} x_n & 1 \le n \le N \\ y_n & n > N \end{cases}$$

then

$$\sum_n \|x_n - z_n\|^2 = \lambda^2 < 1.$$

Because of

$$\left\| \sum_{n=1}^{m} \alpha_n(x_n - z_n) \right\|^2 \le \sum_{n=1}^{m} |\alpha_n|^2 \sum_{n=1}^{m} \|x_n - z_n\|^2 \le \lambda^2 \left\| \sum_{n=1}^{m} \alpha_n x_n \right\|^2$$

for all finite sequences of numbers $\alpha_1, \ldots, \alpha_m$, so by [3], Ch. I, Th. 9.1, (b)δ) and [3], Ch. II, Th. 17.1, $1^0. \Leftrightarrow 2^0.$, $\{z_n\}$ is also an unconditional basis of B.

Now by $\Sigma_n \|z_n - y_n\| < +\infty$, and 1) of this Lemma, $\{y_n\}$ is also an unconditional basis of B. □

THEOREM 6. *Let T be a linear operator of (D) type in Banach space B, $\{x_n\}$, $\{\lambda_n\}$ and N as in Definition* 1, $0 \in \rho(T)$. *Let V be a linear operator in B, such that $A = VT^{-\alpha}$ bounded, where $0 \le \alpha < 1$. Let $\nu_n = \min_{m \ne n} |\lambda_m - \lambda_n|$ and we have one of the following conditions*:

(1) $\Sigma_{n>N}(|\lambda_n| + \nu_n)^\alpha / \nu_n < +\infty$;

(2) *If B is a Hilbert space, and* $\Sigma_{n>N}(|\lambda_n| + \nu_n)^{2\alpha}/\nu_n^2 < +\infty$;

(3) $\lim_n(|\lambda_n| + \nu_n)^\alpha/\nu_n = 0$, *and* $\Sigma_{i,j}|a_{ij}| < \infty$, *where* $a_{ij} = f_j(Ax_i)$, *and* $f_j \in B^*$, $f_j(x_i) = \delta_{i,j}$;

(4) $(|\lambda_n| + \nu_n)^\alpha/\nu_n \le G$, $\forall n$, *and* $\Sigma_{i,j}|a_{ij}| \le \beta$, *where a_{ij} as in* (3), *and β is sufficiently small*;

(5) *If B is a Hilbert space*, $\langle x_n, x_m \rangle = \delta_{n,m}$, $\forall n, m$, $\lim_n(|\lambda_n| + \nu_n)^\alpha/\nu_n = 0$ *and A is a Hilbert-Schmidt operator*;

(6) *If B is a Hilbert space*, $\langle x_n, x_m \rangle = \delta_{n,m}$, $\forall n, m$, $(|\lambda_n| + \nu_n)^\alpha/\nu_n \le G$, $\forall n$, $\|A\|_2 \le \beta$, *where* $\| \ \|_2$ *is Hilbert-Schmidt norm, and β is sufficiently small, then $(T + V)$ is also (D) type in B.*

Proof. We can write $T = T_s + F$ such that $T_s x_n = \lambda_n x_n$, $n = 1, 2, \ldots$, and $F[x_1, \ldots, x_N] \subset [x_1, \ldots, x_N]$, $Fx_n = 0$, $\forall n > N$. Using $(F + V)$ instead of V, we can assume that $Tx_n = \lambda_n x_n$, $n = 1, 2, \ldots$. However, we can also assume $\|x_n\| = 1$, $\forall n$. By [3], Ch. II, Th. 17.1, there exists a constant $M\ (\ge 1)$, such that

$$\left\| \sum_n \beta_n \alpha_n x_n \right\| \le M\|x\|, \qquad \forall x = \sum_n \alpha_n x_n \in B \text{ and } |\beta_n| \le 1, \forall n.$$

(1) Let N_1 sufficiently large $(> N)$ such that

$$0 < M\frac{a_n}{1 - a_n} < 1, \quad \forall n > N_1$$

where $a_n = 2M\|A\|(|\lambda_n| + \nu_n)^\alpha/\nu_n$.

