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SOME INEQUALITIES FOR PRODUCTS OF POWER SUMS

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We study the asymptotic behavior of the range of the ratio of products of power sums. For $x = (x_1,\dots,x_n)$, define $M_p = M_p(x) = \Sigma x_i^p$. As two representative and explicit results, we show that the maximum and minimum of the function M_1M_3/M_2^2 are $\pm 3\sqrt{3}/16\, n^{1/2} + 5/8 + \mathcal{O}(n^{-1/2})$ and that $n \geq M_1M_3/M_4 > -n/8$, where "1/8" is the best possible constant. We give readily computable, if less explicit, formulas of this kind for $M_{p_1}^{a_1} \cdots M_{p_r}^{a_r}/M_q^b$, $\Sigma a_i p_i = bq$. Applications to integral inequalities are discussed. Our results generalize the classical Hölder and Jensen inequalities. All proofs are elementary.

1. Introduction and background. In this paper I shall discuss some inequalities involving power sums which build upon, and generalize, the Hölder and Jensen inequalities. Since the proofs, although elementary, involve lengthy and cumbersome computation, I shall indicate the main results and spirit of the paper in this introduction.

For $x = (x_1,\dots,x_n) \in \mathbf{R}^n$ and $p > 0$ define

$$M_p(x) = \sum_{i=1}^{n} x_i^p; \tag{1.1}$$

we exclude the possibility that some x_i is negative in (1.1) when p is not integral and set $M_0(x) \equiv n$.

MAIN THEOREM (*see* (3.5) *and* (3.17)). *Suppose*

$$f(x) = M_{p_1}^{a_1}(x) \cdots M_{p_r}^{a_r}(x)/M_q^b(x),$$

where $\Sigma a_i p_i = bq$ *and all parameters are positive. Let* M *denote the maximum value of* f (*M depends on* n, *the number of variables*). *Then there exist readily computable constants* c_i *so that* $M = c_1 n^{c_2} + \mathcal{O}(n^{c_3})$. *The minimum,* $\overline{m}$, *defined when all parameters are integers, in many cases satisfies* $\overline{m} = c_4 n^{c_2} + o(n^{c_2})$, *where* c_4 *is not always readily computable.*

Hölder's inequality (1.2) and Jensen's inequality (1.3) — see [**3**], p. 28 — state that for all x with $x_i \geq 0$ ($x \geq 0$),

(1.2) $M_p^a(x)M_r^c(x) \geq M_q^b(x)$ if $ap + cr = bq$ and $a, b, c, p, q, r \geq 0$

$$M_p^{1/p}(x) \geq M_q^{1/q}(x) \text{ if } p > q > 0. \tag{1.3}$$

The inequality in (1.2) is strict unless x is some permutation of $(t,\ldots,t,0,\ldots,0)$; the inequality in (1.3) is strict unless x is some permutation of $(t,0,\ldots,0)$. In §2, after making some basic definitions and conventions, we combine (1.2) and (1.3) to prove that the maximum of M_p^a/M_q^b, $ap = bq$, is equal to $n^{\max(0,a(1-p/q))}$. The constant c_2 from the Main Theorem turns out to be the best exponent derivable from repeated application of this result.

The Main Theorem is proved in §3. Section 4 is devoted to some partial results when M_q^b is replaced by $M_{q_1}^{b_1}\cdots M_{q_s}^{b_s}$. In §5, we present a "notebook" of certain special cases in which a more detailed analysis is possible. To be specific, the maximum and minimum of M_1M_3/M_2^2 are $\pm 3\sqrt{3}/16\, n^{1/2} + 5/8 + \mathcal{O}(n^{-1/2})$, (indeed, they are given by a Laurent series in $n^{-1/2}$). For $M_1^3M_3/M_2^3$, the maximum and minimum are computed exactly and are $\pm(\sqrt{n-1} \pm 1)^4/8\sqrt{n-1}$. The maximum value for M_1M_3/M_4 is n by the Hölder and Jensen inequalities. We show that $M_1M_3/nM_4 > -1/8$, where $-1/8$ is best possible, first directly and then through an analysis of the classical moment problem. Finally we discuss the role of integral inequalities and compute the asymptotics for $M_1^rM_3^s/M_4^{(r+3s)/4}$.

The methods of proof are elementary and rest on these observations. If f has an extreme value at y then $\partial f/\partial x_i(y) = 0$ for $i = 1,\ldots,n$. When f is symmetric, this can drastically reduce the set of y on which f needs to be considered and provide an upper bound on the extreme value of f. A judicious choice of x's, on the other hand, can provide a lower bound on the extreme value of f. When we are lucky, the difference between these bounds is the error term. We can also use (1.2) and (1.3) to make *a priori* estimates which are often achieved.

This paper sits between two problems already analyzed in the literature. Ursell [**6**] has studied the mapping T: $x \to (M_{p_1}(x),\ldots,M_{p_r}(x))$ for $x \geq 0$ and determined those y for which $T(y)$ is on the boundary of the range of T. Also, if we restrict our attention to integer exponents, we can embed our situation into the classical moment problem.

The importance of [**6**] is immediately obvious and an appeal to it would save some space in the proof of the Main Theorem. Those omitted arguments would have to be repeated in detail in §5. In any case, the presentation of [**6**] is rather opaque and the major result is nowhere isolated as a theorem. I hope to discuss Ursell's work, without his restriction $x \geq 0$, in a future publication [**5**].

Let μ be a measure with n unit point masses at which g attains the values $x_1,\ldots,x_n$. Then $\sum x_i^p = \int g^p\, d\mu$. Thus, any inequality on the ratio of products of moments automatically applies to power sums. In fact, we show in §5 that

$$1 \geq \frac{\int g\,d\mu \int g^3\,d\mu}{\int d\mu \int g^4\,d\mu} \geq -\frac{1}{8}.$$

As $\int d\mu = n$, this implies the aforementioned inequalities for M_1M_3/M_4. However, the expression

$$\frac{\left(\int g\,d\mu\right)\left(\int g^3\,d\mu\right)}{\left(\int d\mu\right)^{1/2}\left(\int g^2\,d\mu\right)^2}$$

is unbounded over (g, μ) with non-negative μ, let alone bounded by $\pm 3\sqrt{3}/16$.

All the empirical evidence suggests that the extreme values of $M_{p_1}^{a_1} \cdots M_{p_r}^{a_r}/M_{q_1}^{b_1} \cdots M_{q_s}^{b_s}$ grow asymptotically like $c_1 n^{c_2}$ and I am willing to make this a conjecture.

2. Notations and preliminaries. The following definitions and restrictions apply for the next several sections and will be referred to collectively as "the usual conditions."

$$0 < p_1 < \cdots < p_r, \quad 0 < q_1 < \cdots < q_s, \quad p_i \neq q_j \tag{2.1}$$

$$a_i > 0, \quad 1 \leq i \leq r; \quad b_j > 0, \quad 1 \leq j \leq s \tag{2.2}$$

$$f = f(p, q; a, b)(x) = \prod_{i=1}^{r} M_{p_i}^{a_i}(x) \Big/ \prod_{j=1}^{s} M_{q_j}^{b_j}(x) \tag{2.3}$$

$$M = M(p, q; a, b) = \sup_{x_i \geq 0} f(p, q; a, b)(x) \tag{2.4}$$

$$m = m(p, q; a, b) = \inf_{x_i \geq 0} f(p, q; a, b)(x) \tag{2.5}$$

$$w = a \cdot p = \sum_{i=1}^{r} a_i p_i = \sum_{j=1}^{s} b_j q_j = b \cdot q. \tag{2.6}$$

From (2.6), $f(\lambda x) = f(x)$ for any $\lambda > 0$ so that, in (2.4) and (2.5), we may assume $\sum_{i=1}^{n} x_i^2 = c$. This restricts our attention to a compact set, so that M and m are realized as values of f. (Without (2.6), $f(\lambda x) = \lambda^t f(x)$ so that $M = \infty$ and $m = 0$.) Occasionally we are interested in allowing negative x_i. This entails some additional restrictions:

$$a_i, b_j, p_i \in \mathbf{Z}, \qquad q_j \in 2\mathbf{Z}. \tag{2.7}$$

(If q is odd then $M_q(1,-1,0,\ldots,0) = 0$ and this is bad for the denominator of f.) If the usual conditions and (2.7) hold, we make two more definitions:

$$\overline{M} = \overline{M}(p, q; a, b) = \sup_x f(p, q; a, b)(x) \tag{2.8}$$

$$\overline{m} = \overline{m}(p, q; a, b) = \inf_x f(p, q; a, b)(x). \tag{2.9}$$

As before, $\overline{M}$ and $\overline{m}$ are realized as the values of f.

