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ON THE SEARCH FOR WEIGHTED NORM INEQUALITIES FOR THE FOURIER TRANSFORM

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B. Muckenhoupt posed in [1] the problem of characterizing those non-negative functions u and v, which for some p, $1 \le p < \infty$, the inequality

$$\int_{-\infty}^{+\infty} |\hat{f}(x)|^p u(x)\,dx \le C \int_{-\infty}^{+\infty} |f(x)|^p v(x)\,dx$$

holds for any f, where $\hat{f}$ denotes the Fourier transform of f. In this paper we deal only with the case where either $u \equiv 1$ or $v \equiv 1$, finding that when $v \equiv 1$, $1 < p < 2$, a necessary condition is that for any $r > 0$,

$$\left[\sum_{k=-\infty}^{+\infty} \left(\int_{rk}^{r(k+1)} u(x)\,dx\right)^b\right]^{1/b} \le Cr^{p-1}$$

where $b = 2/(2-p)$, and that a sufficient condition ($v \equiv 1$, $1 \le p$) is that for any measurable set E,

$$\int_E u(x)\,dx \le C|E|^{p-1}.$$

Similar conditions are obtained for the case $u \equiv 1$. Although we will show that the sufficient condition is not necessary (in §4, Corollary 1 and again in §6, Corollary 3 and Remark 4), we were unable to obtain any conclusions on our necessary condition.

1. As far as we know, only sufficient conditions had been considered before, in the case where both u and v are powers of $|x|$ (see e.g. [**2**] for the trigonometric case), although our sufficient condition is somehow a restatement of a generalized Hausdorf-Young inequality (see e.g. [**4**, page 200]). In this connection we must point out the work of Hardy, Littlewood and Paley (see e.g. [**5**, Chapter VII, §8 and Chapter XII, §§3, 5 and 6]). By the way, both of our conditions may be easily translated to the trigonometric setting, where relationships between f and its Fourier coefficients are considered, f appearing either on the left or right hand side of the inequality. For a similar inequality to that of our necessary condition see, for instance, [**5**, Chapter XVI, Example 8, page 298] where, however, the exponents are less than 2.

In §2 we give some introductory ideas which give some "feeling" for the subject and use these in §3 to show a simple necessary condition when $v \equiv 1$ which is quite similar to the sufficient condition treated in §4 where we also give several equivalent conditions. In §5 we give some examples showing the gap between the conditions of the two previous sections so

that the necessary condition of §6 comes as no surprise. In this latter section we make several observations and give some examples, showing the implications between all three conditions. Finally, in §7 we briefly consider the case $u \equiv 1$.

We will denote by $\hat{f}$ the Fourier transform of f defined by

$$\hat{f}(x) = \int_{-\infty}^{+\infty} e^{-ixy} f(y)\, dy.$$

The corresponding inverse Fourier transform of f will be denoted by $\check{f}$. C, as usual, stands for a constant which need not be the same at each occurrence, which may depend on p but not on the general functions considered. We will work in just one dimension and omit limits of integration or summation when it is clear what they should be. Throughout the paper u and v will be non-negative measurable functions.

2. Preliminaries. When considering inequalities of the type

$$(*) \qquad \int |\hat{f}|^p u\, dx \le C \int |f|^p v\, dx$$

it is apparent that the particular behaviour of translations and dilations under the Fourier transform will be reflected on properties of u and v. Let us recall that

(1) if $f_\varepsilon(x) = \frac{1}{\varepsilon} f(x/\varepsilon)$, then $(f_\varepsilon)\hat{}\,(x) = \hat{f}(\varepsilon x)$.

(2) if $g(x) = f(x + a)$, then $\hat{g}(x) = e^{iax}\hat{f}(x)$ and the "reciprocal" to (2).

(3) if $g(x) = e^{iax} f(x)$, then $\hat{g}(x) = \hat{g}(x - a)$.

