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SURJECTIVE EXTENSION OF THE REDUCTION OPERATOR

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In this paper it is shown that there exists a Riemann surface R and a nonnegative 2-form P on R such that the space of energy finite solutions of $d * du = uP$ on R is properly contained in the space of Dirichlet finite solutions yet the subspaces of bounded functions in these two spaces coincide.

Consider a nonnegative locally Hölder continuous 2-form P on a hyperbolic Riemann surface R. Let $PX(R)$ denote the space of solutions of $d * du = uP$ on R satisfying a certain boundedness property X, e.g. D (finite Dirichlet integral $\int_R du \wedge *du$), E (finite energy integral $\int_R du \wedge *du + u^2P$), B (finite supremum norm) or the combinations BD and BE. The reduction operator T_X is defined to be the linear injection of the space $PX(R)$ into the space $HX(R)$ such that for each $u \in PX(R)$ there is a potential p_u on R with $|u - T_X u| \leq p_u$. The unique existence of T_X for the cases $X = B, D, E$ was established in [5] together with the representations

$$T_X u(z) = u(z) + \frac{1}{2\pi}\int_R G_R(z, \zeta)u(\zeta)P(\zeta).$$

where $G_R(\cdot, \zeta)$ is the Green's function for T with pole at ζ.

One of the central questions concerning reduction operators is whether

(1) $\qquad T_{BX}$ is surjective implies that T_X is surjective,

$X = D, E$. Since $PBX(R)$ is dense in $PX(R)$ in the same fashion as $HBD(R)$ is dense in $HD(R)$ (cf. [1], [4]), it is natural to conjecture that the implication (1) holds. Surprisingly, in [12] and [7] it was shown that (1) is false for $X = D, E$. Even the stronger conditions $\int_R P < +\infty$, $\int_{R\times R} G_R(z, \zeta)P(z)P(\zeta) < +\infty$ do not imply the surjectiveness of T_E and T_D respectively as was shown in [8], [9], [10].

In this connection we raise the question whether the fact that (1) does not hold for $X = E$ by itself implies that (1) does not hold for $X = D$. This is closely related to the following: Is it true that $PBD(R) = PBE(R)$ implies that $PD(R) = PE(R)$? We shall show here that the answer to the latter question is no even under the stronger assumption that $PBD(R) = PBE(R) \cong HBD(R)$ which is a consequence of the surjectiveness of T_{BE}. Therefore the former question will also be settled in the negative.

1. We freely use the machinery of the Royden ideal boundary theory and the equation $d{*}du = uP$, as well as some of the techniques introduced in [2]. Let $W \subset R$ be an open set with ∂W consisting of analytic curves. The extremization operator μ_X^P, $X = D, E, BD, BE$, is the linear injection of the relative class $PX(W; \partial W)$ into $PX(R)$ characterized by the property that for each $u \in PX(W; \partial W)$ there is a potential p_u on R with $|u - \mu_X^P u| \leq p_u$. An alternate characterization of μ_X^P is the condition that $u - \mu_X^P u \,|\, \Delta = 0$ for each $u \in PX(W; \partial W)$. We also consider the reduction operator $T_{X,W}$: $PX(W) \cap \tilde{M}(R) \to HD(W)$, which can be characterized by the condition $u - T_{X,W}u \,|\, \partial W \cup bW = 0$, where $bW = (\overline{W} \cap \Delta) \setminus \overline{\partial W}$. In particular, $T_{X,W}$: $PX(W; \partial W) \to HD(W; \partial W)$ is determined by $u - T_{X,W}u \,|\, bW = 0$.

In [2] we completely characterized the functions in $HD(R)$ that are in the image of μ_D. In view of the following result, which we will make essential use of here, the corresponding problem for $PX(R)$ is also settled.

THEOREM. *Let u be in $PX(R)$ and set $h = T_X u$. Then $u \in \mu_X^P(PX(W; \partial W))$ if and only if $h \in \mu_D(HD(W; \partial W))$, $X = D$ or E.*