The first lemma collects a number of fairly obvious, but useful, observations.

Lemma 2.10. *Suppose that the usual conditions hold, as do* (2.7), (2.8) *and* (2.9) *when appropriate. Then*

(i) $M \geq 1 \geq m > 0$

(ii) $m(p, q; a, b) = (M(q, p; b, a))^{-1}$

(iii) $\overline{M} = M$

(iv) $\overline{m} > -M$

(v) $M(\lambda p, \lambda q; a, b) = M(p, q; a, b)$ *for* $\lambda > 0$

(vi) $M(p, q; \lambda a, \lambda b) = (M(p, q; a, b))^{\lambda}$ *for* $\lambda > 0$.

(vii) *If* $n \geq 3$ *and* $a_i p_i$ *is odd for some* i *then* $m(p, q; a, b) < 0$.

(viii) *For fixed* a, p, b, q *and increasing* n, M *is non-decreasing in* n, *and* m *and* $\overline{m}$ *are non-increasing*.

Proof. (i) The first two inequalities follow from $f(1,\ldots,1) = 1$, the third from f realizing its infimum and $f(x) > 0$ for $x \geq 0$, $x \neq 0$.

(ii) Note that $f(p, q; a, b)(x) f(q, p; b, a)(x) = 1$. The relation need not hold for $\overline{M}$ and $\overline{m}$ as (2.7) might not be satisfied by both functions.

(iii) Let $|x| = (|x_1|,\ldots,|x_n|)$ and assume (2.7). Then $M_p(|x|) = M_p(x)$ if p is even and $M_p(|x|) \geq |M_p(x)|$ if p is odd with strict inequality iff x has components of opposite sign; thus $f(|x|) \geq |f(x)|$. Since $|x| \geq 0$, $M \geq \overline{M}$.

(iv) If $\overline{m} \geq 0$, this is immediate. If $0 > \overline{m}$, then $\overline{m} = f(x)$ for some x with components of opposite sign. By the last proof, $|\overline{m}| < f(|x|) \leq M$.

(v) For fixed $\lambda > 0$, if $y_i = x_i^{\lambda}$, $1 \leq i \leq n$ then $M_p(y) = M_{\lambda p}(x)$. Hence $f(\lambda p, \lambda q;\ a, b)(x) = f(p, q;\ a, b)(y)$. Since $x \mapsto y$ is one-to-one and invertible on the set $\{x \geq 0\}$, the suprema are identical.

(vi) Observe that $f(p, q; \lambda a, \lambda b) = (f(p, q; a, b))^{\lambda}$.

(vii) If $x_t = (1, 1, t, 0,\ldots,0)$ then $M_p(x_t) = (2 + t^p)$. Thus if $a_i p_i$ is odd then $M_{p_i}^{a_i}$ (and f) will change sign at $t = -2^{1/p_i}$. Since q_j is even by (2.7) this condition is necessary as well as sufficient for $\overline{m}$ to be negative.

(viii) In an abuse of notation, equate $M_p(x_1,\ldots,x_n)$ and $M_p(x_1,\ldots,x_n,0)$. As n increases, the suprema and infima of f are then taken over ever larger sets. □

One final notation is convenient. Suppose $x = (x_1,\ldots,x_n)$ has n components, n_1 of which are c_1, n_2 of which are c_2, etc., then $x = (c_1, c_2,\ldots; n_1, n_2,\ldots)$. Since all functions here are symmetric, the order of the components is immaterial. In this notation, (1.2) is sharp at $(t, 0; k, n-k)$ for $1 \le k \le n$ and (1.3) is sharp at $(t, 0; 1, n-1)$.

In the special case $r = s = 1$, $M = M(p, q;\ a, b)$ and $m = m(p, q;\ a, b)$ can be deduced from (1.2), (1.3) and (2.10)(iii), but it is more instructive to approach the problem directly first. Assume the usual conditions for $f = M_p^a/M_q^b$ — that is, $ap = bq$. If $f(y) = M$ then y is a local maximum for f and $(\partial f/\partial x_i)(y) = 0$ for $1 \le i \le n$. As $M \ge 1$ we can assume that $M_p(y) \ne 0$ and $M_q(y) \ne 0$. By taking the logarithmic derivative of f,

$$(2.11)\qquad \frac{ap}{M_p(y)} y_i^{p-1} - \frac{bq}{M_q(y)} y_i^{q-1} = 0 \quad \text{for } i = 1,\ldots,n \text{ and extremal } y.$$

(For $g = \prod g_i$, $g'/g = \sum g_i'/g_i$ and for $h = M_r^c$, $\partial h/\partial x_i = crM_r^{c-1}x_i^{r-1}$ so $h'/h = crx_i^{r-1}/M_r$.) From (2.11) we see that there can be at most one non-zero value attained by the y_i's; that is, $y = (r, 0; k, n-k)$ for some k and r. A direct computation shows that $f(y) = k^{a-b}$ (independent of r since $f(\lambda y) = f(y)$). As $1 \le k \le n$ and $b = ap/q$,

$$(2.12)\qquad M = n^{\max(0, a(1-p/q))}.$$

On the other hand, $f = (M_p^{1/p}/M_q^{1/q})^w$, so if $p > q$ then $f(x) \le 1$ by (1.3) and $M = 1$ by (2.10)(i) with equality at $(r, 0;\ 1, n-1)$. If $p < q$ then $M_0^{a-b}M_q^b \ge M_p^a$ by (1.2) so $n^{a-b} \ge f(x)$ with equality at (r, n). The method of (2.11) is generalized in the next section.

M. D. Choi, T. Y. Lam and the author **[2]** will study symmetric positive semi-definite quartic forms in n variables and have been interested in finding those (α, β) so that $\alpha \sum_{i=1}^n x_i \sum_{i=1}^n x_i^3 + \beta(\sum_{i=1}^n x_i^2)^2 \ge 0$. This is equivalent to finding M and $\overline{m}$ for the function $g = M_1M_3/M_2^2$. Suppose $M = g(y)$ and $\overline{m} = g(z)$. Then by methods outlined in §5,

$$(2.13)\qquad y = (1, u;\ n-1, 1), \qquad z = (1, v;\ n-1, 1)$$

$$(2.14)\qquad \begin{aligned} u &= 1 + 2n^{1/2}\cos(1/3 \arccos n^{-1/2}) \\ v &= 1 + 2n^{1/2}\cos(1/3 \arccos n^{-1/2} + 2\pi/3) \end{aligned}$$

$$(2.15)\qquad \begin{aligned} M &= 3\sqrt{3}/16\, n^{1/2} + 5/8 + \mathcal{O}(n^{-1/2}) \\ \overline{m} &= -3\sqrt{3}/16\, n^{1/2} + 5/8 + \mathcal{O}(n^{-1/2}). \end{aligned}$$

As (2.14) suggests, the trigonometric solution to the cubic equation is critical. What might one have expected? Application of (2.12) gives

$$(2.16)\qquad |g| = \left|\frac{M_1M_3}{M_2^2}\right| = \left|\frac{M_1}{M_2^{1/2}}\right|\cdot\left|\frac{M_3}{M_2^{3/2}}\right| \le n^{1/2}\cdot 1 = n^{1/2}$$

with no sharpness since the ingredient inequalities are sharp at different places. Comparison with (2.15) shows that this crude maximum is only off by a constant factor. Further, the growth of the leading term in $\overline{m}$ is equal and opposite to the growth of the leading term of M. This is counterintuitive: it is hard to find x with $M_1(x)$ and $M_3(x)$ of opposite sign (so that g is negative). As we shall see in the next several sections, each of the above remarks is valid more generally.

