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HOMOMORPHISMS OF MINIMAL FLOWS AND GENERALIZATIONS OF WEAK MIXING

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In this paper we are concerned with generalizations of weakly mixing. Let ϕ: $(X, T) \to (Y, T)$ be a homomorphism of metric minimal flows and let $S(\phi)$ denote the relativized equicontinuous structure relation. The main result is that if ϕ has a RIM, λ, and $z \in Z$ such that the support of λ_z equals the fiber $X_0 = \phi^{-1}(z)$, then:

$$oc(V_1 \times \cdots V_n) \supseteq S(\phi)(V_1) \times \cdots \times S(\phi)(V_n),$$

and also there exists a dense set of points $x_1, x_2, x_3, \ldots$ in X_0 such that $oc(x_1, x_2, x_3 \ldots) \supseteq S(\phi)(x_1) \times S(\phi)(x_2)x \ldots$

0. Introduction. This paper is chiefly concerned with homomorphisms of minimal flows (on compact Hausdorff spaces by a discrete phase group) having relative invariant measures (RIM's). If ϕ: $X \to Z$ has a RIM, λ, we will frequently restrict our attention to points z in Z with the support of λ_z equal to $\phi^{-1}(z)$ since otherwise the results would be substantially more difficult to state (and prove).

The major motivation for this paper is a generalization of weakly mixing — if (X, T) is a metric minimal flow having an invariant measure, then it is well known that $Q = X \times X$ implies $\text{cls}(x, x')T = X \times X$ for some x, x' in X; we show that even when $Q \neq X \times X$ a similar statement holds, that $\text{cls}(x, x')T \supseteq Q(x) \times Q(x')$ for some x, x' in X. The main results of this paper are generalizations of this idea. Some may also be viewed as a study of the recurrence properties of various subsets of X. We will now mention some special cases of the main results.

Suppose ϕ: $X \to Z$ has a RIM, λ, and X, Z are minimal and metric. Then $S(\phi) = \{(x, x')\colon (xu, x'u) \in Q(\phi) \text{ for some } u \in J\}$ (see 2.1). Suppose $z \in Z$ such that the support of λ_z equals the fiber $X_0 = \phi^{-1}(z)$. If $N = oc(V_1 \times \cdots V_n)$ where V_i is an open set in X_0, then $N \supseteq S(\phi)(V_1) \times \cdots \times S(\phi)(V_n)$ (see 1.1). Also there exists a dense set of points x_1, x_2, $x_3 \cdots$ in X_0 such that $oc(x_1, x_2, x_3, \ldots) \supseteq S(\phi)(x_1) \times S(\phi)(x_2) \times \ldots$, (see 1.5). If R is the smallest closed invariant equivalence relation containing (x_1, x_2), x_1, x_2 as above, then ϕ': $X/R \to Z$ is almost automorphic, that is, $Q(\phi')(y) = \{y\}$ for some y in X/R (see 1.4). If $N = oc(\{x\} \times V_1 \times \cdots \times V_n)$ where $V_1, \ldots, V_n$ are open sets in X_0, then $N \supseteq S(\phi)(x) \times V_1 \times \cdots \times V_n$ (see 2.9) and $N \supseteq S(\phi)(x)V \times S(\phi)(V_1)v \times \cdots \times S(\phi)(V_n)v$ for every v in J (see 2.11). In part we showed the last statement as a possible start in determining whether or not for each x in

X_0 there exists x' in X_0 with $\operatorname{cls}(x, x')T \supseteq S(\phi)(x) \times S(\phi)(x')$. If $x_\iota, y_\iota \in X_0$ and $x = (x_\iota) \in \prod X$, $y = (y_\iota) \in \prod X$, then $(x, y) \in Q(\prod\phi)$ iff $(x_\iota, y_\iota) \in Q(\phi)$ for every i, (see 2.13).

DEFINITIONS AND NOTATION. Let (X, T) be a flow with compact Hausdorff phase space X and discrete phase group T. We will write X for both the flow and the phase space. Suppose Φ: $X \to Z$ is a homomorphism of flows. We will assume ϕ is onto. We denote the orbit closure of x by $oc(x)$ $(= \operatorname{cls}(xT))$. We let X_m denote the set of transitive points (points with dense orbit), $R_m(\phi) = \{(x, x') \in X_m \times X_m\colon\ \phi(x) = \phi(x')\}$, $Q_m(\phi) = \{(x, x')$: there exist nets t_n in T and $(x_n, x_n') \in R_m(\phi)$ such that $(x_n, x_n') \to (x, x')$ and $(x_n, x_n')t_n \to (x_0, x_0)\}$ for any x_0 in X_m, $S_m(\phi)$ is the smallest closed (in $R_m(\phi)$) invariant equivalence relation containing $Q_m(\phi)$.

When X is minimal $X_m = X$, $R_m(\phi) = R(\phi)$, $Q_m(\phi) = Q(\phi)$ is the relativized regionally proximal relation $S_m(\phi) = S(\phi)$ is the relativized equicontinuous structure relation. If X is minimal and Z is the singleton flow, we denote $Q(\phi)$ by Q and $S(\phi)$ by S. Let P denote the proximal relation on any minimal flow.

Neighborhoods are assumed to be open, we denote the set of neighborhoods of x by $\mathfrak{N}_x$. The Stone-Čech compactification of T is denoted by βT, $M \subseteq \beta T$ denotes the universal minimal set (a minimal right ideal in βT), $J \subseteq M$ denotes the set of idempotents in M.

The set of closed subsets of X is denoted by 2^X and is given the usual Hausdorff topology. For $A \in 2^X$, $p \in \beta T$, we denote the limit in 2^X of At_n by $A \circ p$, where $t_n \to p$ in βT; $Ap = \{ap\colon a \in A\}$. A homomorphism of minimal flows, ϕ: $X \to Y$, is relatively incontractible (RIC) iff for every $p \in M$, $\Phi^{-1}(yp) = (\phi^{-1}(y)u) \circ p$ where $y \in Y$, $u \in J$ with $yu = y$ (see [**5**$_\mathbf{b}$] for details).

Let $\mathfrak{M}(X)$ be the set of Borel probability measures on X. For μ in $\mathfrak{M}(X)$ define μt by $\mu(At^{-1})$ for every measurable set A. A RIM (relative invariant measure — also called a section) λ for ϕ: $X \to Z$ is a homomorphism λ: $Y \to \mathfrak{M}(X)$ such that the support of λ is contained in the fiber $\phi^{-1}(z)$. If z is fixed, then for any RIM, λ, S_λ denotes the support of λ_z. Also we define $\hat{\varphi}$: $\mathfrak{M}(X) \to \mathfrak{M}(Z)$ by $\hat{\alpha}(\mu)(A) = \mu(\phi^{-1}(A))$, A a measurable subset of Z. For $B \subseteq \mathfrak{M}(X)$ we denote the closed convex hull of B by $\overline{\operatorname{co}}(B)$.

Given φ: $X \to Z$, θ: $Y \to Z$, $X \circ^Z Y = \{(x, y)\colon \varphi(x) = \theta(y)\}$. If N is a subset of $X \times Y$, $N_x = N(x) = \{y \in Y\colon (x, y) \in N\}$ is a set such that $\{x\} \times N_x = N \cap (\{x\} \times Y)$. For our purposes we will consider sets N contained in $X \circ^Z Y$ and thus $N_x \subseteq \theta^{-1}(\phi(x))$.

For the convenience of the reader we will now state some simplified results of [**6**] that we will be using. First we note that the assumption that Y be point-transitive in [**6**] was not needed.

COROLLARY 1.4 of **[6]**. *Suppose X is point-transitive, φ: $X \to Z$, θ: $Y \to Z$ are surjective homomorphisms, and θ has a* RIM, λ. *If $x_0 \in X_m$, A a Borel set contained in $\theta^{-1}(\phi)(x_0)$, and $N = \mathrm{cls}((\{x_0\} \times A)T)$, then for $x \in S(\phi)(x_0)$, $\lambda_{\phi(x_0)}(A \cap N_x) = \lambda_{\phi(x_0)}(A)$ (that is, $\lambda_{\phi(x_0)}(A \setminus N_x) = 0$). If in addition $A = B \cap C$ where B is a Borel set contained in the support of $\lambda_{\phi(x_0)}$ with $\lambda_{\phi(x_0)}(B) = 1$ and C is a non-empty open set, then $A \subseteq N_x$.*

Compare this with Lemma 2.6 below.

THEOREM 1.5 of **[6]**. *Suppose X is minimal and Q: $X \to Z$ has a* RIM, λ. *Then for x in the support of $\lambda_{\phi(x)}$, $Q(\phi)(x) = S(\phi)(x)$.*

1. A main consequence of this section is that if ϕ: $(X, T) \to (Z, T)$ has a RIM, λ, then for some z in Z there exist $x_1, x_2 \in \phi^{-1}(z)$ such that $oc(x_1, x_2) \supseteq Q(\phi)(x_1) \times Q(\phi)(x_2)$. This holds for all z that satisfy three types of conditions, $z \in Z_1 \cap Z_2 \cap Z_3$, where z_1, z_2, z_3 are as follows.