Let us take for instance the case where $(*)$ holds and $u = v$ is locally integrable. Using (2) and (3) above and $(*)$ twice it is not difficult to show that

$$\int |f(-x + a)|^p u(x)\, dx \le C \int |f(x + b)|^p u(x)\, dx$$

for any choice of a and b, since $\hat{\hat{f}}(x) = 2\pi f(-x)$. If f is the characteristic function of the interval $(-\varepsilon, \varepsilon)$ we may conclude that for any $\varepsilon > 0$ and any a and b,

$$\frac{1}{2\varepsilon} \int_{|x-a|<\varepsilon} u(x)\, dx \le \frac{C}{2\varepsilon} \int_{|x-b|<\varepsilon} u(x)\, dx.$$

Since we are assuming that u is locally integrable, for some $b = b_0$ we must have both

$$M(b) = \sup_{0<\varepsilon<1} \frac{1}{2\varepsilon} \int_{|x-b|<\varepsilon} u(x)\, dx < \infty$$

and

$$u(b) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|x-b|<\varepsilon} u(x)\, dx.$$

Therefore for any a we must have

$$M(a) = \sup_{0<\varepsilon<1} \frac{1}{2\varepsilon} \int_{|x-a|<\varepsilon} u(x)\, dx \leq CM(b_0)$$

and for almost all a,

$$u(a) \leq Cu(b_0).$$

So we conclude that if u is not identically zero (i.e. the set where u is different from zero has positive measure), there exist non-zero constants A and B so that

$$A \leq u(x) \leq B$$

for almost all x. Hence, u may be replaced by the funciton identically equal to 1 in (*).

Now if $f(x) = e^{-x^2/2}$, we know that $\hat{f}(x) = \sqrt{2\pi} f(x)$, taking $f_\varepsilon(x)$ as in (1) above, we see that

$$\int |\hat{f}_\varepsilon|^p\, dx = \frac{C_1}{\varepsilon}$$

and

$$\int |f_\varepsilon|^p\, dx = \frac{C_2}{\varepsilon^{p-1}}.$$

Applying (*) with $u \equiv v \equiv 1$, we must have for all $\varepsilon > 0$,

$$\frac{C_1}{\varepsilon} \leq C \frac{C_2}{\varepsilon^{p-1}}$$

so that if $u \not\equiv 0$, necessarily $p = 2$.

This shows how different our weights must be from those considered when $\hat{f}$ is replaced in (*) by the Hardy-Littlewood maximal function or the Hilbert transform (see e.g. [**1**]).

We will treat now the case $v \equiv 1$.

3. A simple necessary condition. Here we want to obtain properties on the non-negative function u if the inequality

$$(**) \qquad \int |\hat{f}|^p u\, dx \leq C \int |f|^p\, dx$$

holds, where $1 \leq p < \infty$. Recalling the behaviour of translations and dilations under the Fourier transform, we let

$$f(x) = \chi_{(-1/r,1/r)}(x)e^{iax}$$

where χ_E denotes the characteristic function of the set E.

Then

$$f(x) = 2\frac{\sin((x-a)/r)}{x-a}$$

so that $|\hat{f}(x)| \geq 1/r$ if $|x-a| \leq r$, and we must have, with $I = \{x: |x-a| \leq r\}$,

$$\int_I u(x)\,dx \leq r^p \int |\hat{f}|^p u\,dx \leq Cr^p \int |f|^p\,dx = Cr^{p-1}$$

i.e. u must be locally integrable and, denoting by $|E|$ the measure of the (measurable) set E,

$$\int_I u(x)\,dx \leq C|I|^{p-1}$$

for any interval I.

We observe that for $p = 1$ this condition implies that u is integrable over all of the real line and, by using Lebesgue's dominated convergence theorem, we see that u must be bounded for $p = 2$ and u must vanish identically for $p > 2$. So from now on we will restrict our attention to the case $1 \leq p \leq 2$.