3. The main theorem. In this section we assume the usual conditions and $s = 1$, so that

$$(3.1)\qquad f(x) = M_{p_1}^{a_1}(x)\cdots M_{p_r}^{a_r}(x)/M_q^b(x).$$

As in (2.16) we have a crude estimate for M, combining (3.1) and (2.12):

$$(3.2)\qquad |f| = \left|\prod_{i=1}^{r} M_{p_i}^{a_i}M_q^{-a_ip_i/q}\right| \le \prod_{i=1}^{r} n^{\max(0,a_i(1-p_i/q))}.$$

If $q < p_1$ then each estimate in (3.2) is sharp at $x = (r,0;\ 1, n-1)$ and $M = 1$; if $p_r < q$ then each estimate is sharp at $(r;\ n)$ so that $M = n^{\Sigma a_i - b}$. Otherwise,

$$(3.3)\qquad p_1 < \cdots < p_j < q < p_{j+1} < \cdots < p_r,$$

and

$$(3.4)\qquad M \le n\sum_{i=1}^{j} a_i(1-p_i/q) = n^E.$$

THEOREM 3.5. *For f as in* (3.1) *with the usual conditions,* (3.3), (3.4), *and with rational a_i,*

$$(3.6)\qquad M = \alpha^b n^E + \mathcal{O}(n^{E-\delta})$$

where

$$(3.7)\qquad u = \left(\sum_{i=j+1}^{r} a_ip_i\right)/w,\quad \alpha(u) = u^u(1-u)^{1-u}\quad \textit{and}$$

$$\delta = \min_i |1 - p_i/q|\ .$$

Several disconnected remarks are appropriate. To prove (3.6) we must establish it as an upper bound and realize it as a value of f. As $\delta < 1$, it suffices to prove (3.6) with $(n-1)^E$, and this is what we do. The two cases discussed before (3.3) correspond to $u = 0$ or 1; the limiting value of α in these cases is 1, so with suitable redefinition they could be included in (3.5). As $1 \geq \alpha \geq .5$, the deviation of (3.6) from (3.4) is well-controlled. Finally, for $g = M_1 M_3/M_2^2$, the parameters are set as follows: $j = 1$, $w = 4$, $u = .25$, $\alpha = 3^{3/4}/4$ so that $\alpha^b = 3\sqrt{3}/16$, reconciling (2.15) with (3.6). The condition that a_i must be rational is regrettable and appears to be unavoidable for purely technical reasons. I am almost certain that the theorem is true without it. To prove Theorem 3.5, we need the following generalization of Descartes' rule of signs, which Ursell [**6**] attributes to Laguerre [**4**].

LEMMA 3.8. *Suppose $h(t) = c_1 t^{r_1} + \cdots + c_m t^{r_m}$ is a "generalized polynomial" with real exponents $r_1 < \cdots < r_m$ and $c_i \neq 0$. If $h(t) = 0$ has k distinct positive roots and the sequence $(c_1, \ldots, c_m)$ has l changes of sign $(c_i c_{i+1} < 0)$, then $l \geq k$.*

Proof (after [**6**]). If $l = 0$ then clearly $h(t) > 0$ or $h(t) < 0$ for all $t > 0$ so $k = 0$. Assume the result for $l - 1$ changes of sign and suppose $(c_1, \ldots, c_m)$ has l changes of sign with one occurring between c_j and c_{j+1}. Choose β so that $r_j < \beta < r_{j+1}$. If $h(t) = 0$ has k positive roots, then so does $g(t) = t^{-\beta} h(t)$. By Rolle's Theorem, $g'(t)$ has $k' \geq k - 1$ roots, as does $t^{\beta+1} g'(t)$. But

$$t^{\beta+1} g'(t) = c_1 (r_1 - \beta) t^{r_1} + \cdots + c_m (r_m - \beta) t^{r_m},$$

and the sequence $(c_1(r_1 - \beta), \ldots, c_m(r_m - \beta))$ has $l - 1$ changes of sign $(c_j(r_j - \beta) c_{j+1}(r_{j+1} - \beta) > 0)$. By the induction hypothesis, $l - 1 \geq k' \geq k - 1$ so $l \geq k$. □

Proof of Theorem 3.5. We first make a technical remark. Suppose Theorem 3.5 is established for integral a_i. Then by (2.10)(vi) and the shape of (3.6), the theorem will hold for rational a_i. Assume now that $a_i \in \mathbf{Z}$ and suppose $f(y) = M$, write $M_p(y) = M_p$ for short. Then $(\partial f/\partial x_i)(y) = 0$ for $1 \leq i \leq n$ and by logarithmic differentiation (cf. (2.11)),

$$(3.9) \quad 0 = \frac{1}{f} \frac{\partial f}{\partial x_i}(y) = \frac{a_1 p_1}{M_{p_1}} y_i^{p_1 - 1} + \cdots + \frac{a_r p_r}{M_{p_r}} y_i^{p_r - 1} - \frac{bq}{M_q} y_i^{q-1}.$$

This suggests a generalized polynomial $h(t)$:

$$h(t) = \frac{a_1 p_1}{M_{p_1}} t^{p_1 - 1} + \cdots + \frac{a_j p_j}{M_{p_j}} t^{p_j - 1} - \frac{bq}{M_q} t^{q-1} + \cdots + \frac{a_r p_r}{M_{p_r}} t^{p_r - 1}. \tag{3.10}$$

By (3.3) $h(t)$ has two changes of sign in its coefficients ($M_p > 0$ since $y \geq 0$) and by Lemma 3.8, $h(t) = 0$ has at most two positive roots. Since $f(y) = M$ implies $h(y_i) = 0$ for $1 \leq i \leq n$ (compare (3.9) and (3.10)), $y = (a, c, 0;\, l, k, n - (k + l))$, for some positive a, c and integers k and l with $k + l \leq n$. Without loss of generality, suppose $l \geq k$ and, as $f(\lambda y) = f(y)$, set $a = 1$. Under the peculiar parametrization $l = ks$ ($s \geq 1$) and $c = s^{1/q}t$, we can now say that f achieves its maximum at a point of shape (3.11):

$$y = \left(1, s^{1/q}t, 0;\, ks, k, n - k(s + 1)\right). \tag{3.11}$$

In (3.11), t ranges over the nonnegative reals, $1 \leq k \leq n/2$ and s is rational with a finite range. For any p,

$$M_p = ks + ks^{p/q}t,$$

so we may write the factors of f in increasing powers of s

$$\begin{cases} M_{p_i}^{a_i} = k^{a_i}(s + t^{p_i}s^{p_i/q})^{a_i} & i \leq j \\ M_{p_i}^{a_i} = k^{a_i}(t^{p_i}s^{p_i/q} + s)^{a_i} & i \geq j + 1 \\ M_q^b = k^b(1 + t^q)^b s^b. \end{cases} \tag{3.12}$$

Accordingly,

$$f(y) = k^{\Sigma_1^r a_i - b} \frac{\prod_1^j (s + t^{p_i}s^{p_i/q})^{a_i} \prod_{j+1}^r (t^{p_i}s^{p_i q} + s)^{a_i}}{s^b(1 + t^q)^b}. \tag{3.13}$$

By hypothesis, all a_i's are integral, so the numerator in (3.13) is a generalized polynomial in s whose coefficients are polynomials in t with degree at most $\Sigma_{i=1}^r a_i p_i = bq = w$. Thus $f(y)$ can be written as a generalized polynomial in s whose coefficients are rational functions in t which are uniformly bounded for real t. (This argument uses the integrality of a_i in an essential way.) The highest order term in (3.13) is

$$k^{\Sigma_1^r a_i - b} \frac{t^{\Sigma_{j+1}^r a_i p_i}}{(1 + t^q)^b} s^{(\Sigma_1^j a_i + \Sigma_{j+1}^r a_i p_i / q - b)}. \tag{3.14}$$

As $b = (\Sigma_1^r a_i p_i)/q$, the exponent of s in (3.14) is $\Sigma_1^j a_i(1 - p_i/q) = E$ (from (3.4)). An easy calculus exercise shows that for $c < bq$, $\phi(t) = t^c(1 + t^q)^{-b}$ achieves its maximum at $t_0 = (c/(bq - c))^{1/q}$; for $c = \Sigma_{j+1}^r a_i p_i$, $\phi(t_0) = \alpha^b$ in the notation of (3.7). Now replace the rational functions of t in the lower powers of s in (3.13) by their uniform bounds:

$$f(y) \leq k^{\Sigma_1^r a_i - b}\left(\alpha^b s^E + \sum_l d_l s^{w_l}\right). \tag{3.15}$$

The summation in (3.15) is over at most $\Pi_1^r(a_i + 1) - 1$ terms, s has a bounded range and the largest exponent w_l is $E - \delta$. Further,

$$\begin{aligned}\sum_1^r a_i - b &= \sum_1^r a_i(1 - p_i/q) \\ &\leq \sum_1^j a_i(1 - p_i/q) - a_{j+1} \,|\, 1 - p_{j+1}/q \,|\leq E - \delta\end{aligned}$$

so that (3.15) can be further simplfied to

$$f(y) \leq \alpha^b(ks)^E + d(ks)^{E-\delta}, \tag{3.16}$$

for some d. Since $ks + k \leq n$, $ks \leq n - 1$; thus $M \leq \alpha^b(n - 1)^E + d(n - 1)^{E-\delta}$ and one direction of (3.6) is established.