First consider π: $X \to X/S(\phi)$, with X metric. Then by Lemma 3.1 of **[9]** there exist a residual subset X_1 of X such that π is open at each x in X_1. By Proposition 3.1 of **[10]** the set $Z_1 = \{z \in Z\text{: } X_1 \cap \phi^{-1}(z) \text{ is residual in } \phi^{-1}(z)\}$ is residual in Z. Note for $x \in X_1$, every neighborhood V of x has $\pi(V)$ a neighborhood of $\pi(x)$ and $V' = V \cap \pi^{-1}(\mathrm{int}(\pi V))$ a neighborhood of x with $\pi^{-1}(\pi(V')) = \pi^{-1}(\mathrm{int}(\pi V))$ open, that is $S(\phi)(V') = \pi^{-1}(\pi(V'))$ open. As noted in **[10]**, $V \subseteq \mathrm{cls}(V')$.

More importantly, for fixed $z \in Z_1$ every open set V^* in $\phi^{-1}(z)$ contains an open set $V^{*\prime}$ in $\phi^{-1}(z)$ such that $S(\phi)(V^{*\prime})$ is open — indeed $V^* = V \cap \phi^{-1}(z)$ where V is open, and $V^* \cap X_1 \neq \varnothing$ and so there is an open set V' such that $V' \cap \phi^{-1}(z) \neq \varnothing$ and $S(\phi)(V')$ is open, thus $V^{*\prime} = V' \cap \phi^{-1}(z)$ has $S(\phi)(V^{*\prime}) = S(\phi)(V' \cap \phi^{-1}(z)) = S(\phi)(V') \cap \phi^{-1}(z)$ open in $\phi^{-1}(z)$. Also $V \subseteq \mathrm{cls}(V')$, so $V^* = V \cap \phi^{-1}(z) \subseteq (\mathrm{cls}\, V') \cap \phi^{-1}(z)$.

REMARK. Note that in the non-metric case if Y is a singleton we have that every open set V contains an open set V^* such that $S(V^*)$ is open, $V \subseteq \mathrm{cls}(V^*)$, and $A \cap V \subseteq V^*$ where $A = \{x\text{: } \phi\text{: } X \to X/S \text{ is open at } x\}$. To prove this consider ϕ: $X \to X/S$. Then $\phi(V)$ has non-empty interior W. Take $V^* = V \cap \phi^{-1}(W)$. Then $S(V^*) = \phi^{-1}(\phi(V \cap \phi^{-1}(W))) = \phi^{-1}(W)$. Let $x \in V$, then for any neighborhood U of x, $U \cap V \neq 0$ and $\phi(U \cap V)$ has non-empty interior. So $\phi(U \cap V) \subseteq W$ and $U \cap V \subseteq \phi^{-1}(W)$. Thus $V \subseteq \mathrm{cls}(V \cap \phi^{-1}(W)) = \mathrm{cls}(V^*)$.

Recall that given a function f from a metric space X onto a metric space Z, if f is a Borel map (in particular, when it is either upper semi-continuous or lower semi-continuous), then f is continuous at a residual subset of X.

Define Φ: $X \to 2^X$ by $\Phi(x) = S(\phi)(x)$. Then Φ is upper semi-continuous. There exists a residual set X_2 of X such that Φ is continuous at each $x \in X_2$. Then there exists a residual set Z_2 of Z such that $X_2 \cap \phi^{-1}(z)$ is residual in $\phi^{-1}(z)$ for $z \in Z_2$.

Finally, if ϕ: $X \to Z$ admit a relative invariant measure (RIM), λ, then the function g: $Z \to 2^X$ defined by $g(z) =$ support of λz is lower semi-continous. Then one can show that there exists a residual set Z_3 of Z such that the support of λz equals $\Phi^{-1}(z)$ for z in Z_3 (Proposition 3.3 of [**4**]).

1.1. PROPOSITION. *Suppose for* $i = 1, 2, \ldots, n$ ϕ_i: $X_i \to Z$ *has a relative invariant measure* λ_i, $z \in Z$ *with* $\phi_i^{-1}(z) =$ *support of* λ_{iz}, *and* V_i *are open subsets of* $\phi_i^{-1}(z)$ *with* $Q(\phi_i)(V_i)$ *open in* $\phi_i^{-1}(z)$. *Then*

$$N = oc(V_1 \times \cdots \times V_n) \supseteq Q(\phi_1)(V_1) \times \cdots \times Q(\phi_n)(V_n)$$
$$= S(\phi_1)(V_1) \times \cdots \times S(\phi_n)(V_n).$$

Proof. The last equality follows from 1.5 of [**6**].

Now $N \supseteq Q(\phi_1)(V_1) \times V_2 \times \cdots \times V_n$ by Corollary 1.4 of [**6**] since $V_2 \times \cdots \times V_n$ is open and $X_2 \times \cdots \times X_n$ has a relative invariant measure. So $N \supseteq Q(V_1) \times Q(V_2) \times V_3 \times \cdots \times V_n$ by 1.4 of [**6**] since $Q(V_1) \times V_3 \times \cdots \times V_n$ is open, the proposition follows by induction.

1.2. PROPOSITION. *For* $i = 0, 1, \ldots, n$. *Suppose* (X_i, T) *is a minimal flow with* T *abelian. Then for any* x_0 *in* X_0 *and for any open sets* V_i *in* X_i, $i = 1, 2, \ldots, n$ *there exist open sets* $U_0, U_1, \ldots, U_n$ *such that* $x_0 \in \mathrm{cls}(U_0)$; $U_i \subseteq V_i$, $i = 1, 2, \ldots, n$; *and*

$$oc(\{x_0\} \times V_1 \times \cdots \times V_n)$$
$$\supseteq oc(U_0 \times U_1 \times \cdots \times U_n) \supseteq \mathrm{cls}[Q(U_0) \times Q(U_1) \times \cdots \times Q(U_n)]$$
$$\supseteq \{x_0\} \times Q(U_1) \times \cdots \times Q(U_n).$$

If $X_0 \to X_0/Q$ *is open at* x_0, *then we can take* U_0 *with* $x_0 \in U_0$.

Proof. Let $V(t_1, t_2, \ldots, t_n)$ denote $V_1 t_1 \times V_2 t_2 \times \cdots \times V_n t_n$ where $t_i \in T$. Then there exist finite many n-tuples $s_1, s_2, \ldots, s_m$ in $\prod_1^n T$ such that $\cup V(s_i) = \prod_1^n X_i$. Let $Y = \prod_1^n X_i$. Then $\cup_1^n [\{x_0\} \times V(s_i)] = \{x_0\} \times Y$. So $\cup_1^n ([\{x_0\} \times V(s_i)]t) = (\cup [\{x_0\} \times V(s_i)])t = (\{x_0\} \times Y)t = \{x_0 t\} \times Y$. Therefore $\cup_1^n \mathrm{cls}([\{x_0\} \times V(s_i)]T) = X_0 \times Y$ and thus $\mathrm{cls}([\{x_0\} \times V(s_i)]T)$ has non-empty interior, I, for some i. Then for some t in T, $[\{x_0\} \times V(s_i)]t \cap I \neq \varnothing$. So there exist open sets U_0', $U_1', \ldots, U_n'$ with $x_0 \in U_0'$, and $U_1' \times \cdots \times U_n' \subseteq V(s_i)$ such that $(U_0' \times U_1' \times \cdots \times U_n')t \subseteq I$ and so $U_0' \times U_1' \times \cdots \times U_n' \subseteq \mathrm{cls}([\{x_0\} \times V(s_i)]T)$.

Let $s_i = (t_1, \dots, t_n)$, then since T is abelian,

$$U_0' \times U_1' t_1^{-1} \times \cdots \times U_n' t_n^{-1} \subseteq \operatorname{cls}([\{x_0\} \times V_1 \times \cdots \times V_n] T).$$

Then by the remarks above there exist open sets $U_i \subseteq U_i' t_i^{-1}$ such that $Q(U_i)$ is open $i = 0, 1, \dots, n$, and $x_0 \in U_0' \subseteq \operatorname{cls} U_0$. When $X_0 \to X_0/Q$ is open at x_0 we have $x_0 \in U_0$. Then the proposition follows by 1.1.