For $n > N_1$, let Γ_n be a circle whose center is at λ_n and radius is $\nu_n/2$. When $\lambda \in \Gamma_n$, because

$$\left|\frac{\lambda_m^\alpha}{\lambda_m - \lambda}\right| \le 2\frac{(|\lambda_n| + \nu_n)^\alpha}{\nu_n} \quad \forall\, m$$

so that $\|R(\lambda, T)\| \le 2M/\nu_n$, $\|VR(\lambda, T)\| \le a_n$. By **[2]**, $(T + V)$ has one and only one single eigenvalue $\lambda_n(V)$ in Γ_n. Since $0 < Ma_n/(1 - a_n) < 1$, we can take

$$x_n(V) = \sum_{l=0}^{\infty} \frac{-1}{2\pi i} \int_{\Gamma_n} (-1)^l R(\lambda, T)[VR(\lambda, T)]^l \, d\lambda x_n$$

as corresponding eigenvector. By

$$\|x_n(V) - x_n\| \le \sum_{l=1}^{\infty} \frac{1}{2\pi} \int_{\Gamma_n} \|R(\lambda, T)\| \, \|VR(\lambda, T)\|^l \, d|\lambda| \le M \sum_{l=1}^{\infty} a_n^l$$
$$< 4M^2\|A\|(|\lambda_n| + \nu_n)^\alpha/\nu_n$$

and condition 1, so that

$$\sum_{n > N_1} \|x_n(V) - x_n\| < +\infty.$$

However, since

$$|\lambda_n(V)| \ge |\lambda_n| - \frac{\nu_n}{2} \ge |\lambda_n| - \frac{|\lambda_n - \lambda_1|}{2}$$
$$\ge |\lambda_n| - \frac{1}{2}(|\lambda_n| + |\lambda_1|) = \frac{|\lambda_n|}{2} - \frac{|\lambda_1|}{2}$$

so that $\lim_n |\lambda_n(V)| = +\infty$.

Let Γ be a closed road, containing the points $\lambda_1, \ldots, \lambda_{N_1}$, and such that $\operatorname{dist}(\Gamma, \{\lambda_1, \ldots, \lambda_{N_1}\}) = \nu/2$, where $\nu = \min_{n > N_1} \nu_n$. With the aid of **[2]**, page 34 Lemma 4.10 and page 178 Theorem 6.17 we are able to show that $(T + V)$ has different eigenvalues $\lambda_1(V), \ldots, \lambda_k(V)$ $(k \le N_1)$ in Γ, and there exist linearly independent elements $x_1(V), \ldots, x_{N_1}(V)$ of B such that

$$(T + V)\big[x_1(V), \ldots, x_{N_1}(V)\big] \subset \big[x_1(V), \ldots, x_{N_1}(V)\big]$$

and

$$\sigma\big((T + V)\,|\big[x_1(V), \ldots, x_{N_1}(V)\big]\big) = \{\lambda_1(V), \ldots, \lambda_k(V)\}.$$

Now it is sufficient to prove that $\{x_n(V) \mid n = 1, 2, \ldots\}$ is an unconditional basis of B. Because of Lemma 5, we only need to prove that $\{x_n(V) \mid n = 1, 2, \ldots\}$ is ω-linearly independent.

Let

$$\hat{P}_n = \frac{-1}{2\pi i}\int_{\Gamma_n} R(\lambda, T+V)\, d\lambda \quad \forall\, n > N_1$$

then

$$\hat{P}_n x_m(V) = \delta_{n,m} x_m(V) \quad \forall\, m \in \mathbf{N} \text{ and } n > N_1.$$

If $\Sigma_m \beta_m x_m(V) = 0$, then $0 = \hat{P}_n \Sigma_m \beta_m x_m(V) = \beta_n x_n(V)$ and $\beta_n = 0$, $\forall\, n > N_1$, and $\Sigma_{m=1}^{N_1} \beta_m x_m(V) = 0$. But $\{x_m(V) \mid 1 \le m \le N_1\}$ is linearly independent, so that $\beta_n = 0$, $\forall\, n$. This shows $\{x_n(V) \mid n = 1, 2, \ldots\}$ is ω-linearly independent.

(2) Similarly to (1), let N_1 sufficiently large ($> N$) such that

$$0 < M\frac{a_n}{1 - a_n} < 1 \quad \forall\, n > N_1$$

where $a_n = 2M\|A\|(|\lambda_n| + \nu_n)^\alpha / \nu_n$.