To get the reverse inequality, put $k = 1$, $s = n - 1$ and $t = t_0$ into (3.13); that is, evaluate f at the point $y = (1, t_0(n - 1)^{1/q}; n - 1, 1)$. The foregoing analysis, applied to (3.13) as an *exact* formula, shows that $f(y) \geq \alpha^b(n - 1)^E + d'(n - 1)^{E-\delta}$, and this completes the proof. □

For $g = M_1 M_3/M_2^2$, this suggested maximum occurs at $(1, \sqrt{3(n - 1)}; n - 1, 1)$, which is close to (2.13) and (2.14). Now an appeal to [U] would have allowed us to say that the maximum of $M_{p_1}^{a_1} M_{p_2}^{a_2}/M_q^b$, $a_1 p_1 + a_2 p_2 = bq$, is achieved at $y = (1, r; n - 1, 1)$ for some r, but we would still need the parametrization of this proof in order to determine M. In any event, (3.9) is used in §5.

We now look at $\overline{m}$ in some cases.

THEOREM 3.17. *If f satisfies the hypotheses of Theorem* 3.5 *and, in addition, $\Sigma_{i=1}^j a_i p_i$ is odd, then*

$$\overline{m} = -\alpha^b n^E + \mathcal{O}(n^{E-\delta}). \tag{3.18}$$

Proof. Since $\overline{m} \geq -M$, $\overline{m} \geq -\alpha^b n^E + \mathcal{O}(n^{E-\delta})$. On the other hand, evaluate f at $y = (1, -t_0(n - 1)^{1/q}; n - 1, 1)$. As in the proof of the last theorem, because $\Sigma_{i=1}^j a_i p_i$ is odd, $f(y) \geq -\alpha^b(n - 1)^E - d'(n - 1)^{E-\delta}$. □

Note here the connection with (2.15). If $\Sigma_{i=1}^{j} a_i p_i$ is even, I can find no non-obvious bounds on $\overline{m}$. As one simple case, let $g = M_1 M_3 M_8 / M_4^3$. A direct application of Theorem 3.5 shows that $M = (4/27)n + \mathcal{O}(n^{3/4})$, (for $E = 1 \cdot (1 - 1/4) + 1 \cdot (1 - 3/4) = 1$, $j = 2$, $u = 8/12$, $b = 3$, so $\alpha^b = (2/3)^2(1/3) = 4/27$ and $\delta = \min \mid 1 - p_i/4 \mid = 1/4$). Thus, $\overline{m} \geq -(4/27)n + \mathcal{O}(n^{3/4})$. When you compute $f(y)$ for $y = (1, A(n-1)^{1/4}; n-1, 1)$, it is asymptotically positive, since $\Sigma_{i=1}^{j} a_i p_i$ is *even* in this case. In fact, the best attainable value from a point with shape $(1, t(n-1)^{1/q}; n-1, 1)$ comes from setting $q = 3$ and $t = -2^{2/3}$. From this, we obtain $\overline{m} \leq -3 \cdot 2^{8/3} n^{2/3} + \mathcal{O}(n^{1/3})$. There is, however, no proof that g attains its extreme values at points of this shape, because Lemma 3.8 only applies to positive roots.

4. More general upper bounds. Theorem 3.5 generalizes somewhat, but at a loss in precision. Given f as in (2.3) we can always factor it into "increasing" weight-zero pieces. To be precise

$$(4.1) \qquad \begin{cases} f = \prod_{i=1}^{t} \left(M_{r_i}^{\alpha_i} / M_{s_i}^{\beta_i} \right), \ \alpha_i r_i = \beta_i s_i = w_i, \ r_i \leq r_{i+1}, \ s_i \leq s_{i+1} \\ \prod M_{r_i}^{\alpha_i} = \prod M_{p_i}^{a_i}, \ \prod M_{s_i}^{\beta_i} = \prod M_{q_i}^{b_i}. \end{cases}$$

For example,

$$M_2 M_8 M_{14} / M_4 M_{10}^2 = \left(M_2 / M_4^{.5} \right) \left(M_8^{.25} / M_4^{.5} \right) \left(M_8^{.75} / M_{10}^{.6} \right) \left(M_{14} / M_{10}^{1.4} \right).$$

For f as in (4.1) let h be the number of changes of sign in the sequence $(r_1 - s_1, \ldots, r_t - s_t)$. We shall find asymptotic estimates for M if h is 0 or 1. The hypothesis (3.3) and $s = 1$ insure that $h = 1$ for those f covered by Theorem 3.5.

THEOREM 4.2. *If $h = 0$ then $M = 1$ or $M = n^{\Sigma a_i - \Sigma b_i}$ depending on whether $r_i > s_i$ or $r_i < s_i$ for $i = 1, \ldots, t$.*

Proof. Application of (2.12) to each factor of (4.1) provides the given values as upper bounds for M; evaluation at $(1, 0; 1, n-1)$ or $(1; n)$ shows that they are sharp. □

Theorem 4.2 subsumes the remarks made before Theorem 3.5. If $h = 1$ there are two fundamentally different cases, depending on whether $r_i - s_i$ goes from negative to positive (of which (3.3) is a special case) or from positive to negative. There will be a distinction in the first case depending on whether $r_i - s_i$ "pivots" on one particular value of s_i or not. First we dispose of the second case, which has an unsurprising answer, but requires a lemma on a fundamental special case.

LEMMA 4.3. *Suppose that* $f = M_{p_1}^{a_1} M_{p_2}^{a_2} / (M_{q_1}^{b_1} M_{q_2}^{b_2})$ *with the usual restrictions and* $q_1 < p_1 < p_2 < q_2$. *Then* $M = n^{\max(0, a_1 + a_2 - (b_1 + b_2))}$.

Proof. Upon evaluating f at $(1; n)$, $m \geq n^{a_1 + a_2 - (b_1 + b_2)}$ and $M \geq 1$ in any case. To obtain the reverse inequality, we apply (1.2) twice. Indeed, $M_{p_1}^{q_2 - q_1} \leq M_{q_1}^{q_2 - p_1} M_{q_2}^{p_1 - q_1}$ and $M_{p_2}^{q_2 - q_1} \leq M_{q_1}^{q_2 - p_2} M_{q_2}^{p_2 - q_1}$. Upon combining with the definition of f (and recalling that $a_1 p_1 + a_2 p_2 = b_1 q_1 + b_2 q_2$), this becomes

$$f^{q_2 - q_1} \leq M_{q_1}^{q_2(a_1 + a_2 - (b_1 + b_2))} M_{q_2}^{-q_1(a_1 + a_2 - (b_1 + b_2))}. \tag{4.4}$$

We can apply (2.12) to the right hand side of (4.4) to find $f^{q_2 - q_1} \leq n^{\max(0, (q_2 - q_1)(a_1 + a_2 - b_1 - b_2))}$, completing the proof. □

THEOREM 4.5. *Suppose* f *satisfies* (4.1) *with* $h = 1$ *and* $r_i - s_i$ *goes from positive to negative. Then* $M = n^{\max(0, \Sigma a_i - \Sigma b_j)}$.