1.3. THEOREM. *Suppose for* $i = 1, 2, \dots, \varphi_i \colon X_i \to Z$ *has a* RIM, λ_i, *and* X_i *is a metric minimal flow. Let* $z_0 \in \bigcap_{i=1}^{\infty} (Z_1^i \cap Z_2^i \cap Z_3^i)$, $(Z_j^i = Z_j$ *as above for* $\phi_j)$. *Given* x_i *in* $X_0^i = \phi_i^{-1}(z_0)$ *and dense* G_δ *subsets* G^i *of* X_0^i *there exist points* x_i' *in* G^i *such that* $x_i' \in B(x_i, 1/i)$ *and* $oc(x_1', x_2', \dots) \supseteq Q(\phi_1)(x_1') \times Q(\phi_2)(x_2') \times \cdots$. *(Recall* $Q(\phi)(x) = S(\phi)(x)$ *for* $x \in \phi^{-1}(Z_3)$.*)*

Proof. Let $B(y_1, \dots, y_n; \varepsilon)$ denote $B(y_1, \varepsilon) \times \cdots \times B(y_n, \varepsilon)$. Fix $\varepsilon > 0$. Consider any set $\{x_i\}$, $x_i \in X_0^i \cap X_2^i$. For each i use the continuity of Φ_i at x_i to associate a neighborhood $U_i^* = U_i^*(x_i, \varepsilon)$ of x_i with x_i and ε such that if $x \in U_i^*$ and $y_i \in Q(\phi_i)(x_i)$, then $Q(\phi_i)(x) \cap B(y_i, \varepsilon) \neq \varnothing$; and if $\varepsilon' > \varepsilon$, then $U_i^*(x_i, \varepsilon^*) \supseteq U_i^*(x_i, \varepsilon)$. Let $U_i = U_i(x_i, \varepsilon) = U_i^* \cap X_0^i$ and $y_i \in Q(\phi_i)(x_i)$; note $x_i \in U_i$. Now for each n consider the set $W_n = W(y_1, \dots, y_n; \varepsilon) = \{w \in U_1 \times \cdots \times U_n \colon wt \in B(y_1, \dots, y_n; \varepsilon)$ for some t in $T\}$. Clearly W_n is open (in $X_0^1 \times \cdots \times X_0^n$). Also W_n is a dense subset of $U_1 \times \cdots \times U_n$; since for any basic open subset $V = V_1 \times \cdots \times V_n$ of $U_1 \times \cdots \times U_n$ in $X_0^1 \times \cdots \times X_0^n$ take an open subset $V^* = V_1^* \times \cdots \times V_n^*$ with $Q(\phi_i)(V_i^*)$ open. Then for any point $(x_1^*, \dots, x_n^*)$ in V^*, there exists $y_i^* \in Q(\phi_i)(x_i^*) \cap B(y_i, \varepsilon)$ for $i = 1, \dots, n$, and by 1.1, we have $oc(V^*) \supseteq Q(\phi_1)(V_1^*) \times \cdots \times Q(\phi_n) \ni (y_1^*, \dots, y_n^*)$; so there exists t in T with $V^* t \cap B(y_1, \dots, y_n; \varepsilon) \neq 0$, and thus W_n is dense.

Consider a cover of $Q(\phi_1)(x_1) \times \cdots \times Q(\phi_n)(x_n)$ by sets of the form $B(y_1, \dots, y_n; \varepsilon)$ where $y_i \in Q(\phi_i)(x_i)$. Take a finite subcover and the (finite) intersection B_n of the corresponding W_n's, then B_n is open (in $X_0^1 \times \cdots \times X_0^n$) and is dense in $U_1 \times \cdots \times U_n$. By continuity, for each b in B_n, there is a neighborhood $E(b)$ of b contained in B_n such that any given open set in the finite subcover contains $E(b)t$ for some t in T. From this it is clear that for any $(y_1, \dots, y_n)$ in $Q(\phi_1)(x_1) \times \cdots \times Q(\phi_n)(x_n)$, $E(b)t \subseteq B(y_1, \dots, y_n; 2\varepsilon)$ for some t in T.

Now consider a given collection $x_i \in X_0^i$, $i = 1, 2, \dots$. We may assume $x_i \in X_0^i \cap X_2^i$. Let H_j^i, $i, j = 1, 2, \dots$, be dense open subsets of X_0^i such that $H_{j+1}^i \subseteq H_j^i$ and $\bigcap_{j=1}^{\infty} H_j^i = X_2^i \cap X_0^i \cap G^i$ for $i = 1, 2, \dots$. Start an induction with x_1, x_2, $n = 2$ and $\varepsilon = \frac{1}{2}$. Take B_2 as above and $b_2 \in (X_2^1 \times X_2^2) \cap (G^1 \times G^2) \cap B_2 \cap [B(x_1, 1) \times B(x_2, \frac{1}{2})]$. Let $E_2 = (H_1^1 \times H_2^2) \cap E(b_2) \cap B(b_2, \frac{1}{2}) \cap [B(x_1, 1) \times B(x_2, \frac{1}{2})]$; note it is a neighborhood of b_2 in $X_0^1 \times X_0^2$. Now consider $b_2 \times \{x_3\}$, $n = 3$, $\varepsilon = \frac{1}{3}$, and take B_3 as above and $b_3 \in (X_2^1 \times X_2^2 \times X_2^3) \cap (G^1 \times G^2 \times G^3) \cap B_3 \cap [E_2 \times B(x_3, \frac{1}{3})]$. Take a neighborhood E_3 of b_3 with $\text{cls}(E_3) \subseteq (H_2^1 \times H_2^2 \times H_2^3) \cap E(b_3) \cap B(b_3, \frac{1}{3}) \cap [E_2 \times B(x_3, \frac{1}{3})]$. Consider $b_3 \times \{x_4\}$, $n = 4$, $\varepsilon = \frac{1}{4}$, take B_4 as above and $b_4 \in (X_2^1 \times X_2^2 \times X_2^3 \times X_2^4) \cap (G^1 \times G^2 \times G^3 \times G^4) \cap [E_3 \times B(x_4, \frac{1}{4})] \cap B_4$. Continue in this way.

Note $\bigcap_{n=b2}^{\infty} (E_n \times \Pi_{n+1}^{\infty} X_i)$ is a singleton, say $\{(x_1', x_2', \ldots)\}$, and note $(x_1', x_2', \ldots) \in [(X_2^1 \cap X_0^1 \cap G^1) \times (X_2^2 \cap X_0^2 \cap G^2) \times (X_2^3 \cap X_0^3 \cap G^3) \times \cdots] \cap [B_1(x_1, 1) \times B(x_2, \frac{1}{2}) \times B(x_3, \frac{1}{3}) \times \cdots]$. We claim $oc(x_1', x_2', \ldots) \supseteq Q(\phi_1)(x_1') \times Q(\phi_2)(x_2') \times \cdots$. For any $(y_1, y_2, \ldots)$ in $Q(\phi_1)(x_1') \times Q(\phi_2')(x_2') \times \ldots$, a basic neighborhood of it is of the form $B(y_1, \ldots, y_n; \lambda) \times \Pi_{n+1}^{\infty} X$ for some n and $\lambda > 0$. Let $U_i' = U(x_i', \lambda)$ for $i = 1, 2, \ldots, n$. Take j such that $b_j \in U_1' \times \cdots \times U_n' \times \Pi_{n+1}^{j} X$ and $1/(j+1) < \lambda$. Then

$$\left[Q(\phi_1)(b_{j1}) \times \cdots \times Q(\phi_n)(b_{jn})\right] \cap B(y_1, \ldots, y_n; \lambda) \neq \varnothing,$$

where $b_j = (b_{j1}, \ldots, b_{jn})$, (since $b_{ji} \in U_i'$). Let $(y_1^*, \ldots, y_n^*)$ be a point in this intersection. Then there exists t in T such that $(x_1', x_2', \ldots)t \in E_{j+1}t \subseteq B(y_1^*, \ldots, y_n^*;\ 2/(j+1)) \subseteq B(y_1, \ldots, y_n;\ 3\lambda)$. Thus $(y_1, y_2, \ldots) \in oc(x_1', x_2', \ldots)$.

1.4. COROLLARY. *Suppose X is metric, minimal flow and ϕ: $X \to Z$ has a* RIM. *Then there exists $(x_0, x_1) \in X \times X$ such that ϕ': $Y = X/R(x_0, x_1) \to Z$ is an almost automorphic extension of Z* (i.e., *there is a point y in Y with $Q(\phi')(y) = \{y\}$*) *where $R(x_0, x_1)$ is the smallest closed invariant equivalence relation containing (x_0, x_1).*

Proof. This is clearly the case if we take (x_0, x_1) such that $oc(x_0, x_1) \supseteq Q(\phi)(x_0) \times Q(\phi)(x_1)$.

2. In this section we develop some connections of a RIM on ϕ: $X \to Y$ to the relativized equicontinuous structure relation, $S(\phi)$, and apply them to study the orbit closures of sets of the form $\{x\} \times A^2 \times \cdots \times A^n$ in a product space and to give a special characterization $S(\phi)$ in the case when $(R(\phi), T)$ has a dense set of almost periodic points.