4. A sufficient condition when $v \equiv 1$, $1 \leq p \leq 2$. A modification of the proof given in Zygmund's book [**5**, vol. 2, page 121] of a theorem of Paley will show that the following theorem holds:

THEOREM 1. *Let $1 \leq p \leq 2$. If the locally integrable function u satisfies the inequality*

$$\int_E u(x)\,dx \leq C|E|^{p-1}$$

for any measurable set E, then

$$\int |\hat{f}|^p u\,dx \leq C\int |f|^p\,dx.$$

Before giving the proof of Theorem 1, however we will give different equivalent conditions on u:

THEOREM 2. *Let* $1 < p < 2$ *and let* $b = 2/(2-p)$. *Suppose* u *is locally integrable, then the following are equivalent*:

(i) $|\{x: u(x) > \lambda\}| \leq C/\lambda^{b/2}$ *for all* $\lambda > 0$.

(ii) $\int_E u\,dx \leq C|E|^{p-1}$ *for all measurable sets* E.

(iii) *For* $\alpha > b/2$,

$$\int_{\{x:\, u^{b/2}(x) \leq \lambda\}} u^{\alpha}(x)\,dx \leq C_{\alpha}\lambda^{(2\alpha/b)-1} \quad \text{for all } \lambda > 0.$$

(iv) *For any* $r > 0, \lambda > 0$,

$$\#\left(\left\{k: \int_{kr}^{(k+1)r} u\,dx > \lambda\right\}\right) \leq C\left[\frac{r^{p-1}}{\lambda}\right]^{2/b}$$

where $\#(A)$ *denotes the number of elements of the set* A.

Proof of Theorem 2.

(i) implies (ii). Let $u^*(t)$ be defined for $t > 0$ as the non-increasing rearrangement of u. Observe that since $\lambda^{-b/2}$ is essentially the distribution function of $|x|^{p-2}$, for some constant $C > 0$ we must have

$$u^*(t) \leq Ct^{p-2}$$

and then

$$\int_E u\,dx \leq \int_0^{|E|} u^*\,dx \leq C\int_0^{|E|} t^{p-2}\,dt = C|E|^{p-1}$$

since $p > 1$. (For the first inequality see for instance [**5**, vol 1, page 31].)

(ii) implies (i). Let $E = \{x: u(x) > \lambda\}$, $E_n = E \cap [-n, n]$. Then

$$|E_n| \leq \frac{1}{\lambda}\int_{E_n} u\,dx \leq \frac{C}{\lambda}|E_n|^{p-1}$$

so that

$$|E_n| \leq \frac{C}{\lambda^{b/2}}$$

and letting n go to infinity, we obtain

$$|E| \leq \frac{C}{\lambda^{b/2}}.$$

(i) implies (iii).

$$\int_{\{x:u^{b/2}(x)\le\lambda\}} u^\alpha(x)\,dx \le \sum_{n=0}^{\infty} \int_{\{x:2^{-n-1}\lambda<u^{b/2}(x)\le 2^{-n}\lambda\}} u^\alpha(x)\,dx$$

$$\le C \sum_{n=0}^{\infty} (2^{-n}\lambda)^{2\alpha/b} \cdot |\{x: u^{b/2}(x) > 2^{-n-1}\lambda\}|$$

$$\le C \sum_{n=0}^{\infty} \frac{\lambda^{2\alpha/b-1}}{2^{n(2\alpha/b-1)}} = C^{(2\alpha/b)-1}.$$

(iii) implies (i).

$$|\{x: u(x) > \lambda\}| \le \sum_{n=0}^{\infty} |\{x: 2^{n+1}\lambda \ge u(x) > 2^n\lambda\}|$$

$$\le \sum_{n=0}^{\infty} \frac{1}{(2^n\lambda)^b} \int_{\{x:\, u(x)\le 2^{n+1}\lambda\}} u^\alpha(x)\,dx$$

$$\le C \sum_{n=0}^{\infty} \frac{1}{(2^n\lambda)^\alpha} (2^n\lambda)^{-b/2} \le C\lambda^{-b/2}.$$