The necessity is simple. Let $s \in PX(W; \partial W)$ such that $\mu_X^P s = u$. Define $v = T_{X,W}s$. Since $v \,|\, \Delta = s \,|\, \Delta = u \,|\, \Delta = h \,|\, \Delta$, we conclude that $h = \mu_D v$. Conversely, assume that $h = \mu_D v$, $v \in HD(W; \partial W)$. It suffices to establish the sufficiency in case $v \geq 0$. We begin by showing that the assertion holds for $X = D$. For each positive integer k, set $\psi_k = (h \cap k) \cup k^{-1}$ and $v_k = \Pi_{\overline{R \setminus W}}(\psi_k - k^{-1})$, the harmonic projection of $\psi_k - k^{-1}$ on W. The sequence $\{v_k\}$ is easily seen to have the following properties (cf. [2]): $v_k \in HBD(W; \partial W)$, $v_k \leq v_{k+1} \leq v$, $v = CD\text{-}\lim v_k$, $\operatorname{Supp}(v_k \,|\, \Delta) \subset \{p \in \Delta \,|\, v(p) > 0\}$, $\lim(v_k \,|\, \Delta) = v \,|\, \Delta$ and

$$D_W(v_k) \leq D_W(v). \tag{3}$$

In view of $v \,|\, \Delta = h \,|\, \Delta = u \,|\, \Delta$ and $u \in PD(R)$ we have that $v \,|\, \Delta \setminus \Delta_P = 0$. Thus $\operatorname{Supp}(v_k \,|\, \Delta) \subset bW \cap \Delta_P$ and consequently there is a function $s_k \in PBD(W; \partial W)$ such that $s_k \,|\, \Delta = v_k \,|\, \Delta$. By the maximum principle, $s_k \leq s_{k+1} \leq v_{k+1} \leq v$ on R and then by the Harnack principle $s = C\text{-}\lim s_k$ exists on W. Since $v \,|\, R \setminus W = 0$, we actually have $s = C\text{-}\lim s_k$ on R with $s \,|\, R \setminus W = 0$, in particular, $s \in P(W; \partial W)$.

We show that in fact $s \in PD(W; \partial W)$. In view of the identity $D_W(u) = D_W(T_{D,W}u) + \langle u, u\rangle_W^P$ (cf. [5]), we have $\langle u, u\rangle_W^P < +\infty$. Comparing boundary values shows that $s_k \leq u$ on W. This together with (3) gives the following bound on $D_W(s_k)$:

$$D_W(s_k) = D_W(v_k) + \langle s_k, s_k\rangle_W^P \leq D_W(v) + \langle u, u\rangle_W^P < +\infty.$$

By Fatou's lemma we obtain $D_W(s) < +\infty$.

In view of $s \le u$, we have $s \mid \Delta \le u \mid \Delta$. On the other hand,

$$s \mid \Delta \ge \lim(s_k \mid \Delta) = \lim(v_k \mid \Delta) = v \mid \Delta = u \mid \Delta.$$

We conclude that $\mu_D^P s = u$, establishing the sufficiency in case $X = D$. If in addition $u \in PE(R)$, then $s \le u$ gives $\int_W s^2 P \le \int_W u^2 P < +\infty$, i.e. $s \in PE(W; \partial W)$, which completes the proof.

2. Let T be a hyperbolic Riemann surface such that $HBD(R)$ consists only of the constant functions. Examples of such surfaces are the Tôki surface and the Tôki covering surfaces (cf. **[11]**). The harmonic boundary of T consists of a single point which we denote by p^*. Fix $q_0 \in T$ and consider the polar coordinate differentials $(dr, d\theta)$ on T defined by

$$\frac{dr(z)}{r(z)} = -dG_T(z, q_0),\ d\theta(z) = -{*}dG_T(z, q_0).$$

Fix α such that $\overline{U}_\alpha$ is homeomorphic to a closed disk, where $U_\alpha = \{p \in T \mid r(p) < e^{-\alpha}\}$. Also fix $\beta \le \alpha/2$. For each $\lambda > 0$ define a rotation free 2-form $P_\lambda = \varphi_\lambda(r) r dr \wedge d\theta$, where φ_λ is the nonnegative Hölder continuous function on [0, 1) determined by the following conditions:

$$\varphi_\lambda(t) = \begin{cases} (1-t)^{-\lambda}, & \text{if } e^{-\beta} \le t < 1, \\ \text{linear}, & \text{if } e^{-2\beta} \le t < e^{-\beta}, \\ 0, & \text{if } 0 \le t < e^{-2\beta}. \end{cases}$$