Suppose ϕ: $X \to Y$ has a RIM, λ, X is minimal and N is a closed invariant set in $R(\phi)$. Then ϕ_N: $R(\phi) \to [0, 1]$ defined by $\phi_N(x, x') = \lambda_{\phi(x)}(N(x) \Delta N(x')) = 2\lambda_{\phi(x)}(N(x) \setminus N(x'))$ is continuous, **[6]** where $\{x\} \times N(x) = N \cap (\{x\} \times X)$ and Δ is the symmetric difference. So for each N, $\phi_N(x, x')$ is a pseudo-metric on each fiber that is invariant, $\phi_N(xt, x't) = \phi_N(x, x')$. Defining $R_N = \{(x, x') \in R(\phi)$: $\phi_N(x, x') = 0\}$, we have $X \to X/R_N \xrightarrow{\psi_N} Y$ and ψ_N is an isometric homomorphism (and thus almost periodic).

Consider $S^*(\phi) = \{(x, x'): \phi_N(x, x') = 0$ for all closed invariant subsets N of $R(\phi)\}$. Then by 1.2 of [6] $S(\phi) \subseteq S^*(\phi)$. We wish to show that $S^*(\phi) \subseteq S(\phi)$. Note by 1.2 of [6] $S^*(\phi)$ is closed and invariant. Suppose $(x, x') \in S^*(\phi)$. Let $\phi(x) = z_0$, let $x_1 \in S_\lambda$ = support of λ_{z_0}, and let $pu \in M$ such that $xpu = x_1$. Note $(xpu, x'pu) \in S^*(\phi)$. For any $V \in \mathfrak{N}_{x_1}$ consider $N = oc(\{x'pu\} \times V \cap S_\lambda)$. By 1.4 of [6] $N \supseteq \{x_1\} \times (V \cap S_\lambda)$, so $N \cap V \times (V \cap S_\lambda) \neq \varnothing$, so there exists t_V in T and x_V in $V \cap S_\lambda$, such that $x_V t_V \in V$ and $x'put_V \in V$. Thus $x' = xpu \in Q(\phi)(x'pu)$ and so $(xu, x'u) \in Q(\phi)$ and $(x, x') \in S(\phi)$. Thus we have the following proposition.

2.1. PROPOSITION. *If* ϕ: $X \to Z$ *has a* RIM, λ, *then* $\{(x, x') \in R(\phi)$: $\lambda_{\phi(x)}(N(x)\Delta N(x')) = 0$ *for all closed invariant sets* N *in* $R(\phi)\} = S(\phi) = \{(x, x') \in R(\phi)$: $(xu, x'u) \in Q(\phi)$ *for some (and thus every)* $u \in J\}$.

2.2. PROPOSITION. *Suppose* Φ: $X \to Y$ *has a* RIM, λ, *and* X *and* Y *are minimal. If* ϕ *is open and* $S(\phi) = R(\phi)$, *then* $Q(\phi) = S(\phi)$.

Proof. Let $(x, x') \in R(\phi) = S(\phi)$, we will show$(x, x') \in \overline{Q(\phi)} = Q(\phi)$. Let U and V be open neighborhoods of x and x' respectively. Let x_0 be any point in the support of $\lambda_{\phi(x)}$. Since ϕ is an open map, $\phi(V) \cap \phi(U)$ is an open neighborhood of $\phi(x)$. There exist t_0 in T with $x_0t_0 \in V$ and $\phi(x_0t_0) \in \phi(V) \cap \phi(U)$. So there is $x_1 \subset U$ with $\phi(x_1) = \phi(x_0t_0)$; then $(x_1t_0^{-1}, x_0) \in R(\phi) = S(\phi)$ and by 1.5 of [6], $x_1t_0^{-1} \in S(\phi)(x_0) = Q(\phi)(x_0)$. Therefore $(x_1, x_0t) = (x_1t_0^{-1}, x_0)t_0 \in Q(\phi)$ and $(x, x') \in \overline{Q(\phi)} = Q(\phi)$.

2.3. LEMMA. *Given* ϕ: $X \to Y$, θ: $Y \to Z$, X *minimal. Let* $x \in X$ *and* $y = \phi(x)$. *Then for any* $y' \in S(\theta)(y)$ *there exists* $x' \in S(\theta \circ \phi)(x)$ *with* $y' = \phi(x')$. (*Note this is somewhat stronger than the statement* $\phi \times \phi(S(\theta \circ \phi)) = S(\theta)$.)

Proof. By 14.2 of [2_b], $\phi \times \phi(Q(\theta \circ \phi)) = Q(\theta)$. Consider $M \overset{\psi}{\to} X$ with $\psi(m) = xm$. Then $\phi \times \phi(\psi \times \psi(Q(\theta \circ \phi \circ \psi))) = Q(\theta)$. Let $u \in J$ with $xu = x$. Note $Q(\theta \circ \phi \circ \psi)$ is left invariant under $G = M_0u$, $M_0 = (\theta \circ \phi \circ \psi)^{-1}(y)$; and so $S(\theta \circ \phi \circ \psi)$ is also, since $g \times g(S(\theta \circ \phi \circ \psi))$ is a closed invariant equivalence relation containing $g \times g(Q(\theta \circ \phi \circ \psi)) = Q(\theta \circ \phi \circ \psi)$, for $g \in G$. Let R denote $\phi \times \phi(\psi \times \psi(S(\theta \circ \phi \circ \psi)))$. Also $S(\theta) \supseteq \phi \times \phi(S(\theta \circ \phi)) \supseteq R$. To show the reverse inclusion first note $Q(\theta) = \phi \times \phi(\psi \times \psi(Q(\theta \circ \phi \circ \psi))) \subseteq R$. Also R is closed and invariant; we will now show that R is an equivalence relation and thus $S(\theta) \subseteq R$ and the lemma will follow. We only need to show that if $(y_1, y_2) \in R$ and

$(y_2, y_3) \in R$, then $(y_1, y_3) \in R$. Let $m_1, m_2, m'_2, m_3 \in M$ with (m_1, m_2), $(m'_2, m_3) \in S(\theta \circ \phi \circ \psi)$ and $\phi \circ \psi(m_i) = y_i$, $i = 1, 2, 3$ $\phi \circ \psi(m'_2) = y_2$. Choose $m \in M$ so that $mm'_2 \in m_2 J$. Then $(mm'_2, mm_3) \in S(\theta \circ \phi \circ \psi)$ and $(m_2, mm'_2) \in S(\theta \circ \phi \circ \psi)$, and so $(m_1, mm_3) \in S(\theta \circ \phi \circ \psi)$. Also $\phi \circ \psi(mm_3) = y_3$ since $\phi \circ \psi(mm'_2) = \phi \circ \psi(m'_2)$; so $(y_1, y_3) \in R$. Thus we have that $S(\theta) = \phi \times \phi(S(\theta \circ \phi)) = R$.

Now suppose $y' \in S(\phi)(y)$, and $(m, m') \in S(\theta \circ \phi \circ \psi)$ with $\phi \circ \psi(m) = y$, $\phi \circ \psi(m') = y'$. We may assume $m = mu$ since $S(\theta \circ \phi \circ \psi)$ is an equivalence relation. Then $(u, m^{-1}m') \in S(\theta \circ \phi \circ \psi)$ and $\psi(u) = x$. Let $x' = \psi(m^{-1}m')$. Then $(x, x') \in S(\theta \circ \phi)$ and $\phi(x') = y'$. Thus the lemma is proved.

2.4. LEMMA. *Let M be the universal minimal set, Z a minimal flow, z a fixed element of Z, $u \in J$ with $zu = z$, and ψ: $M \to Z$ be defined by $\psi(p) = zp, p \in M$.*

If $p \in S(\psi)(u)$ and $pv = p$, $v \in J$, then $[S(\psi)(u)]v = [S(\psi)(u)]p$.

Proof. If $m \in S(\psi)(u)$, then $mp \in S(\psi)(up) = S(\psi)(u)$ since $up = p$ and $S(\psi)$ is a closed invariant equivalence relation. So $S(\psi)(u)p \subseteq S(\psi)(u)$ and so $S(\psi)(u)p \subseteq S(\psi)(u)v$.

Let p^{-1} be the inverse of p in the group Mv. Then $S(\psi)(u)p^{-1} \in S(\psi)(u)$ and $S(\psi)(u)v = S(\psi)(u)p^{-1}p \subseteq S(\psi)(u)p$.

2.5. COROLLARY. *Using the same notation as in Lemma 2.4 and $v \in J$; if $p \in S(\psi)(u)$, $pv = p$, and ϕ: $X \to Z$, then $S(\phi)(x)p = S(\phi)(x)v$ for all x in X with $\phi(x) = z$ and $xu = x$.*

Proof. Straightforward.

The following lemma is a variation of Corollary 1.4 of [**6**].