Proof. The basic idea is to decompose f into a product of factors to which the lemma can be applied. Suppose $1 \leq i \leq j$ and $j + 1 \leq k \leq t$, $r_i > s_i$ and $r_k < s_k$. Let $g_i = M_{r_i}^{\alpha_i} M_{s_i}^{-\beta_i}$, $h_k = M_{r_k}^{\alpha_k} M_{s_k}^{-\beta_k}$, $v_i = \alpha_i - \beta_i$ and $z_k = \alpha_k - \beta_k$. Then $\alpha_i r_i = \beta_i s_i = w_i$ and $r_i > s_i$ implies $v_i < 0$ and similarly $z_k > 0$. Finally, let $\gamma_i = v_i / \Sigma v_i$ and $\delta_k = z_k / \Sigma z_k$ then $0 < \gamma_i, \delta_k$ and $\Sigma_i^j \gamma_i = \Sigma_{j+1}^t \delta_k = 1$. Thus, in view of (4.1),

$$f = \prod_{i=1}^{j} \prod_{k=j+1}^{t} g_i^{\delta_k} h_k^{\gamma_i}. \tag{4.6}$$

On the other hand, $g_i^{\delta_k} h_k^{\gamma_i} = M_{r_i}^{\alpha_i \delta_k} M_{r_k}^{\alpha_k \gamma_i} / (M_{s_i}^{\beta_i \delta_k} M_{s_k}^{\beta_k \gamma_i})$, $s_i < r_i < r_k < s_k$ (by the order of i and k), and this factor has weight 0, so we can apply Lemma 4.3:

$$| g_i^{\delta_k} h_k^{\gamma_i} | \leq n^{\max(0, \delta_k(\alpha_i - \beta_i) + \gamma_i(\alpha_k - \beta_k))}. \tag{4.7}$$

The exponent in (4.7) is

$$w_i \cdot z_k / \sum z_k + z_k \cdot w_i / \sum w_i = w_i z_k \left(1 / \sum z_k + 1 / \sum w_i \right)$$

and so has uniform sign as i and k traverse their ranges ($w_i z_k < 0$). Since $\Sigma_i \Sigma_k w_i z_k = (\Sigma w_i)(\Sigma z_k)$, the exponents in (4.7) can be combined by adding in (4.6) to make

$$| f | \leq n^{\max(0, \Sigma w_i + \Sigma z_k)}. \tag{4.8}$$

But $\Sigma w_i + \Sigma z_k = \Sigma \alpha_i - \Sigma \beta_i$, so $M \leq n^{\max(0, \Sigma a_i - \Sigma b_j)}$. As in the lemma, this bound is achieved for $x = (1, 0; 1, n - 1)$ or $(1; n)$. □

The remaining case occurs when $r_i - s_i < 0$ for $1 \le i \le j$ and $r_i - s_i > 0$ for $j + 1 \le k \le t$. This, in turn, splits into two cases: $r_i - s_i$ *pivots* if $s_j = s_{j+1}$, otherwise, it *jumps*.

THEOREM 4.9. *In the remaining case, if $r_i - s_i$ jumps,*

$$M = n^{\Sigma_1^j(\alpha_i - \beta_i)} + \mathcal{O}\left(n^{\Sigma_1^j(\alpha_i - \beta_i) - \delta}\right), \qquad \delta = \frac{s_{j+1} - s_j}{s_{j+1} + s_j}. \tag{4.10}$$

Proof. Since $M_{r_i}^{\alpha_i} M_{s_i}^{-\beta_i} \le n^{\max(0, \alpha_i - \beta_i)}$, repeated application of (4.1) gives $M \le n^{\Sigma_1^j(\alpha_i - \beta_i)}$; this establishes one direction of (4.10). Taking a cue from Theorem 3.5, we will find y so that $f(y) \ge (n-1)^{\Sigma_1^j(\alpha_i - \beta_i)} - d(n-1)^{\Sigma_1^j(\alpha_i - \beta_i) - \delta}$. Since $\delta < 1$, we can replace $n - 1$ by n in the asymptotics.

Let $s = (s_j + s_{j+1})/2$ then $r_i < s_i < s < s_k < r_k$ for $i \le j < k$. Further, let $y = (1, (n-1)^{1/s}; n - 1, 1)$, then $M_p(y) = (n-1) + (n-1)^{p/s}$. For i, we have

$$\begin{aligned}\left(M_{r_i}^{\alpha_i} M_{s_i}^{-\beta_i}\right)(y) &= \left((n-1) + (n-1)^{r_i/s}\right)^{\alpha_i} \\ &\quad \times \left((n-1) + (n-1)^{s_i/s}\right)^{-\beta_i} \\ &= (n-1)^{\alpha_i - \beta_i} + \mathcal{O}\left((n-1)^{\alpha_i - \beta_i - \delta}\right),\end{aligned} \tag{4.11}$$

since the true power of the error term is $\alpha_i - \beta_i - (1 - s_i/s) < \alpha_i - \beta_i - \delta$. Similarly,

$$\begin{aligned}\left(M_{r_k}^{\alpha_k} M_{s_k}^{-\beta_k}\right)(y) &= \left((n-1)^{r_k/s} + (n-1)\right)^{\alpha_k} \\ &\quad \times \left((n-1)^{s_k/s} + (n-1)\right)^{-\beta_k} \\ &= 1 + \mathcal{O}(n-1)^{-\delta},\end{aligned} \tag{4.12}$$

since the true power of the error term is $1 - r_k/s$ and $\delta \le |1 - r_k/s|$. These estimates are now combined into (4.10) and the other direction of this inequality is established. □

Note that the careful analysis of Theorem 3.5 in establishing the upper bound is unnecessary here because of the *a priori* (2.12) estimates. As an illustration, for $f = (M_1^2 M_8)/(M_2 M_4^2) = (M_1^2/M_2) \cdot (M_8/M_4^2)$, $n \ge M \ge n - \mathcal{O}(n^{2/3})$. The final case, where $r_i - s_i$ pivots, includes Theorem 3.5 — without the condition of rational a_i — but with weaker conclusions. The trouble seems to be that Lemma 3.8 is not very helpful and the equivalent of (3.13) cannot be reduced to a generalized polynomial because of its denominator.

THEOREM 4.13. *If $r_i - s_i$ pivots and $s_j = s_{j+1} = s$, then*

$$n^{\Sigma_1^t(\alpha_i-\beta_i)} \geq M \geq \alpha^d n^{\Sigma_1^t(\alpha_i-\beta_i)} + \mathcal{O}\left(n^{\Sigma_1^t(\alpha_i-\beta_i)-\delta}\right), \tag{4.14}$$

where $d = \Sigma_{s_i=s}\beta_i$, $u = (\Sigma'_{s_i=s} w_i)/(\Sigma_{s_i=s} w_i)$, Σ' being the summation over $i \geq j+1$, $\alpha = u^u(1-u)^{1-u}$ and $\delta = \min_{s_i=s}(|1 - r_i/s|, |1 - s_i/s|)$.

Proof. The upper bound in (4.14) is found, as in the last theorem, by repeated application of (2.12). To get the lower bound, we use the natural substitution $y = (1, t(n-1)^{1/s};\ n-1, 1)$, so $M_p(y) = (n-1) + t^p(n-1)^{p/s}$ (t will be chosen later and fixed now). Asymptotically, there are four cases of $(M_{r_i}^{\alpha_i} M_{s_i}^{-\beta_i})(y)$ depending on whether $i \leq j, j+1 \leq i$ and whether $s_i = s$ or $s_i \neq s$. We omit the intermediate arguments, which should be familiar by now, so that

$$f(y) = \left(t^c/(1+t^s)^d\right)(n-1)^{\Sigma_1^t(\alpha_i-\beta_i)} + \text{lower terms in } (n-1), \tag{4.15}$$

where $c = \Sigma\alpha_i r_i = \Sigma w_i$, the summation over $s_i = s$; $i \geq j+1$ and $d = \Sigma_{s_i=s}\beta_i$. As in the proof of (3.5), the maximum value of $t^c(1+t^s)^{-d}$ can be computed, and for *this* value of t, we may replace the "lower terms in $(n-1)$" by the $\mathcal{O}$-term in (4.14). □

It seems likely that the lower bound in (4.14) is sharp, but I can't prove it.

Just like Theorem 3.5, Theorems 4.5, 4.9 and 4.13 can be generalized with results contingent on a certain sum being odd, but we omit the details. Theorem 4.2, however, does generalize fully with a weaker (and non-effective) constant.