(i) implies (iv). Given $\lambda > 0$ and $r > 0$, set $\beta = (\lambda/4r)^{b/2}$ and let $E = \{x:\ u^{b/2}(x) > \beta\}$. Let I_k denote the interval $(kr,(k+1)r)$ and assume $\int_{I_k} u\,dx > \lambda$. Then either $\int_{I_k\cap E} u\,dx > \lambda/2$ or $\int_{I_k\cap \mathcal{C}E} u\,dx > \lambda/2$, where $\mathcal{C}E$ is the complement of E. However, $\int_{I_k\cap \mathcal{C}E} u\,dx > \lambda/2$ implies $\lambda/2 < \int_{I_k\cap \mathcal{C}E} u\,dx \le \beta^{2/b} r = \lambda/4$, which is impossible. So

$$\#\left(\left\{k: \int_{I_k} u\,dx > \lambda\right\}\right) \le \#\left(\left\{k: \int_{I_k\cap E} u\,dx > \frac{\lambda}{2}\right\}\right).$$

Now

$$\frac{\lambda}{2}\#\left(\left\{k: \int_{I_k\cap E} u\,dx > \frac{\lambda}{2}\right\}\right) \le \sum_{k=-\infty}^{+\infty} \int_{I_k\cap E} u\,dx \le \int_E u\,dx$$

which, by (ii) and (i) is bounded by $C|E|^{p-1} \le C\beta^{1-p}$. So

$$\#\left(\left\{k: \int_{I_k} u\,dx > \lambda\right\}\right) \le C\left(\frac{r}{\lambda}\right)^{(p-1)/(2-p)} \frac{1}{\lambda} = C\left(\frac{r^{p-1}}{\lambda}\right)^{2/b}.$$

(iv) implies (i). Let

$$u_n(x) = 2^n \int_{k/2^n}^{(k+1)/2^n} u(t)\,dt \quad \text{for } \frac{k}{2^n} \le x < \frac{k+1}{2^n}.$$

Then u_n converges to u almost everywhere and hence in measure. Therefore

$$|\{x: u(x) > \lambda\}| \leq \underline{\lim} |\{x: u_n(x) > \lambda/2\}|$$

but

$$|\{x: u_n(x) > \lambda\}| = \frac{1}{2^n} \# \left(\left\{ k: 2^n \int_{k/2^n}^{(k+1)/2^n} u(t)\, dt > \lambda \right\} \right)$$

$$\leq \frac{C}{2^n} \left(\frac{(1/2^n)^{p-1}}{\lambda/2^n} \right)^{1/(2-p)} = \frac{C}{\lambda^{2/p}}. \qquad \square$$

We should observe that in Theorem 2, if $p = 1$ we have (i) implies (ii) but not the converse. Actually, in the proof of Theorem 1 we will only use condition (iii) when $1 < p < 2$. However it is important to put the condition in the form (i) which says that the distribution function of u is bounded by that of $|x|^{2-p}$, showing the connection with the Hausdorf-Young theorem and the Hardy, Littlewood and Paley results mentioned in the introduction. Condition (iii) is stated because of its remarkable similarity with the necessary condition found in §3. Condition (iv) will be useful for comparing the sufficient condition and the necessary one in §6.

We turn now to the proof of Theorem 1:

Proof of Theorem 1. As already mentioned, we will use Marcinkiewicz' interpolation theorem to interpolate between $p = 1$ and $p = 2$. Let T be defined for $f \in L^p \cap L^1$ by

$$Tf(x) = \begin{cases} u^{-b/2}(x) f(x) & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Then

$$\int |\hat{f}|^p u\, dx = \int |Tf|^p u^b\, dx$$

where $b = 2/(2 - p)$.