2.6. LEMMA. *Suppose ϕ: $X \to Z$, θ: $Y \to Z$, Z minimal and θ has a* RIM *(section), λ. Let $r \in X$ and $z = \phi(r)$, let V be an open set in the support of λ_z, and let $N = oc(\{r\} \times V)$ and $v \in J$, with $zv = z$. Then $N \supseteq \{rv\} \times v$. (Note X and Y are not required to be minimal, otherwise it would be trivial in view of* 1.4 *of* [**6**] *since rv and r are proximal and so (rv, r) would be in $S(\phi)$.)*

Proof. We will assume the reader is familiar with the notation and definitions in [**6**]. Let $W \in \mathfrak{N}(N_{rv})$ with $\lambda_z(W) < \lambda_z(N_{rv}) + \varepsilon$. Then there exists t in T for which $N_r t \subseteq W$ and $N_{rv} t \subseteq W$ and $|\lambda_{zt}(W) - \lambda_z(W)| < \varepsilon$.

Then

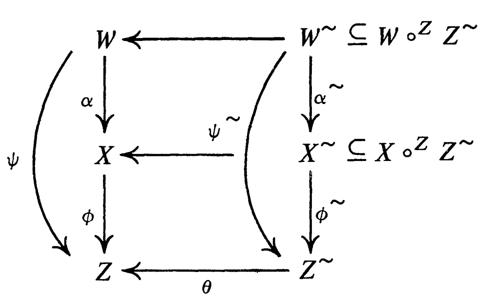
$$\begin{aligned}\lambda_z(N_r \setminus N_{rv}) &= \lambda_{zt}(N_r t \setminus N_{rv} t) \le \lambda_{zt}(W \setminus N_{rv} t)\\ &= \lambda_{zt}(W) - \lambda_{zt}(N_{rv} t)\\ &= \lambda_{zt}(W) - \lambda_z(N_{rv})\\ &\le |\lambda_{zt}(W) - \lambda_z(W)| + |\lambda_z(W) - \lambda_z(N_{rv})|\\ &< 2\varepsilon.\end{aligned}$$

Thus $\lambda_z(N_r \setminus N_{rv}) = 0$. Now $\lambda_z(V \setminus N_{rv}) \le \lambda(N_r \setminus N_{rv}) = 0$, and so $V \setminus N_{rv} = \varnothing$ since $V \setminus N_{rv}$ is open in the support of λ_z. Thus $V \subseteq N_{rv}$.

2.7 LEMMA. *Suppose* ϕ: $X \to Z$, θ: $Y \to Z$, Z *minimal, and* θ *has a* RIM. *Let* $x_0 \in X$, $z_0 = \phi(x_0)$, *and let* Λ *be a non-empty subset of the set* $\{\lambda: \lambda$ *is a* RIM *for* $\theta\}$. *Let* S_λ *be the support of* λ_{z_0} *and* $S = \text{cls}(\bigcup_{\lambda \in \Lambda} S_\lambda)$. *Let* C *be an open set in* $\theta^{-1}(z_0)$ *and* $A = S \cap C$. *Consider* $N = oc(\{x_0\} \times A)$. *Then* $N_x \supseteq A$ *for all* $x \in S_m(\phi)(x_0)$. *Note if* X *is minimal,* $S_m(\phi) = S(\phi)$. *In addition if* $v \in J$ *with* $z_0 v = z_0$, *then* $N \supseteq \{x_0 v\} \times A$.

Proof. By 1.4 of [6], $A \cap S_\lambda = C \cap S_\lambda \subseteq N_x$ for every λ in Λ. So $A \cap (\bigcup S_\lambda) = C \cap (\bigcup S_\lambda) \subseteq N_x$ and $\text{cls}(C \cap (\bigcup S_\lambda)) \subseteq N_x$. If $y \in C \cap S$, then for every open neighborhood V of y in $\theta^{-1}(z_0)$ with $V \subseteq C$, there exists y_λ in S_λ for some λ in Λ with $y_\lambda \in V \subseteq C$; thus $A = C \cap S \subseteq \text{cls}(C \cap (\bigcup S_\lambda)) \subset N_X$. The additional statement follows similarly from 2.6.

2.8. PROPOSITION. *Given homomorphisms* α: $W \to X$, ϕ: $X \to Z$ *where* ϕ *has* RIM λ *and* W *is minimal, let* $\psi = \phi \circ \alpha$. *Then there exists a strongly proximal extension* θ: $Z^\sim \to Z$ *such that the following diagram commutes*



where $W^\sim$ *and* $X^\sim$ *are the unique minimal sets in* $W \circ^Z Z^\sim$ *and* $X \circ^Z Z^\sim$ *respectively. And* $\psi^\sim$ *has a* RIM γ, *that induces the* RIM $\lambda_{\theta(\nu)} \times \delta_\nu$, $\nu \in Z^\sim$ *on* $\phi^\sim$. ($\delta\nu$ *is the point mass at* $\nu \in Z^\sim$).

REMARK. (a) Compare this with 5 of [7].

(b) When W is the universal minimal set M, we have $M^{\sim}$ which is in fact isomorphic with M through the map $(p, \gamma) \to p$.

(c) If Z is a universal strongly proximal flow, then θ: $Z^{\sim} \to Z$ is an isomorphism and any RIM on ϕ can be lifted to a RIM on ψ.

Proof. We assume the reader is familiar with the contents of [4]. Let $x_0 \in X_0$, $w_0 \in W$ with $\alpha(w_0) = x_0$, and $u \in J$ with $w_0 u = w_0$. Consider $\hat{\psi}$: $\mathfrak{M}(W) \to \mathfrak{M}(Z)$, $\hat{\alpha}$: $\mathfrak{M}(W) \to \mathfrak{M}(X)$. Let $P = \overline{\mathrm{co}}(oc(\lambda_{z_0}))$ and note $\hat{\phi}$: $P \to \mathfrak{M}(Z)$ is $\mathfrak{M}(Z)$ irreducible since λ is a RIM. Let Q be a P-irreducible subset of $\hat{\alpha}^{-1}(P)$ and note that Q is also $\mathfrak{M}(Z)$ irreducible. Let $Z^{\sim} = \overline{\mathrm{ex}}(Q)$ and $\theta = \hat{\psi} \mid Z^{\sim}$ — the restriction of $\hat{\psi}$ to $Z^{\sim}$; we identify z in Z with δ_z and consider θ: $Z^{\sim} \to Z$. Let $X^{\sim}$, $W^{\sim}$ be the unique minimal sets in $\{(x, \nu) \in X \times Z^{\sim} : \phi(x) = \theta(\nu)\}$, $\{(p, \nu) \in W \times Z^{\sim} : \phi \circ \alpha(p) = \theta(\nu)\}$ respectively. Let $\phi^{\sim}$ be the projection of $X^{\sim}$ onto $Z^{\sim}$ and $\psi^{\sim}$ be the projection of $W^{\sim}$ onto $Z^{\sim}$. For each ν in $Z^{\sim}$ the measure $\nu \times \delta_\nu$ on $W \times Z^{\sim}$ is supported in $W^{\sim}$ and the map γ: $Z^{\sim} \to \mathfrak{M}(W^{\sim})\gamma_\nu = \nu \times \delta_\nu$ is a RIM for $\psi^{\sim}$: $W^{\sim} \to Z^{\sim}$. Also $(\hat{\alpha} \times \mathrm{id})$: $W^{\sim} \to X^{\sim}$ induces a RIM $\beta = (\hat{\alpha} \times \mathrm{id})(\gamma)$ on $\phi^{\sim}$: $X^{\sim} \to Z^{\sim}$ by $\beta_\nu = (\hat{\alpha} \times \mathrm{id})(\gamma_\nu) = \hat{\alpha}(\nu) \times \delta_\nu$; note $\hat{\alpha}(\nu_0) = \lambda_{z_0}$ for some ν_0 in $Z^{\sim}$ and so $\theta(\nu_0) = z_0$ and $\beta_\nu = \lambda_{\theta(\nu)} \times \delta_\nu$ for ν in $Z^{\sim}$.

2.9. THEOREM. *Suppose for $i = 1, 2, 3, \ldots, n$, ϕ_i: $X^i \to Z$ are homomorphisms and X^i is a minimal flow. Suppose ϕ_1 has a* RIM, λ. *Let $z_0 \in Z$, $X_0^i = \phi_i^{-1}(z_0)$. Then, there exist non-empty subsets $J^* \subseteq J'$ of J such that $X_0^i J^*$, $X_0^i J'$ are compact subsets of X_0^i and such that given $A^i = (X_0^i J^*) \cap V^i$ where V^i is an open subset of X_0^i, $x, x' \in X_0^1 J'$ with $(x, x') \in S(\phi_1)$, and $N = oc(\{x\} \times A^2 \times \cdots \times A^n)$ we have $N \supseteq \{x'\} \times A^2 \times \cdots \times A^n$.*

REMARK (a) If $u, v \in J'$, $x_1 u \in X_0^1$, $x_2 v \in X_0^2$, and

$$N = \mathrm{oc}\big(\{(x_1 u, x_2 v)\}\big) \times A^3 \times \cdots \times A^n;$$

then $N \supseteq \{(x_1 u, x_2 u)\} \times A^3 \times \cdots \times A^n$.

(b) $X_0^i J^* \supseteq \bigcup \{X_0^i u$: $u \in J$ for which $x'u \in S_\mu$ for some x' in X_0^i and some RIM, μ, for $\phi_i\}$ where S_μ is the support of μ_{z_0}.