THEOREM 4.16. *If f satisfies the hypotheses of Theorem 4.2 and $a_i p_i$ is odd for some i, then there exists c so that*

$$\overline{m} = -cn^{\max(0,\Sigma a_i - \Sigma b_j)} + o\left(n^{\max(0,\Sigma a_i - \Sigma b_j)}\right). \tag{4.17}$$

Proof. If $\Sigma a_i - \Sigma b_j \leq 0$ then $M = 1$ by Theorem 4.2, so $\overline{m} \geq -1$. As n increases, $\overline{m} = \overline{m}(n)$ is non-increasing ((2.10)(viii)) and bounded below and so must approach a limit, establishing (4.16) in this case. Otherwise, let $w = \Sigma a_i - \Sigma b_j > 0$, let $g(n) = -\overline{m}(n)$ and $h(n) = n^{-w}g(n)$. In this notation, (4.16) is equivalent to: $\lim h(n) = c$. Since $g(n+i) \geq g(n)$ for integral i and n, $h(n+i) \geq (n/(n+i))^w h(n)$, and from Theorem 4.2, $h(n) \leq 1$. Any $x = (x_1, \ldots, x_n)$ can be "stuttered" k times into $x^k = (x_1, \ldots, x_n;\ k, \ldots, k)$; $M_p(x^k) = kM_p(x)$ so that $f(x^k) = k^w f(x)$. Choose

x_n so that $f(x_n) = -g(n)$. Then $f(x_n^k) = -k^w g(n) \geq -g(kn)$; that is, $h(kn) \geq h(n)$. Let $\beta = \overline{\lim}\, h(n)$ and $\gamma = \underline{\lim}\, h(n)$. Pick n so that $h(n) \geq \beta - \varepsilon$, any m can be written as $kn + i$ with $0 \leq i < n$. Combining the above,

$$(4.18)\quad h(m) = h(kn+i) \geq \left(\frac{kn}{kn+i}\right)^w h(kn) \geq \left(\frac{k}{k+1}\right)^w h(n).$$

Upon taking $\underline{\lim}$ of both sides of (4.18), $\gamma \geq \beta - \varepsilon$ so $\gamma = \beta$ and $\lim h(n)$ exists. □

5. Illustrations and integral inequalities. We start this section with the study of M and $\overline{m}$ in three simple situations in which rather more explicit information is possible: M_1M_3/M_2^2, $M_1^3M_3/M_2^3$ and M_1M_3/M_4. These will serve, I hope, to illuminate the theorems of the last several sections.

First, let $g = M_1M_3/M_2^2$ and suppose $g(y) = M$ or $\overline{m}$, then, as before, $(\partial g/\partial x_i)(y) = 0$ for $i = 1,\ldots,n$. As in (3.9),

$$(5.1)\qquad \frac{1}{M_1} - \frac{4}{M_2}y_i + \frac{3}{M_3}y_i^2 = 0 \quad \text{for } i = 1,\ldots,n.$$

If y has only one distinct component then $y = (r, n)$, $g(y) = 1$. As $0 > \overline{m}$ and $g(1,2,0,0,\ldots) = 1.08$, this case can be ignored. Otherwise $y = (r, s;\ k, l)$ and r and s are both roots of the quadratic in (5.1)[1]. This leads to two equations:

$$(5.2)\qquad r + s = \frac{4(kr^3 + ls^3)}{3(kr^2 + ls^2)}, \qquad rs = \frac{kr^3 + ls^3}{3(kr + ls)}.$$

Both equations in (5.2) lead to the same cubic in r and s: $kr^3 - 3kr^2s - 3lrs^2 + ls^3 = 0$. Scale so that $l \geq k$ and $s = 1$; this cubic becomes

$$(5.3)\qquad r^3 - 3r^2 - 3rw + w = 0, \qquad w = l/k \geq 1.$$

Equation (5.3) is readily solved by the trigonometric method:

$$(5.4)\qquad r = 1 + 2(w+1)^{1/2}\cos\theta, \qquad \cos 3\theta = (w+1)^{-1/2}.$$

For $y = (r, 1;\ k, wk)$, $g(y) = (r+w)(r^3+w)/(r^2+w)^2$ and one can substitute (5.4) into this to determine the dependence on w (keeping in mind that r is triple-valued). It is easier computationally to view w as a function of r (remembering that w has a finite range and this, in turn, gives r a finite range). Indeed, $r = 1/3$ is never a root of (5.3) and,

[1] It should be remembered that the previous analysis was purely *asymptotic* and we cannot assume *a priori* that $k = 1$ or $l = 1$, etc.

otherwise, $w = (r^3 - 3r^2)/(3r - 1)$. After some reduction, we find that

$$g(y) = \frac{3(r+1)^2}{16r} = \frac{3}{16}\left(r + \frac{1}{r}\right) + \frac{3}{8}. \tag{5.5}$$

Thus the extreme values of g, M and $\overline{m}$, are achieved when $r + 1/r$ is maximized and minimized in the finite range of r.

Elementary curve-sketching techniques applied to (5.3) show that one value of r is less than -1, another is between 0 and $1/3$ and the third is greater than 3. Further, on any branch of $r = r(w)$, $dr/dw = (dw/dr)^{-1}$ and $dw/dr = 6r(r-1)^2/(3r-1)^2 \geq 0$ so r increases for increasing w. Thus $\overline{m}$ is achieved by the minimum of $r + 1/r$ with $0 > r > -1$; this is the largest such r, which comes from the largest w, $n - 1$. Similarly M is achieved by the maximum of $r + 1/r$—if $0 < r < 1/3$, this is the smallest r and if $r > 3$, this is the largest. In the former case $w = 1$ and (5.3) becomes $(r-1)(r^2 - 4r + 1) = 0$, so $r = 2 - \sqrt{3}$. In the latter case, again $w = n - 1$ and the largest r from (5.4) with $w = n - 1$ is larger than that from (5.4) with $w = 1$, namely, $2 + \sqrt{3} = (2 - \sqrt{3})^{-1}$. Thus M is achieved when $w = n - 1$ in (5.4), defining $(r, s;\ k, l)$ as in (2.13) and (2.14). The asymptotics in (2.15) are most easily found by using Taylor series and (5.5). As (2.15) suggests, M and $\overline{m}$ can be written as series in $n^{1/2}$ whose coefficients agree on the full powers of n and are opposite on the half-powers.

For the second example, we change g somewhat into $h = M_1^3 M_3/M_2^3$. As before, if $h(y)$ is extreme then $(\partial h/\partial x_i)(y) = 0$ so

$$\frac{3}{M_1} - \frac{6}{M_2} y_i + \frac{3}{M_3} y_i^2 = 0 \quad \text{for } i = 1, \ldots, n. \tag{5.6}$$

If $y = (r;\ n)$ then $h(y) = n$. Otherwise, from (5.6), we have $y = (r, s;\ k, n-k)$, $r \neq s$, $k \leq n - k$, and, as in (5.2),

$$r + s = 2\frac{kr^3 + (n-k)s^3}{kr^2 + (n-k)s^2}, \qquad rs = \frac{kr^3 + (n-k)s^3}{kr + (n-k)s}. \tag{5.7}$$

The alterations in coefficients from (5.2) to (5.7) are crucial, for now the derived cubic is degenerate:

$$\begin{aligned} kr^3 - kr^2 s - (n-k)rs^2 + (n-k)s^3 \\ = (r-s)\left(kr^2 - (n-k)s^2\right) = 0. \end{aligned} \tag{5.8}$$

The case $r = s$ was discussed above; accordingly scale $r = (n-k)^{1/2}$, $s = \pm k^{1/2}$. A slight computation shows that

$$h(y) = \pm\frac{\left(k^{1/2} \pm (n-k)^{1/2}\right)}{8k^{1/2}(n-k)^{1/2}}. \tag{5.9}$$

Another slight computation shows that the numerator in (5.9) is maximized, and the denominator minimized, by choosing $k = 1$. This leads to the *exact* formulas

$$M = \frac{\left(\sqrt{n-1}+1\right)^4}{8\sqrt{n-1}}, \qquad \overline{m} = -\frac{\left(\sqrt{n-1}-1\right)^4}{8\sqrt{n-1}}. \tag{5.10}$$

A small check is needed to show that the given value for M is greater than n; it is for $n \geq 2$, (let $u = \sqrt{n-1}$, $n = u^2+1$). The same degeneracy as (5.8) occurs for functions $M_{2m-a}^{m(2m+a)} M_{2m+a}^{m(2m-a)} / M_{2m}^{(2m-a)(2m+a)}$; the amusing details are left to the reader.

The final example, $f = M_1M_3/M_4$, involves greater difficulties. The maximum M equals n as a straightforward application of Theorem 4.2. (Indeed, $nM_4 - M_1M_3 = \Sigma(x_i - x_j)^2(x_i^2 + x_ix_j + x_j^2) \geq 0$.) We concentrate on the minimum, $\overline{m}$. By Theorem 4.16, $\lim \overline{m}/n$ exists, and we shall show that it is $-1/8$. In fact, we shall show that $\overline{m} > -n/8$, without equality, because $7 + 4\sqrt{3}$ is irrational!