In L^1 we have the weak-type estimate

$$\int_{\{x: |Tf(x)| > \lambda\}} u^b(x)\, dx < \frac{C}{\lambda} \int |f|\, dx$$

since $|\hat{f}(x)| \leq \int |f(t)|\, dt$ and (iii) of Theorem 2 is satisfied. On the other hand, in L^2, by Plancherel's identity we have the inequality

$$\int |Tf|^2 u^b\, dx \leq C \int |f|^2\, dx.$$

By interpolating, we obtain the result when $1 < p < 2$. If $p = 1$, the condition $\int_E u \leq C$ for any measurable set E implies $u \in L^1$, and since $|\hat{f}(x)| \leq \int |f(t)|\, dt$, the result is obvious. On the other hand, when $p = 2$, $\int_E u\, dx \leq C|E|$ implies $|u(x)| \leq C$ and the result follows by Plancherel's identity. □

It is interesting to observe that for $p = 1$ or $p = 2$ the simple necessary condition of §3 and the sufficient condition of Theorem 1 are equivalent, in fact we used the former in the last proof when $p = 1$ or $p = 2$. However we cannot "interpolate" and obtain that they are equivalent for $1 < p < 2$, as we show in the following section.

5. Some examples. Let p be strictly between 1 and 2. Let F be a measurable set such that $|F| < \infty$. It is not difficult to show that if we take $u = \chi_F$ then

$$\int |\hat{f}|^p u\, dx = \int_F |\hat{f}|^p\, dx \leq C|F|^{2-p} \int |f|^p\, dx$$

where C may depend on p. (In fact it follows from the previous theorem, keeping track of the constants, or directly by using interpolation.) However, for some sets we can sharpen the inequality:

Let $E = \bigcup_{k=1}^{N} (2^k, 2^k + 1)$, we will show that in this case the mapping $f \to \chi_E \hat{f}$ has norm essentially equivalent (depending on p) to $N^{(2-p)/2p}$. To prove this fact we will use the equivalence

$$\int \Big(\sum_{I \in \Delta} |S_I f|^2 \Big)^{p/2} dx = \int |f|^p\, dx$$

valid for $1 < p < \infty$, where Δ stands for the collection of all dyadic intervals and $S_I f$ is defined by $(S_I f)\hat{} = \chi_I \hat{f}$. (A precise statement and proof may be found in [3, page 104].)

Let $A_k = (2^k, 2^k + 1)$ and let $I_k = (2^k, 2^{k+1})$ so that $E = \bigcup_1^N A_k$ and $A_k \subset I_k$. Then

$$\int_E |\hat{f}|^p\, dx = \sum_1^N \int_{A_k} |\chi_{I_k} \hat{f}|^p\, dx \leq C \sum_1^N \int |S_{I_k} f|^p\, dx$$

where the last inequality follows by taking $A_k = F$ at the beginning of this paragraph. Using Hölder's inequality we derive

$$\int_E |\hat{f}|^p\, dx \leq \int \Big(\sum_1^N |S_{I_k} f|^2 \Big)^{p/2} N^{(2-p)/2}\, dx$$

$$\leq C N^{(2-p)/2} \int |f|^p\, dx.$$

To see that indeed the norm is not much smaller than $N^{(2-p)/2p}$, we take f so that $\hat{f} = \chi_E$. Then

$$\int_E |\hat{f}|^p \, dx = \int_E dx = N$$

and

$$\int |f|^p \, dx \simeq \int \left(\sum_1^N |S_{I_k} f|^2 \right)^{p/2} dx$$

but

$$|S_{I_k} f| = |\check{\chi}_I| \text{ , where } I = (0, 1).$$

Therefore

$$\int |f|^p \, dx \simeq N^{p/2}$$

and so

$$\int_E |\hat{f}|^p \, dx \simeq N^{1-p/2} \int |f|^p \, dx.$$

As an application of this example we have the following.