(c) J^* and J depend on Z but not on the ϕ_i's.

(d) For $n = 2$ compare this with 1.4 of [6], where ϕ_2 has a RIM and ϕ_1 is not required to have a RIM.

2.10. COROLLARY. *If $X = X^i$, $i = 1, 2$, $x \in X_0 J^*$, $x' \in X_0 J'$, and $(x, x') \in S(\phi)$, then there exist x_n in $X_0 J^*$ and t_n in T with $x_n \to x$, $x_n t_n \to x$, $x' t_n \to x$; in particular $(x, x') \in Q(\phi)$.*

Proof of 2.9. Let $u \in J$ with $z_0 u = z_0$. Define ψ: $M \to Z$ by $p \to z_0 p$. Let $M_0 = \psi^{-1}(z_0)$. Fix $x_0^i \in \phi_i^{-1}(z_0) = X_0^i$ with $x_0^i u = x_0^i$ and define α_i: $M \to X^i$ by $\alpha_i(p) = x_0^i p$. Note $\psi = \alpha_i \circ \phi_i$. By 4.1 of [4], there is a strongly proximal extension $\tilde{Z}$ of Z, $\tilde{Z}$ is minimal, $\theta: \tilde{Z} \to Z$ is strongly proximal, such that the projection $\tilde{\psi}$ of $\tilde{M}$ onto $\tilde{Z}$ has a RIM where $\tilde{M}$ is the unique minimal set in $M \circ^Z \tilde{Z} = \{(m, z) \in M \times Z: \psi(m) = \theta(z)\}$. By 2.7 we see that we are interested that the union of the supports of the RIM's on $\tilde{\psi}$ be as large as possible. We will now determine one aspect of the size of this union by "translating" measures. Given a RIM γ on $\tilde{\psi}$ define the translation $p\gamma$ by $p\gamma_\nu(A) = \gamma_\nu(pA)$ for $p \in \tilde{M_0} = \tilde{\psi}^{-1}(\psi(u))$ and $\nu \in \tilde{Z}$. It is easy to see that $p\gamma$ is again a RIM on $\tilde{\psi}$. Let $\gamma_0 = \tilde{\psi}(u)$, Γ be the set of all RIM's on $\tilde{\psi}$, and S_γ, $\gamma \in \Gamma$, be the support of γ_{ν_0}. From the above it is easy to see that $\mathrm{cls}\{\bigcup S_\gamma: \gamma \in \Gamma\}$ is of the form $M_0 J^* \subseteq \tilde{M_0} \subseteq M_0 \times \{\nu_0\}$ for some subset J^* of J.

Now to prove the theorem we first show a similar result for $\tilde{\phi_1}$ and then reduce it to ϕ_1. Suppose $A^i = (X_0^i J^*) \cap V^i$ where V^i is open in X^i and $((x, \nu_0), (x', \nu_0)) \in S(\tilde{\phi_1})$. Let $\tilde{N} = oc(\{x_0, \nu_0\}) \times (A^2 \times \{\nu_0\}) \times \cdots \times (A^n \times \{\nu_0\})$. Then if $(p, \nu_0) \in \tilde{M_0}$ with $\alpha_1(p) = x$, there exist $(p', \nu_0) \in \tilde{M_0}$ with $\alpha_1(p') = x'$ and $((p, \nu_0), (p', \nu_0)) \in S(\tilde{\psi})$. Consider

$$N^* = oc\big\{\{(p, \nu_0)\} \times \big(\alpha_2^{-1}(V^2) \times \{\nu_0\} \cap \tilde{M_0} J^*\big) \times \cdots \times \big(\alpha_n^{-1}(V^n) \times \{\nu_0\} \cap \tilde{M_0} J^*\big)\big\}.$$

For $i = 2, 3, \ldots, n$, let $\gamma_i \in \Gamma$, then $\Pi\gamma_i$ is a RIM and $S_{\pi\gamma_i} = \Pi S_{\gamma_i}$ so $\mathrm{cls}(\bigcup\{S_{\Pi\gamma_i}: \gamma_i \in \Gamma, i = 2, \ldots, n\}) = \Pi_2^n \tilde{M_0} J^*$. So by 2.7

$$N^* \supseteq \{(p', \nu_0)\} \times \big[\alpha_2^{-1}(V^2) \times \{\nu_0\} \cap \tilde{M_0} J^*\big] \times \cdots \times \big[a_n^{-1}(V^n) \times \{\nu_0\} \cap \tilde{M_0} J^*\big],$$

and $\tilde{N} \supseteq \{(x', \nu_0)\} \times (A^2 \times \{\nu_0\}) \times \cdots \times (A^n \times \{\nu_0\})$, since if $\alpha_i(p_i) = a_i \in A^i$ then $a_i u_i^* = a_i$ for some $u_i^* \in J^*$ and $(p_i u_i^*, \nu_0) \in (\alpha_i^{-1}(V^i) \times \{\nu_0\}) \cap \tilde{M_0} J^*$. Thus

$$N = oc\big(\{x\} \times A^2 \times \cdots \times A^n\big) \supseteq \{x'\} \times A^2 \times \cdots \times A^n.$$

We will now show that if $(x, x') \in S(\phi_1)$ and $(x, \nu_0), (x', \nu_0) \in \tilde{X^1}$, then $((x, \nu_0), (x', \nu_0)) \in S(\tilde{\phi_1})$, where $\tilde{X^1}$ is the unique minimal set in $X^1 \circ^Z \tilde{Z}$. (We let $J' = \{v \in J: \nu_0 v = \nu_0\}$ and note for $x \in X_0^1$, $(x, \nu_0) \in \tilde{X^1}$ iff $x \in X_0^1 J'$.) First suppose $x \in S_\lambda$, then there exist x_n in S_λ and t_n in T with $x_n \to x$, $x_n t_n \to x$, $x' t_n \to x$ by 1.5 of [6]. Now $S_\lambda \times \{\nu_0\} \subseteq \tilde{X^1}$ since $\lambda_{\theta(\nu)} \times \delta_\nu$, $\nu \in \tilde{Z}$ is a RIM on $\tilde{\phi_1}$ by 5 of [7]. So $(x_n, \nu_0) \in \tilde{X^1}$ and we have $((x, \nu_0), (x', \nu_0)) \in S(\tilde{\phi_1})$. Now suppose $x \notin S_\lambda$, let $x_1 \in S_\lambda$ and $w \in J^*$ such that $(x_1, \nu_0)w = (x_1, \nu_0)$. Let $pw \in M$ with $xpw = x_1$; then $(x'pw, xpw) \in S(\phi_1)$ and $((x'pw, \nu_0), (xpw, \nu_0)) \in S(\tilde{\phi_1})$. Multiplying on the right by $(pw)^{-1} \in Mw$, we get $((x'w, \nu_0), (xw, \nu_0)) \in S(\tilde{\phi_1})$ and therefore $((x', \nu_0), (x, \nu_0)) \in S(\tilde{\phi_1})$. Remark (a) is easily proved as above applying Lemma 2.7 to $\phi^*: oc(x_1 u, x_2 v) \to Z$. Remark (b) follows from 5 to [7] and 2.8.

2.11. THEOREM. *Suppose for* $i = 1, 2, \ldots, n$ *that* $\phi_i\colon X_i \to Z$ *has a* RIM, μ_i, *and* X_i *is minimal flow. Suppose* $z \in Z$ *such that the support* S_{μ_i} *of* μ_{iz} *equals the fiber* $\phi_i^{-1}(z)$, *for* $i = 1, \ldots, n$. *Suppose* X_0 *is a minimal flow and* $\phi_0\colon X_0 \to Z$ *is a homomorphism. Given* x *in* $\phi_0^{-1}(z) \subseteq X_0$ *and open sets* V_i *in* $\phi_i^{-1}(z) \subseteq X_i$, *the set* $N = oc(\{x\} \times V_1 \times \cdots \times V_n) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times \cdots \times Q(\phi_n)(V_n)v$ *for every* v *in* J, *and thus* $N \supseteq [S(\phi_0)(x)v] \circ v \times [Q(\phi_1)(V_1)v] \circ v \times \cdots \times [Q(\phi_n)(V_n)v] \circ v$.