According to the main results of [6] we have

$$P_\lambda BD(T) = \{0\} \text{ if and only if } \lambda \in [3/2, +\infty),$$

$$P_\lambda BE(T) = \{0\} \text{ if and only if } \lambda \in [1, +\infty).$$

By our choice of T here it follows that $\dim P_\lambda X(T) = \dim P_\lambda BX(T) \le 1$ for $X = D, E$. Furthermore, it can be seen from **[6]** that

$$\int_T G_T(q_0, \cdot) P_\lambda < +\infty, \text{ if } \lambda < 2, \tag{4}$$

$$\langle 1, 1\rangle_T^{P_\lambda} < +\infty, \text{ if } \lambda < \frac{3}{2}, \tag{5}$$

$$\int_T P_\lambda < +\infty, \text{ if } \lambda < 1. \tag{6}$$

Set $W_\alpha = T \setminus \overline{U}_\alpha$. Then $P_\lambda X(W_\alpha; \partial W_\alpha) = P_\lambda BX(W_\alpha; \partial W_\alpha)$ is isometric to $P_\lambda X(T) = P_\lambda BX(T)$, $X = D, E$. So for each $\lambda \in (0, 1]$ we may choose $w_\lambda \in P_\lambda D(W_\alpha; \partial W_\alpha)$ with $w_\lambda(p^*) = 1$. Then w_λ spans $P_\lambda D(W_\alpha; \partial W_\alpha)$. For

$\lambda \in (0, 1)$, $w_\lambda \in P_\lambda E(W_\alpha; \partial W_\alpha)$ whereas $\int_{W_\alpha} w_1^2 P_1 = +\infty$. Define

$$\Phi(\lambda) = E_T^{P_\lambda}(w_\lambda).$$

LEMMA. *Φ is a continuous function on* $(0, 1)$ *and* $\liminf_{\lambda \uparrow 1} \Phi(\lambda) = +\infty$.

Fix $\lambda_0 \in (0, 1)$. We first show that Φ is continuous from the right at λ_0. Let $\lambda_0 \leq \lambda < \nu < 1$. Since $P_\lambda \leq P_\nu$, we see that w_λ is a supersolution with respect to $d * du = uP_\nu$ and thus $w_\nu \leq w_\lambda \leq w_{\lambda_0} < 1$. The function $\hat{w} = \lim_{\lambda \downarrow \lambda_0} w_\lambda$ exists and $\hat{w} \leq w_{\lambda_0}$ on T. The Harnack inequality applied to (4) implies $\int_T G_T(\zeta, \cdot) P_1 < +\infty$, for any $\zeta \in W_\alpha$ and hence $\int_{W_\alpha} G_{W_\alpha}(\zeta, \cdot) P_1 < +\infty$. Set

$$\tau^{P_\lambda} \varphi(\zeta) = \frac{1}{2\pi} \int_{W_\alpha} G_{W_\alpha}(\zeta, z) \varphi(z) P_\lambda(z),$$

for a suitable function φ on W_α. Since $w_\lambda P_\lambda < P_1$ on W_α, the Lebesgue dominated convergence theorem gives $\lim_{\lambda \downarrow \lambda_0} \tau^{P_\lambda} w_\lambda = \tau^{P_{\lambda_0}} \hat{w}$. Note that for any λ, $T_{D, W_\alpha} w_\lambda$ is the same function $v \in HBD(W_\alpha; \partial W_\alpha)$. Therefore,

$$\begin{aligned} v &= w_{\lambda_0} + \tau^{P_{\lambda_0}} w_{\lambda_0} \geq \hat{w} + \tau^{P_{\lambda_0}} \hat{w} \\ &= \lim_{\lambda \uparrow \lambda_0} \left(w_\lambda + \tau^{P_\lambda} w_\lambda \right) = v \end{aligned}$$

on W_α. This implies that $w_{\lambda_0} = \hat{w}$ on W_α and consequently on T. By Dini's theorem we arrive at $w_{\lambda_0} = B\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$ on T.

We continue with $\lambda_0 \leq \lambda < \nu < 1$. Note that the function $w_\lambda - w_\nu$ is P_λ energy finite. Indeed, $w_\lambda - w_\nu$ is clearly Dirichlet finite and the inequality $0 \leq w_\lambda - w_\nu \leq w_\lambda$ gives $\int_T (w_\lambda - w_\nu)^2 P_\lambda \leq \int_T w_\lambda^2 P_\lambda < +\infty$. Since $w_\lambda - w_\nu$ vanishes at p^* we may choose a sequence $\{f_n\} \subset M_0(T)$ such that $w_\lambda - w_\nu = BE^{P_\lambda}\text{-}\lim f_n$. Moreover, the sequence $\{f_n\}$ may be chosen with $f_n \mid U_\alpha = 0$ since $w_\lambda - w_\nu$ has this property. Thus

$$E_T^{P_\lambda}(w_\lambda - w_\nu, w_\lambda) = \lim_n E_{W_\alpha}^{P_\lambda}(f_n, w_\lambda) = 0$$

and consequently

$$\begin{aligned} 0 \leq D_T(w_\nu - w_\lambda) &\leq E_T^{P_\lambda}(w_\nu - w_\lambda) \\ &= E_T^{P_\lambda}(w_\nu) - E_T^{P_\lambda}(w_\lambda) \leq \Phi(\nu) - \phi(\lambda). \end{aligned}$$