COROLLARY 1. *The condition* $\int_E u(x)\, dx \leq C |E|^{p-1}$ *for any measurable set* E *is not necessary for* (**) *to hold if* $1 < p < 2$.

Proof. Let $E(\varepsilon, N) = \cup_{k=1}^N (\varepsilon 2^k, \varepsilon(2^k + 1))$. By using the behaviour of dilations under the Fourier transform and the previous example we get

$$\int_{E(\varepsilon,N)} |\hat{f}|^p \, dx \leq (\varepsilon^2 N)^{1/b} \int |f|^p \, dx.$$

Let now $\varepsilon_n = 2^{-n}$, $N_n = 2^n$,

$$E_n = 2^{N_n+1} + E(\varepsilon_n, N_n) = \bigcup_{k=1}^{2^n} \left(2^{2^n+1} + 2^{k-n}, 2^{2^n+1} + (2^k + 1)2^{-n}\right).$$

Finally let $u = \Sigma_{n=1}^\infty \chi_{E_n}$. Since the E_n's are disjoint, we have

$$\int |\hat{f}|^p u \, dx = \sum_{n=1}^\infty \int_{E_n} |\hat{f}|^p \, dx \leq C \sum_{n=1}^\infty \frac{1}{2^{n/b}} \int |f|^p \, dx \leq C \int |f|^p \, dx.$$

So (**) holds. On the other hand, for $\lambda < 1$,

$$|\{x: u(x) > \lambda\}| = \sum_n |E_n| = \infty$$

and condition (i) of Theorem 2 is not satisfied. □.

COROLLARY 2. *The condition* $\int_I u\,dx \le C\,|I|^{p-1}$ *for any interval* I *is not sufficient for* (**) *to hold if* $1 < p < 2$.

Proof. Take $u = \chi_E$, where $E = \cup_{k=1}^{\infty}(2^k, 2^k + 1)$. It is easy to see that (**) does not hold for u, because of the example given at the beginning of this section letting $N \to \infty$. On the other hand, the condition of §3 is satisfied since

$$2^j < |I| \le 2^{j+1} \quad \text{for } j > 0,$$

implies

$$\int_I u\,dx \le \int_0^{2^{j+1}} u\,dx = j \le C \log |I| \,. \qquad \square$$

REMARK 1. It is a curious fact that in both corollaries, the function u that was found is independent of p.

6. A stronger necessary condition. The preceding paragraph shows that we must get an intermediate condition for $1 < p < 2$. It is possible to obtain the following

THEOREM 3. *If* $1 < p < 2$ *and* u *satisfies*

$$\int |\hat{f}|^p u\,dx \le C \int |f|^p\,dx$$

then u *must satisfy the inequality*

$$\left(\sum_{k=-\infty}^{+\infty} \left(\int_{kr}^{(k+1)r} u(x)\,dx \right)^b \right)^{1/b} \le C r^{p-1}$$

for any $r > 0$, *where* $b = 2/(2 - p)$.

Proof. Let $g(x) = \sum_k a_k \chi_{(k,k+1)}$, where $a_k = 0$ except for finitely many k's. Let f be defined by $\hat{f}(x) = g(x)$, i.e.

$$f(x) = \sum_k a_k e^{ikx} \frac{(e^{ix} - 1)}{2\pi i x}.$$

Let us estimate firstly the L^p-norm of f.

$$\int |f|^p\,dx = \int_{-\pi}^{\pi} |f(x)|^p\,dx + \sum_{n \ne 0} \int_{-\pi + n2\pi}^{\pi + n2\pi} |f(x)|^p\,dx.$$