Proof. Let $v \in J$ with $zv = z$. We will show $N \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times \cdots \times Q(\phi_n)(V_n)v$ by induction. But first some preliminaries. Let $(x_0, x_1, \ldots, x_n)$ be an element in the right hand side. Then we have $x_1 v = x_1$ and for some r_1 in V_1, $x_1 \in Q(\phi)(r_1)$. Suppose $v_1 \in J$ with $r_1 v_1 = r_1$; define $\alpha\colon M \to X_1$ by $\alpha(p) = r_1 p$, then $\phi_1 \circ \alpha = \psi\colon M \to Z$ where $\Psi(p) = zp$. So by 2.3 we see that there exists p_1 in $S(\psi)(v) = S(\psi)(v_1)$ such that $r_1 p = \alpha(p_1) = x_1$ and $p_1 v = p_1$ since $x_1 v = x_1$. By 2.5 $S(\phi_0)(x)p_1 = S(\phi_0)(x)v$ and since $x_0 \in S(\phi_0)(x)v$ there exists r_0 in $S(\phi_0)(x)$ with $r_0 p_1 = x_0$, and we may assume $r_0 v_1 = r_0$. Now (a) $N \supseteq S(\phi_0)(x) \times V_1 \times \cdots \times V_n$ by 1.4 of **[6]** and so $N \supseteq \{(r_0, r_1)\} \times V_2 \times \cdots \times V_n$. Now consider the flow $oc(r_0 v_1, r_1 v_1)$. It is minimal and has an induced map $\hat{\phi}\colon oc(r_0 v_1, r_1 v_1) \to Z$. Thus by 1.4 of **[6]** $N \supseteq S(\hat{\phi})(r_0 v_1, r_1 v_1) \times V_2 \times \cdots \times V_n$ which equals $S(\hat{\phi})(x_0, x_1) \times V_2 \times \cdots \times V_n$ since $(r_0 v_1, r_1 v_1)p_1 = (x_0, x_1)$ and $p_1 \in S(\psi)(v)$.

Now we note that when $n = 1$ we have for any $x_0 \in S(\phi_0)(x)v$ and $x_1 \in Q(\phi_1)(V_1)v$, (b) $oc(\{x\} \times V_1) \supseteq S(\tilde{\phi})(x_0, x_1) \ni (x_0, x_1)$ and so $oc(\{x\} \times V_1) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v$.

Proceeding by induction, assume that the theorem is true for $n = k - 1$ and prove it for $n = k$. With $n = k$, we have for any $x_0 \in S(\phi_0)(x)v$ and $x_1 \in Q(\phi_1)(V_1)v$, (c) $oc(\{x\} \times V_1 \times \cdots \times V_k) \supseteq oc(S(\phi)(x_0, x_1) \times V_2 \times \cdots \times V_k) \supseteq oc(\{x_0, x_1\}) \times V_2 \times \cdots \times V_k) \supseteq S(\tilde{\phi})(x_0, x_1)v \times Q(\phi_2)(V_2)v \times \cdots \times Q(\phi_k)(V_k)v \supseteq \{x_0\} \times \{x_1\} \times Q(\phi_2)(V_2)v \times \cdots \times Q(\phi_k)(V_k)v$ by induction. And so $oc(\{x\} \times V_1 \times \cdots \times V_k) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times Q(\phi_2)(V_2)v \times \cdots \times Q(\phi_k)(V_k)v$; thus the theorem is proved for every $v \in J$ with $zv = z$ and thus for every $v \in J$.

2.12. THEOREM. *Suppose for* $i = 0, 1, \ldots, n$ *that* $\phi_i\colon X^i \to Z$ *has a* RIM, μ_i, *and* X^i *is minimal. Suppose* $z \in Z$ *and* $X_0^i = \phi_i^{-1}(z)$. *Let* J^* *and* J' *be as in* 2.9. *Let* V^i *be open in* X_0^i, $A^i = V^i \cap X_0^i J^*$, *and* $x \in X_0^i$. *Then*

$$N = oc(\{x\} \times A^1 \times \cdots \times A^n)$$
$$\supseteq Q(\phi_0)(x)v \times Q(\phi_1)(A^1)v \times \cdots \times Q(\phi_n)(A^n)v$$

for every v *in* J.

Proof. We indicate where the proof differs from the above. Of course V_i is replaced by A^i and J by J^*. Statement (a) would read "Now $N \supseteq S(\phi_0)(x) \cap X_0^0 J^* \times A^1 \times \cdots \times A^n$ by Proposition 2.9." Note $S(\phi_0)(x) \cap X_0^0 J^* = S(\phi_0)(x) J^*$. Statement (b) would read "$oc(\{x\} \times A^1) \supseteq S(\tilde{\phi})(x_0, x_1) J^* \ni (x_0, x_1)$." Statement (c) would read

$$\text{"}oc\big(\{x\} \times A^1 \times \cdots \times A^n\big) \supseteq oc\big(S(\tilde{\phi})(x_0, x_1) J^* \times A^2 \times \cdots \times A^n\big).\text{"}$$

2.13. Corollary. *Suppose Γ is an index set and for $i \in \Gamma$, ϕ_i: $X_i \to Z$ has a* RIM *and X_i is minimal. Suppose $z \in Z$ and $x_i, y_i \in X_i$ with $x_i, y_i \in X_0^i J^* = \phi_i^{-1}(z) J^*$, $x = (x_i) \in \Pi_{i \in \Gamma} X_i$, $y = (y_i) \in \Pi_{i \in \Gamma} X_i$ where J^* is taken as in 2.9. Then $(x, y) \in Q(\Pi \phi_i)$ iff $(x_i, y_i) \in Q(\phi_i)$ for every i in Γ.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose $u \in J^*$. Let $\Pi_{i \in F} U_i \times \Pi_{i \notin F} X_i$ and $\Pi_{i \in F} V_i \times \Pi_{i \notin F} X_i$ be neighborhoods of x and y respectively, where F is a finite subset of Γ. Let $A_i = U_i \cap X_0^i J^*$ and $B_i = V_i \cap X_0^i J^*$. Then $N = oc(\Pi A_i \times \Pi B_i) \supseteq \Pi Q(A_i) u \times \Pi Q(B_i) u \supseteq \Pi Q(x_i) u \times \Pi Q(y_i) u \ni \Pi\{x_i u\} \times \Pi\{x_i u\}$; and the corollary clearly follows.

Remark. The above was known under various more specialized conditions, see [**1**, **3**].

2.14. Corollary. *Suppose ϕ: $X \to Z$ has a* RIM, λ, *let $z \in Z$, $x_0 \in X_0 J^* = \phi^{-1}(z) J^*$, Γ an index set, and $x_i \in S(\phi)(x_0) J^*$, $i \in \Gamma$. Then there exist nets t_n in T and x_i^n in $X_0 J^*$ with $x_i^n \underset{n}{\to} x_i$, $x_i^n t_n \underset{n}{\to} x_0$ for $i \in \Gamma$, and $x_0 t_n \to x_0$.*

Proof. Let $u \in J^*$ with $x_0 u = x_0$. For any neighborhood V_i of x_i in X_0 let $A_i = V_i \cap X_0 J^*$. Then

$$\begin{aligned} oc\big(\{x_0\} \times \Pi A_i\big) &\supseteq Q(\phi(x_0) u) \times (\Pi Q(\phi)(A_i) u) \\ &\supseteq Q(\phi)(x_0) \times \Pi Q(\phi)(x_i) u \ni \{x_0\} \times \Pi\{x_0\}; \end{aligned}$$

and the corollary clearly follows.

2.15. Lemma. *Suppose Z is a minimal flow and $z_0 \in Z$. Define ψ: $M \to Z$ by $p \to z_0 p$ and let $M_0 = \psi^{-1}(z_0)$. Suppose ψ is* RIC *and has a* RIM, λ. *Then there exist $w \in M_0 \cap J$ such that for p in* cls$(M_0 w)$ *and q in $Q(\psi)(p)$ there exist nets p_n in $M_0 w$ and t_n in T such that $p_n \to p$, $p_n t_n \to p$, $q t_n \to p$. In particular for p in* cls$(M_0 w)$, *$Q(\psi)(p) = \{q$: there exist nets p_n in $M_0 w$ and t_n in T with $p_n \to p$, $p_n t_n \to p$, $q t_n \to p\}$ $= \bigcap\{$*cls$(\beta T(p) \cap$ cls$(M_0 w))$: *$\beta = V \times V$, V an open set in $M\}$. (Recall that $S(\psi) = Q(\psi)$ if ψ in* RIC, *see* [**9**].)

Proof. Let S_λ be the support of λ_{z_0} and $p \in S_\lambda \subseteq M_0$. Suppose $u \in J$ with $pu = p$ and $q \in M_0 u$, then μ defined by $\mu_z(A) = \lambda_z(qp^{-1}A)$ is a RIM and $q \in S_\mu$. So if $S = \text{cls}(\cup\{S_\mu\colon \mu \text{ is a RIM for } \psi\})$, then $S = M_0 J_1$ for some subset J_1 of $J \cap M_0$. Now consider the left flow $(M_0 u, S)$ with the action being multiplication on the left and $M_0 u$ is a group given the discrete topology. Then it contains a minimal set $(M_0 u, \text{cls}(M_0 w))$ for some w in J_1.