Suppose $f(y) = \overline{m}$, then $(\partial f/\partial x_i)(y) = 0$ so that

$$\frac{1}{M_1} + \frac{3}{M_3}y_i^2 - \frac{4}{M_4}y_i^3 = 0, \qquad i = 1,\ldots,n. \tag{5.11}$$

The cubic in (5.11) might have three real roots and probably some contradiction can be wrought from the assumption $y = (r, s, t;\ k, l, m)$, $r < s < t$ and

$$r+s+t = \frac{3(kr^4+ls^4+mt^4)}{4(kr^3+ls^3+mt^3)}, \qquad rs+rt+st = 0, \tag{5.12}$$

$$rst = \frac{kr^4+ls^4+mt^4}{4(kr+ls+mt)}.$$

Rather, I shall take the coward's way out and appeal to a forthcoming theorem [**2**]: if $p(x)$ is a positive semi-definite (psd) symmetric quartic form, not a quadratic in M_1^2 and M_2, and $p(y) = 0$ then y has at most two distinct components. This theorem is applicable to $p(x) = \overline{m}M_4 + M_1M_3$ which is psd and for which $p(y) = 0$ if $f(y) = \overline{m}$. With this in mind, suppose $y = (1, s;\ wk, k)$, $w \geq 1$. Then $M_p(y) = k(w + s^p)$ as before and $n = k(w+1)$, so

$$\frac{1}{n}f(y) = \frac{(s+w)(s^3+w)}{(1+w)(s^4+w)} = F(s, w). \tag{5.13}$$

In the notation of (5.13), $\overline{m} = n \cdot \inf F(s, w)$, where s is real and w has the usual finite range ($w = (n-k)/k$, $k \leq n/2$). We see immediately that

$F(s,w) > 0$ unless $-w \le s \le -w^{1/3}$ thus the infimum is actually a minimum and achieved at some (s,w) for which $(\partial F/\partial s)(s,w) = 0$. After some work, we compute that this derivative vanishes if $s = 1$ ($F(s,w) = n = M$) or

$$w = \frac{s^5 + s^4 + 4s^3}{4s^2 + s + 1} = \phi(s). \tag{5.14}$$

As with $M_1 M_3 / M_2^2$, it will be more profitable to view w as a function of s, rather than s as a function of w. Indeed, $\lim \phi(s) = \mp\infty$ as $s \to \pm\infty$ so the range of ϕ is $\mathbf{R}$ and

$$\phi'(s) = -12s^2(s^2+1)(s^2+s+1)/\left(4s^2+s+1\right)^2 \le 0$$

so that ϕ is one-to-one and $H = \phi^{-1}$ is well defined. Thus, assuming (5.14), and substituting into (5.13),

$$F(s,\phi(s)) = F(H(w),w) = \frac{(s+w)(s^3+w)}{(1+w)(s^4+w)} \tag{5.15}$$

$$= \frac{3}{4}\,\frac{s(1+s)^2}{1+2s+6s^2+2s^3+s^4}.$$

Finally, let $u = s + 1/s$ then (5.15) takes the form

$$F(H(w),w) = \frac{3}{4}\,\frac{u+2}{u^2+2u+4} = k(u). \tag{5.16}$$

To recapitulate, $\overline{m} = n \cdot \min k(u)$, where $u = s + 1/s$ and $\phi(s)$ has the form $(n-k)/k$, $k \le n/2$. A quick analysis shows that $k(u) \ge -1/8$ with equality only when $u = s + 1/s = -4$. But $s^2 + 4s + 1 = 0$ implies $s = -2 \pm \sqrt{3}$ and, from (5.14), $w = 7 \mp 4\sqrt{3}$. *As w is rational, this never occurs* so $k(u) > -1/8$ and $\overline{m} > -n/8$ for all n. On the other hand, as $n \to \infty$ one can easily find acceptable $w_n \to 7 + 4\sqrt{3}$ so that $s_n \to -4$ and $k(u_n) \to -1/8$; that is, $\overline{m}_n/n \to -1/8$. This determines the constant from Theorem 4.16. Since both w and $F(s,\phi(s)) = F$ are rational functions of s, they are algebraically related. In principle this relation would determine F in terms of w so that for any given n, the best w could be found and $\overline{m}_n$ explicitly determined. Unfortunately, as the reader may verify, this relation is

$$\begin{aligned} 64F^3\big((4F-4)(w^2+1) - 8Fw\big)^2 & \\ + (8F-3)^2(16F+3)w\big((4F-4)(w^2+1) - 8Fw\big) & \\ + (16F-6)^3 w^2 = 0. & \end{aligned} \tag{5.17}$$

As a form of Monday-morning calculating, it should be noted that $F(s, w) \geq -1/8$ can be directly checked, as

$$(5.18)\quad 8F(s,w) + 1 = \left(9w^2 + (s^4 + 8s^3 + 8s + 1)w + 9s^4\right)/\left(w + 1\right)(s^4 + w).$$

The numerator in (5.18) would be non-negative for $w \geq 0$ provided $s^4 + 8s^3 + 8s + 1 \geq -18s^2$; that is, $(s^2 + 4s + 1)^2 \geq 0$. This is a short-cut to $\overline{m} \geq -n/8$ but does not as readily lead to $\lim \overline{m}/n = -1/8$ and leaves "$-1/8$" as a mysterious constant. A third approach is discussed below.

We now consider these examples in terms of the classical moment problem (see [**1**] for proofs of the assertions in this paragraph). Given $\{a_i\}$, $0 \leq i \leq 2m$, there exist a real function f and a non-negative measure μ on $\mathbf{R}$ such at $a_i = \int f^i \, d\mu$, $0 \leq i \leq 2m - 1$, $a_{2m} \geq \int f^{2m} \, d\mu$, if and only if the matrix $A_{m+1} = [a_{i+j}]$, $0 \leq i, j \leq m$, is positive semi-definite. As indicated in the introduction, we can embed our previous discussion into the classical moment problem by restricting μ to be a measure with n atoms of unit mass so $\sum x_i^p = \int g^i \, d\mu = a_i$, $n = a_0$. Then any inequality on moments necessarily induces an inequality on power sums. The converse is false, because power sums represent moments for a limited class of measures. The examples of this section only involve M_p for $0 \leq p \leq 4$ so we need consider the 3×3 matrix $A_3 = [a_{i+j}]$, $0 \leq i, j \leq 2$. A necessary condition for A_3 to be positive semi-definite is that the following inequalities hold:

$$(5.19)\quad \begin{array}{ll} \text{(i)} & a_o \geq 0, \quad a_2 \geq 0, \quad a_4 \geq 0 \\ \text{(ii)} & a_0 a_2 \geq a_1^2, \quad a_0 a_4 \geq a_2^2, \quad a_2 a_4 \geq a_3^2 \\ \text{(iii)} & a_0 a_2 a_4 + 2a_1 a_2 a_3 \geq a_2^3 + a_1^2 a_4 + a_0 a_3^2. \end{array}$$

These inequalities are also sufficient provided equality in one implies equality in all inequalities containing it. If A_3 is a positive semi-definite matrix, and the a_i's are a moment sequence, then there exists (f, μ) with $a_i = \int f^i \, d\mu$ where μ has at most three atoms. If (5.19)(iii) is an equality, then μ has at most two atoms and $a_i = \lambda r^i + \mu s^i$ for some $\lambda, \mu \geq 0$. The analogy with our earlier discussion of where M can occur is clear, and deceptive. For A_{m+1}, even when one inequality is "slack", the best one can hope for is a measure with m atoms. The same condition is to be found, in effect, in [**6**]. Theorem 3.5 is much sharper in directing our attention to points with at most two different components.

First, we wish to find the extreme values of $a_1 a_3 / a_0 a_4$ for moments $\{a_i\}$. It is clear that $a_0 = 0$ or $a_4 = 0$ imply $a_1 = a_2 = a_3 = 0$, so we may assume $a_0 a_4 > 0$. Under the change $(f, \mu) \to (\lambda f, c\mu)$, $a_i \to c\lambda^i a_i$, so that

(5.19)(i), (ii), (iii) and the ratio a_1a_3/a_0a_4 are unaltered. We may therefore assume, without loss of generality, that $a_0 = a_4 = 1$; from (5.19)(ii), $|a_1a_3| \le a_0^{1/2}a_2a_4^{1/2} \le a_0a_4 = 1$, and $a_i \equiv 1$ satisfy the inequalities. Thus $1 \ge a_1a_3/a_0a_4$ so that $n \ge M_1M_3/M_4$. Of course, the Hölder and Jensen inequalities apply to $\int |f|^p\, d\mu$ so this result could be foreseen. For the other direction, combine $a_0 = a_4 = 1$, $a_1^2 + a_3^2 \ge -2a_1a_3$ and (5.19)(iii) to get $a_2 - a_2^3 \ge -(1 + a_2)a_1a_3$. Since $1 + a_2 \ge 0$, $-a_1a_3 \le a_2(1 - a_2)/2 \le 1/8$, or $a_1a_3 \ge -1/8$. Further, the choice

$$(5.20) \qquad a_0 = a_4 = 1, a_1 = -a_3 = \pm 1/\sqrt{8}, \qquad a_2 = \tfrac{1}{2}$$

satisfies (5.19) and has $a_1a_3/a_0a_4 = -1/8$ so that this is the true minimum. A return to the theory of moment sequences shows that the pairs (f, μ) with moments (5.20) consist of a measure with two atoms whose mass ratio is $7 + 4\sqrt{3}$; the values of f on these atoms have ratio $-2 + \sqrt{3}$. This checks our earlier analysis of M_1M_3/M_4. We generalize this result below as Theorem 5.27.