For $n \ne 0$,

$$\int_{-\pi + n2\pi}^{\pi + n2\pi} |f(x)|^p\,dx \le \frac{C}{|n|^p} \int_{-\pi}^{\pi} \left| \sum a_k e^{ikx} \right|^p dx$$

and also

$$\int_{-\pi}^{\pi} |f(x)|^p \, dx = \int_{-\pi}^{\pi} \left| \sum a_k e^{ikx} \right|^p dx$$

since $|(e^{ix} - 1)/x|$ remains bounded by above and by below away from zero if $|x| < \pi$. Therefore

$$\int_{-\infty}^{+\infty} |f|^p \, dx \cong \int_{-\pi}^{\pi} \left| \sum a_k e^{ikx} \right|^p dx.$$

On the other hand,

$$\int |\hat{f}|^p u \, dx = \sum_k |a_k|^p \int_k^{k+1} u(x) \, dx.$$

Since we must have

$$\sum_k |a_k|^p \int_k^{k+1} u(x) \, dx \le C \int_{-\pi}^{\pi} \left| \sum a_k e^{ikx} \right|^p dx \le C \left(\sum_k |a_k|^2 \right)^{p/2}$$

for any choice of the (finite) sequence a_k, we obtain our result with $r = 1$.

The case of general $r > 0$ is obtained from the previous one by considering dilations: if $f_\varepsilon(x) = \frac{1}{\varepsilon} f(x/\varepsilon)$, then $\widehat{(f_\varepsilon)}(x) = \hat{f}(\varepsilon x)$ and $\int |f_\varepsilon|^p \, dx = (\varepsilon^{p-1})^{-1} \int |f|^p \, dx$, choosing $\varepsilon = \frac{1}{r}$, the result follows. □

Several remarks are in order:

REMARK 2. It should be clear that this new condition stays in between the earlier two: it implies that of §3 obviously and is strictly stronger by the example in Corollary 2 in §5. It is implied by the sufficient condition of course, but it is weaker by Corollary 1 of §5. This may be also seen by (iv) of Theorem 2, a result which we state as a corollary.

COROLLARY 3. *Let u be locally integrable and let $1 < p < 2$. Suppose for any measurable E,*

$$\int_E u \, dx \le C |E|^{p-1}$$

then for any $\alpha > 1/(2 - p) = b/2$ and any $r > 0$

$$\left(\sum_{k=-\infty}^{+\infty} \left(\int_{kr}^{k(r+1)} u \, dx \right)^\alpha \right)^{1/\alpha} \le C r^{p-1}.$$

Proof. We just observe that for $\alpha = b/2$, we have a weak-type inequality by (iv) in Theorem 2 of §4.

REMARK 3. The necessary condition in Theorem 3 cannot be changed much as for the lengths of the intervals: we cannot consider arbitrary

lengths in the same sum. For suppose $u(x) = |x|^{p-2}$, for which (**) holds, and let $I_k = (2^k, 2^{k+1})$, then

$$\sum_{k=-\infty}^{+\infty} \left(\frac{1}{|I_k|^{p-1}} \int_{I_k} u\,dx \right)^{\alpha} = \infty$$

for *any* α.

REMARK 4. The exponent $b = 2/(2-p)$ cannot be lowered. For let $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \gamma_k^b < \infty$, and let $u = \sum_{k=1}^{\infty} \gamma_k \chi_{(2^k, 2^{k+1})}$, then (**) holds. For let $f \in L^p$,

$$\int |\hat{f}|^p u\,dx = \sum_{k=1}^{\infty} \gamma_k \int_{2^k}^{2^{k+1}} |\hat{f}|^p\,dx = \sum_{k=1}^{\infty} \gamma_k \int_{2^k}^{2^{k+1}} |(S_k f)\hat{}\,|^p\,dx$$

where $S_k f$ is defined by $(S_k f)\hat{} = \hat{f} \cdot \chi_{(2^k, 2^{k+1})}$. By what was said at the beginning of §5 and the result on dyadic decomposition mentioned there, we have