Suppose V is an open subset of M_0 and $V \cap M_0 w \neq \varnothing$. Then there exists a finite set F of f's in $M_0 w$ such that $\cup_{f\in F} F(V \cap \overline{M_0 w}) \supseteq \overline{M_0 w}$. Let $B = B_V = \text{cls}(V \cap \overline{M_0 w}) = \text{cls}(V \cap M_0 w)$. Then $\cup_{f\in F} F(B \circ w) = [\cup_{f\in F} fB] \circ w \supseteq \overline{M_0 w} \circ w = M_0$ since ψ is RIC. So $\cup_{f\in F}(S \cap f(B \circ w)) \supseteq S$. So $\text{int}(S \cap f(B \circ w)) \neq \varnothing$ for some f in F where the interior is with respect to S, and thus $\text{int}(S \cap (B \circ w)) \neq \varnothing$. Let $p \in \overline{M_0 w}$ and $p^* \in \bigcap_{V\in\mathfrak{N}_p} \text{cls int}(S \cap (B_V \circ w))$.

Suppose $q^* \in Q(\psi)(p^*)$ and consider

$$N_V = oc(\{q^* w\} \times \text{int}[S \cap (B_V \circ w)])$$

then by Lemma 2.7

$$N_V \supseteq \{p^*\} \times \text{cls}(\text{int}[S \cap (B_V \circ w)]) \ni (p^*, p^*).$$

Let $U \in \mathfrak{N}_{p^*}$. Then there exist $t = t_{VU}$ in T and $r = r_{VU}$ in $S \cap (B_V \circ w)$ such that $q^* wt \in U$ and $rt \in U$. Then there exist $s = s_{VU}$ and $m = m_{VU}$ in $V \cap M_0 w$ such that $q^* s$ is near $q^* w$ and ms is near r; that is, $q^* s \in Ut^{-1}$ and $ms \in Ut^{-1}$. Thus we have nets m_{VU} in $M_0 w$ and $s_{VU} t_{VU}$ in T with $m_{VU} \to p$, $m_{VU} s_{VU} t_{VU} \to p^*$ and $q^* s_{VU} t_{VU} \to p^*$ thus $(q^*, p) \in Q(\psi)$.

So we have assumed $(p^*, q^*) \in Q(\psi)$ and shown $(q^*, p) \in Q(\psi)$. Now suppose $(p, q) \in Q(\psi)$; we can repeat the preceeding paragraph with q in place of q^* to obtain the lemma.

2.16. PROPOSITION. *Suppose $\phi\colon X \to Z$ is a homomorphism of minimal flows such that the set $D(\phi)$ of almost periodic points in $R(\phi)$ is dense. Let $x_0 \in X$, $\phi(x_0) = z_0$, and $X_0 = \phi^{-1}(z_0)$. Then there exists $w \in J$ with $z_0 w = z_0$ such that for x, y in $\text{cl}(X_0 w)$ with y in $Q(\phi)(x)$ and for $p \in \text{cls}(Mw)$ with $x_0 p = x$, there exist q in M and nets p_n in Mw and t_n in T such that $x_0 q = y$ and $p_n \to p$, $qt_n \to p$, $p_n t_n \to p$.*

Proof. Let $X_0 \in X_0 = \phi^{-1}(z_0)$. Define $\beta\colon M \to X$ by $\beta(p) = x_0 p$. Let $\psi = \phi \circ \beta\colon M \to Z$, $M_0 = \psi^{-1}(z_0)$. Take a proximal extension Z^* of Z, $\theta\colon Z^* \to Z$ such that $\psi^*\colon M^* \subseteq M \circ^Z Z^* \to Z^*$ is RIC and has a RIM. Let $z_0^* \in \theta^{-1}(z_0)$, $M_0^* = \psi^{*-1}(z_0^*)$, and let $w \in J \cap M_0^*$ as in Lemma 2.15. If $x \in \text{cls}(X_0 w)$ and $y \in Q(\phi)(x)$, then by 2.1.4 of **[6]**, $((x, z), (y, z)) \in Q(\phi^*)$ for some z in Z^*, and thus $((xw, z_0), (yw, z_0)) = ((xw, zw), (yw, zw)) \in Q(\phi^*)$. Since $x, y \in \text{cls}(X_0, w)$, (x, z_0), $(y, z_0) \in X^*$ and so

$((x, z_0), (xw, z_0)) \in P$, $((y, z_0), (yw, z_0)) \in P$ and $((x, z_0), (y, z_0)) \in Q(\phi^*)$. Let $p \in \operatorname{cls}(M_0 w)$ with $(x_0, z_0)p = (x, z_0)$. By 14.2 of [**2**$_b$] we can take q in $Q(\psi)(p)$ with $(x_0, z_0)q = (y, z_0)$. The proposition clearly follows from Lemma 2.15.

A stronger result can be obtained if we assume Z is a singleton. Fix x_0 in X and define $\psi: M \to X$ by $p \to x_0 p$. Let $u \in J$. Then Mu is a group. Give it the discrete topology and consider the (left) flow (Mu, M) with the action being multiplication on the left. Then it contains a minimal set $(Mu, \overline{Mw})$ for some w in $J \subseteq M$. Note $(Mw, \overline{Mw})$ is also minimal. See 2.10 of [**8**] for related results.

2.17. THEOREM. *Suppose X is a minimal flow and has an invariant measure. Let $w \in J$ such that $(Mw, \overline{Mw})$ is a minimal (left) flow as above. Let $x \in X$. Suppose $x_0\overline{Mw} \circ w = X$, (that is, X is incompressible). Then for each x in $\overline{Xw} = x_0\overline{Mw}$, p in $\psi^{-1}(x) \cap \overline{Mw}$ and x' in $Q(x)$, there exist nets m_n in Mw and t_n in T with $m_n \to p$, $x_0 m_n \to x_0 p = x$, $x' t_n \to x^*$, $x_0 m_n t_n \to x^*$ for any x^* in X. In particular, for x in $\overline{Xw}$,*

$$\begin{aligned} Q(x) &= \{x' : \text{there exist nets } x_n \text{ in } Xw \text{ and } t_n \text{ in } T \\ &\qquad\qquad \text{with } x_n \to x,\ x_n t_n \to x,\ x' t_n \to x\} \\ &= \bigcap \{\operatorname{cls}(\alpha T(x) \cap Xw) : \alpha = V \times V,\ V \text{ an open set in } X\} \\ &= \bigcap \{\operatorname{cls}(\alpha T(x) \cap \overline{Xw}) : \alpha = V \times V,\ V \text{ an open set in } X\}. \end{aligned}$$

Proof. Suppose $x \in x_0\overline{Mw}$, $p \in \psi^{-1}(x) \cap \overline{Mw}$ and $V \in \mathfrak{N}_p$. Then $V \cap \overline{Mw} \neq \varnothing$ and is open in $\overline{Mw}$. Then since $(Mw, \overline{Mw})$ is minimal, there exists a finite set F of f's in Mw such that $\bigcup_{f \in F} f(V \cap \overline{Mw}) \supseteq \overline{Mw}$. Let $B = B_v = V \cap \overline{Mw}$. Then $\bigcup_{f \in F} x_0 f B \circ w = x_0 \bigcup_{f \in F} fB \circ w = x_0[\bigcup_{f \in F} fB] \circ w \supseteq x_0\overline{Mw} \circ w = X$. So $\operatorname{int}(x_0 fB \circ w) \neq \varnothing$ for some f in F. Then $\operatorname{int}(B \circ w) \neq \varnothing$. Therefore $\operatorname{int}(x_0 B \circ w) \neq \varnothing$. Let $x^* \in \bigcap_{V \in \mathfrak{N}_p} \operatorname{cls} \operatorname{int}(x_0 B_V \circ w)$.

Suppose $x^\# \in Q(x^*)$ and consider $N_V = oc(\{x^\# w\} \times \operatorname{int}(x_0 B_v \circ w))$. Then by 1.4 of [**6**],

$$N_V \supseteq \{x^*\} \times \operatorname{cls}\operatorname{int}[x_0 B_v \circ w] \ni (x^*, x^*).$$

Let $U \in \mathfrak{N}_{x^*}$. Then there exists $t = t_{V,U}$ in T and $y = y_{V,U}$ in $X_0 B_V \circ w$ such that $x^\# wt \in U$ and $yt \in U$. Then there exists $s = s_{V,U}$ in T and $m = m_{V,U}$ in $V \cap Mw$ such that $x^\# s \in Ut^{-1}$ and $x_0 ms \in Ut^{-1}$. Thus we have nets $m_{V,U}$ in Mw and $s_{V,U} t_{V,U}$ in T with $m_{V,U} \to p$, $x_0 m_{V,U} \to x_0 p$, $x_0 m_{U,V} s_{V,U} t_{V,u} \to x^*$, $x^\# s_{V,U} t_{V,U} \to x^*$. Thus $(x^\#, x) = (x^\#, x_0 p) \in Q$.

So we have assumed $(x^*, x^\#) \in Q$ and shown $(x^\#, x) \in Q$. Now suppose $(x, x') \in Q$; then $(x^*, x') \in Q$ and we can repeat the preceding paragraph with x' in place of $x^\#$ to obtain the theorem (note the x^* can be replaced as the limit by any point in X since X is a minimal flow).

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