The other two examples do not generalize in this way to integral inequalities, and show the limitations of the technique. The natural analogue to M_1M_3/M_2^2 is a_1a_3/a_2^2, but this ratio is unbounded among moment sequences, and even $a_1a_3/a_0^ta_2^2$ is unbounded for any t. Indeed, let $a_0 = a_4 = s$, $a_1 = \lambda as$, $a_3 = \pm\lambda as$, $a_2 = a^2s$, subject to the conditions $s > 0$ and $2\lambda^2 + a^2 < 1$. Then (5.19) is strictly satisfied, but $a_1a_3/a_0^ta_2^2 = \pm\lambda^2/a^2s^t$ which is unbounded for fixed λ and s as $a \to 0$. With the same choice of a_i's, $a_1^3a_3/a_0^ta_2^3 = \pm\lambda^4/a^2s^{1-t}$, which is similarly unbounded. The fundamental reason for this failure is that the measures described require atoms with arbitrarily large mass-ratios; this cannot happen in a power sum with fixed $n = a_0$.

We conclude by analyzing a family of functions which generalizes M_1M_3/M_4. If r, s and $(r + 3s)/4$ are integers, let $f = M_1^rM_3^s/M_4^{(r+3s)/4}$; by Theorem 4.2, we have $M = n^{(3r+s)/4}$. If r and s are even then clearly $\overline{m} = 0$; assume henceforth that r and s are odd. By Theorem 4.16, $\overline{m} \simeq -cn^{(3r+s)/4}$ in this case, where c is an unspecified constant. We now consider the ratio $a_1^ra_3^s/a_0^{(3r+s)/4}a_4^{(r+3s)/4}$ and wish to find its minimum subject to (5.19). As before, we may assume that $a_0 = a_4 = 1$, so (5.19) determines a compact set in (a_1, a_2, a_3)-space. Since $a_1^ra_3^s$ is continuous, the minimum occurs at some point on the boundary of the set; that is, where some inequality is an equality. A quick check, of which we omit the details, shows that if any inequality in (5.19)(ii) is an equality then a_1 and a_3 have the same sign so $a_1^ra_3^s \ge 0$. Thus, we can recast our problem as finding the minimum of $a_1^ra_3^s$ subject to $a_2 - a_2^3 - a_1^2 + 2a_1a_2a_3 - a_3^2 = 0$. (Of course, we need to check later that the other inequalities hold.) Let (a, b, c) be a point at which $a_1^ra_3^s$ has an extreme value. After applying

Lagrange multipliers and recalling the constraint, we derive the following four equations:

$$(5.21)\qquad \begin{aligned} &\text{(i)} && \lambda r a^{r-1} c^{s} = -2a + 2bc \\ &\text{(ii)} && 0 = 1 - 3b^2 + 2ac \\ &\text{(iii)} && \lambda s a^{r} c^{s-1} = -2c + 2ab \\ &\text{(iv)} && b - b^3 - a^2 + 2abc - c^2 = 0 \end{aligned}$$

Equations (i) and (iii) are equivalent to

$$s(abc - a^2) = r(abc - c^2)$$

and in view of (iv),

$$abc - a^2 = \frac{r}{r+s}(b^3 - b) \quad \text{and} \quad abc - c^2 = \frac{s}{r+s}(b^3 - b).$$

Finally, by applying (ii) we get a and c in terms of b:

$$(5.22)\qquad \begin{aligned} &\text{(i)} && a^2 = \left(\frac{3}{2} - \frac{r}{r+s}\right)b^3 + \left(\frac{r}{r+s} - \frac{1}{2}\right)b \\ &\text{(ii)} && c^2 = \left(\frac{3}{2} - \frac{s}{r+s}\right)b^3 + \left(\frac{s}{r+s} - \frac{1}{2}\right)b \end{aligned}.$$

But now we have two expressions for a^2c^2, from combining (5.22)(i) and (ii) and from $ac = (3b^2 - 1)/2$. After eliminating the extraneous double root at $b^2 = 1$, we find that

$$(5.23)\qquad b^2 = \frac{(r+s)^2}{3r^2 + 10rs + 3s^2} = \frac{(r+s)^2}{(3r+s)(r+3s)}.$$

We can now substitute (5.23) into (5.22). As $a^r c^s < 0$, $ac < 0$ and by arbitrarily choosing $a > 0$ and $c < 0$,

$$(5.24)\qquad \begin{cases} a = 2^{1/2} r (3r+s)^{-3/4} (r+3s)^{-1/4} \\ b = (r+s)(3r+s)^{-1/2}(r+3s)^{-1/2} \\ c = -2^{1/2} s (3r+s)^{-1/4}(r+3s)^{-3/4}. \end{cases}$$

Compare (5.24) with (5.20), when $r = s = 1$. The suspicious reader should check that (5.19) is satisfied. Using (5.24) we compute $a^r c^s$ and obtain the inequality:

$$(5.25)\qquad \begin{aligned} 1 &\geq \frac{a_1^r a_3^s}{a_0^{(3r+s)/4} a_4^{(r+3s)/4}} \\ &\geq -2^{(r+s)/2} r^r s^s (3r+s)^{-(3r+s)/4} (r+3s)^{-(r+3s)/4}. \end{aligned}$$

Let $w = r/(r+s)$ and $x = (r+3s)/(3r+s)$.The constant on the right-hand side of (5.25) can be rewritten $-(2^{-3/2}\alpha(w)/\alpha(x))^{r+s}$ where $\alpha(u) = u^u(1-u)^{1-u}$ as in §3, and one can show that $1 \leq \alpha(w)/\alpha(x) \leq 4\cdot 3^{-3/4} \simeq 1.755$. Further, (5.24) with $a_0 = a_4 = 1$ is the moment sequence of (f, μ) where μ is a measure with two atoms. Indeed, $a_i = \lambda u^i + (1-\lambda)v^i$ for $0 \leq i \leq 4$, where

$$(5.26) \qquad \lambda = \frac{1}{2} - \frac{1}{2\sqrt{3}}\,\frac{5r+s}{3r+s}, \quad u = -\frac{1+\sqrt{3}}{\sqrt{2}}\left(\frac{3r+s}{r+3s}\right)^{1/4},$$

$$v = \frac{\sqrt{3}-1}{\sqrt{2}}\left(\frac{3r+s}{r+3s}\right)^{1/4}.$$

Observe that the ratio of the values of f at the two atoms is always $-(2-\sqrt{3})$ and that the ratio of masses, $\lambda/(1-\lambda)$ is never rational. However, by approximating $\lambda/(1-\lambda)$ by rationals a_i/b_i and taking $x_i = (u, v; a_i, b_i)$, we obtain a sequence of points at which $M_1^r M_3^s / n^{(3r+s)/4} M_4^{(r+3s)/4}$ approaches the constant in (5.25). We summarize this discussion in the following theorem:

THEOREM 5.27. *If r and s are odd and $(3r+s)/4$ is an integer then*

$$n^{(3r+s)/4} \geq M_1^r M_3^s / M_4^{(r+3s)/4}$$
$$\geq -2^{(r+s)/2} r^r s^s (3r+s)^{-(3r+s)/4}(r+3s)^{-(r+3s)/4} n^{(3r+s)/4},$$

where the constant on the right-hand side is best possible.

REFERENCES

1. N. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, Edinburgh, 1965.
2. M. D. Choi, T. Y. Lam and B. Reznick, *Positive semidefinite quartic forms*, in preparation.
3. G. H. Hardy, J. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1967.
4. E. Laguerre, *Sur la régle des signes de Descartes*, Nouv. Ann. de Math., (2) **18** (1879), 5–13.
5. B. Reznick, *Ursell's Theorem and Katznelson's problem*, in preparation.
6. H. Ursell, *Inequalities between sums of powers*, Proc. London Math. Soc., (3) **9** (1959), 432–450.

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