$$\begin{aligned}
\int |\hat{f}|^p u\,dx &\le C \sum_{k=1}^{\infty} \gamma_k \int_{-\infty}^{+\infty} |S_k f|^p\,dx = C \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{\infty} \gamma_k |S_k f|^p \right) dx \\
&\le C \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{\infty} \gamma_k^b \right)^{1/b} \left(\sum_{k=-\infty}^{+\infty} |S_k f|^2 \right)^{p/2} dx \\
&\le C \left(\sum_{k=1}^{\infty} \gamma_k^b \right)^{1/b} \int_{-\infty}^{+\infty} |f|^p\,dx
\end{aligned}$$

using Hölder's inequality to obtain the next to the last inequality. However, we may further ask $\sum_{k=1}^{\infty} \gamma_k^{\alpha} = \infty$ for $\alpha < b$, from where

$$\sum_{j=-\infty}^{+\infty} \left(\int_{J}^{j+1} u\,dx \right)^{\alpha} = \infty$$

for any such α.

7. The case $u \equiv 1$. Here we want to examine those v's for which, for some p, $1 < p < \infty$,

$$(***) \qquad \int |\hat{f}|^p\,dx \le C \int |f|^p v\,dx.$$

An immediate consequence of this inequality is the fact that the set where v vanishes has measure zero, which follows by taking f supported on that set. There is a duality between this case and the one treated in the previous section. For let q be the dual expoent p, $1/p + 1/q = 1$, and let

$u(x) = v^{-q/p}(x) = v^{1/(p-1)}(x)$ ($u(x) = 0$ if $v(x) = \infty$). Then u satisfies an inequality of the type (**), since

$$\int |f|^q u\, dx = \int |\hat{f} u^{1/q}|^q\, dx = \sup_{g \in \Delta} \int \hat{f} g\, dx$$

where $\Delta = \{g: \int |g|^p v\, dx \leq 1\}$. (To see this we have to use the property of the set of zeros of v.) We have, if $g \in \Delta$,

$$\int \hat{f} g\, dx = \int f \hat{g}\, dx \leq C \left(\int |f|^q\, dx \right)^{1/q}.$$

Conversely, if u satisfies an inequality of the type

$$\int |\hat{f}|^q u\, dx \leq \int |f|^q\, dx$$

for some q, $1 < q < \infty$, then u is locally integrable and the set where u is infinite has measure zero. If $1/p + 1/q = 1$, we define $v(x) = u^{-p/q}(x)$ if $u(x) \neq 0$ and $v(x) = \infty$ if $u(x) = 0$. A similar reasoning shows then that (***) holds for this v.

We thus obtain several properties, similar to those obtained in the previous section, whose proofs are obtained either by duality or by using a reasoning analogous to those of section II, so we will state them without proof.

PROPOSITION 1. *If (***) holds then v satisfies*

$$\int_I v(x)\, dx \geq C |I|^{p-1}$$

for any interval I. In particular for $p < 2$ only the trivial case $v \equiv \infty$ (in the almost everywhere sense) is admissible. For $p = 2$, this condition implies that $v(x) \geq C > 0$ almost everywhere, which is also a sufficient condition.

PROPOSITION 2. *Suppose $2 < p < \infty$ and v satisfies any of the following equivalent conditions*

(a) $\int_E v(x)\, dx \geq C |E|^{p-1}$ *for any measurable set E*

(b) $|\{x: v(x) \leq \delta\}| \leq C\delta^{1/p-2}$ *for any $\delta > 0$*

(c) $\int_{\{x: v(x) \geq \lambda^{p-2}\}} v^{2/(p-2)}(x)\, dx \leq C/\lambda$ *for any $\lambda > 0$*

then (***) holds.

PROPOSITION 3. *If (***) holds, then for any $r > 0$,*

$$\left(\sum_{k=-\infty}^{+\infty} \left(\int_{rk}^{r(k+1)} v^{-1/(p-1)}(x)\, dx \right)^{\alpha} \right)^{1/\alpha} \leq C r^{1/(p-1)}$$

where $\alpha = 2(p-1)/(p-2)$.

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