For $n > N_1$, let Γ_n as (1). When $\lambda \in \Gamma_n$, we also have $\|R(\lambda, T)\| \le 2M/\nu_n$ and $\|VR(\lambda, T)\| \le a_n$. Hence $(T+V)$ has one and only one single eigenvalue $\lambda_n(V)$ in Γ_n, and corresponding eigenvector is

$$x_n(V) = \sum_{l=0}^{\infty} \frac{-1}{2\pi i}\int_{\Gamma_n} (-1)^l R(\lambda, T)[VR(\lambda, T)]^l\, d\lambda\, x_n.$$

It is obvious that $\lim_n |\lambda_n(V)| = +\infty$. We also take Γ and $x_1(V), \ldots, x_{N_1}(V)$ as in (1). For $n > N_1$, as (1)

$$\|x_n(V) - x_n\| \le 4M^2\|A\|(|\lambda_n| + \nu_n)^\alpha / \nu_n.$$

By condition (2)

$$\sum_n \|x_n(V) - x_n\|^2 < +\infty.$$

Similarly as (1), $\{x_n(V) \mid n = 1, 2, \ldots\}$ is ω-linearly independent. By Lemma 5, $\{x_n(V) \mid n \in \mathbf{N}\}$ is an unconditional basis of B, so that $(T+V)$ is still (D) type.

(3) Let

$$\sup_n \|f_n\| = K, \qquad b_n = \frac{1}{\nu_n} 2MK\sum_{ij} |a_{ij}|(|\lambda_n| + \nu_n)^\alpha.$$

Because of $\|A\| < K\Sigma_{ij}|a_{ij}|$, so $b_n > a_n$ (the definition of a_n, see (1)). By assumption, for large n,

$$0 < M\frac{b_n}{1 - b_n} < 1.$$

Let Γ_n as (1), then there exists only one single eigenvalue $\lambda_n(V)$ of $(T+V)$ in Γ_n, and the corresponding eigenvector $x_n(V)$ has also the formula as (1). For $\lambda \in \Gamma_n$

$$\|R(\lambda, T)[VR(\lambda, T)]^l x_n\|$$

$$= \left\| \sum_{k_1} \cdots \sum_{k_l} \frac{\lambda_n^\alpha}{\lambda_n - \lambda} \frac{\lambda_{k_1}^\alpha}{\lambda_{k_1} - \lambda} \cdots \frac{\lambda_{k_{l-1}}^\alpha}{\lambda_{k_{l-1}} - \lambda} \frac{\langle Ax_n, f_{k_1}\rangle \cdots \langle Ax_{k_{l-1}}, f_{k_l}\rangle}{\lambda_{k_l} - \lambda} x_{k_l} \right\|$$

$$\leq \left[\frac{2(|\lambda_n| + \nu_n)^\alpha}{\nu_n}\right]^l \frac{2}{\nu_n} \sum_{k_1, \ldots, k_l} |a_{nk_1} \cdots a_{k_{l-1}k_l}|$$

$$\leq \left[\frac{2(|\lambda_n| + \nu_n)^\alpha}{\nu_n}\right]^l \frac{2}{\nu_n} \left(\sum_{ij} |a_{ij}|\right)^{l-1} \sum_k |a_{nk}|$$

so that

$$\|x_n(V) - x_n\| \leq \sum_{l=1}^{\infty} b_n^l \sum_k |a_{nk}| \Big/ \sum_{ij} |a_{ij}|$$

and $\sum_n \|x_n(V) - x_n\| < +\infty$. The rest part of proof is similar as (1).

(4) Take

$$\beta < (2M(M+1)KG)^{-1}$$

the proof is similar as (3).

(5) Let

$$c_n = 2M\|A\|_2(|\lambda_n| + \nu_n)^\alpha / \nu_n$$

because $\|A\| \leq \|A\|_2$, so that $c_n > a_n$. If n sufficiently large,

$$0 < M\frac{c_n}{1 - c_n} < 1.$$

Let Γ_n as 1), then $\lambda_n(V)$, $x_n(V)$ as (1). For $\lambda \in \Gamma_n$,

$$\|R(\lambda, T)[VR(\lambda, T)]^l x_n\|^2$$

$$\leq \left[\frac{2(|\lambda_n| + \nu_n)^\alpha}{\nu_n}\right]^{2l} \left(\frac{2}{\nu_n}\right)^2 \|A\|_2^{2(l-1)} \sum_k |a_{nk}|^2$$

and

$$\|x_n(V) - x_n\| < \sum_{l=1}^{\infty} c_n^l \frac{1}{\|A\|_2} \left(\sum_k |a_{nk}|^2 \right)^{1/2}$$

so that

$$\sum_n \|x_n(V) - x_n\|^2 < +\infty.$$

The rest part of proof is similar as (2).

(6) Take

$$\beta < (2M(M+1)G)^{-1}$$

the proof is similar as (5).

This completes the proof of Theorem 6.

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