This shows that $\lim_{\lambda \downarrow \lambda_0} \Phi(\lambda)$ exists which in turn implies that $\{w_\lambda\}$ is D-Cauchy. By Kawamura's lemma we arrive at $w_{\lambda_0} = BD\text{-}\lim_{\lambda \downarrow \lambda_0} w_\lambda$, and in particular, $\lim_{\lambda \downarrow \lambda_0} D_T(w_\lambda) = D_T(w_{\lambda_0})$. By (6), $\int_T P_\nu < +\infty$ and we apply the Lebesgue dominated convergence theorem to obtain $\lim_{\lambda \downarrow \lambda_0} \int_T w_\lambda^2 P_\lambda = \int_T w_{\lambda_0}^2 P_{\lambda_0}$. This completes the proof of $\lim_{\lambda \downarrow \lambda_0} \Phi(\lambda) = \Phi(\lambda_0)$.

We now consider $\lambda_0 \in (0, 1]$ and show that $w_{\lambda_0} = BD\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$. Let $0 < \nu < \lambda \leq \lambda_0$ and note that $w_{\lambda_0} \leq w_\lambda < w_\nu < 1$. Thus $\lim_{\lambda \uparrow \lambda_0} w_\lambda$ exists and by an argument analogous to the one above we see that actually $w_{\lambda_0} = B\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$. Since $w_\nu - w_\lambda$ vanishes at p^* we can find a sequence $\{f_n\} \subset M_0(T)$ such that $w_\nu - w_\lambda = BD\text{-}\lim f_n$. We choose $\{f_n\}$ with the additional properties $f_n \geq 0, f_n \mid U_\alpha = 0$. Thus

$$D_T(w_\nu - w_\lambda, w_\nu) = \lim_n D_{W_\alpha}(f_n, w_\nu)$$
$$= -\lim_n \int_{W_\alpha} f_n d * dw_\nu \leq 0,$$

which implies that

$$0 \leq D_T(w_\lambda - w_\nu) \leq D_T(w_\lambda) - D_T(w_\nu).$$

Thus $D_T(w_\lambda)$ increases as λ increases and is bounded above by $D_T(w_{\lambda_0})$. Therefore $\{w_\lambda\}$ is D-Cauchy and by Kawamura's lemma $w_{\lambda_0} = BD\text{-}\lim_{\lambda \uparrow \lambda_0} w_\lambda$.

In case $\lambda_0 \in (0, 1)$, as before we see that $\lim_{\lambda \uparrow \lambda_0} \int_T w_\lambda^2 P_\lambda = \int_T w_{\lambda_0}^2 P_{\lambda_0}$. We arrive at $\lim_{\lambda \uparrow \lambda_0} \Phi(\lambda) = \Phi(\lambda_0)$ and the continuity of Φ at λ_0 is established. In case $\lambda_0 = 1$ we apply Fatou's lemma to conclude that $+\infty = \int_T w_1^2 P_1 \leq \liminf_{\lambda \uparrow 1} \int_T w_\lambda^2 P_\lambda \leq \liminf_{\lambda \uparrow 1} \Phi(\lambda)$.

3. Recall that the definition of P_λ involved a parameter β. We now adopt the notations $P_\lambda^{(\beta)}$, $w_\lambda^{(\beta)}$ to indicate the dependence of P_λ, w_λ on β. Set $a = D_{W_\alpha}(v)$, where v is the function in $HBD(W_\alpha; \partial W_\alpha)$ determined by $v(p^*) = 1$.

LEMMA. *Let b, c be given such that $a < b < c$. It is possible to choose $\beta \in (0, \alpha/2)$, $\lambda \in (0, 1)$ such that*

$$D_{W_\alpha}\left(w_\lambda^{(\beta)}\right) < b, \tag{7}$$

$$E_{W_\alpha}^{P_\lambda^{(\beta)}}\left(w_\lambda^{(\beta)}\right) = c. \tag{8}$$

Note that for $\beta \leq \beta'$ we have $P_\lambda^{(\beta)} \leq P_\lambda^{(\beta')}$ and that $\lim_{\beta \downarrow 0} P_\lambda^{(\beta)} = 0$. Thus in view of (4), (5) we have

$$\lim_{\beta \downarrow 0} \langle 1, 1 \rangle_{W_\alpha}^{P_1^{(\beta)}} = 0, \lim_{\beta \downarrow 0} \int_{W_\alpha} P_{1/2}^{(\beta)} = 0.$$

We therefore may choose β such that

$$\langle 1, 1 \rangle_{W_\alpha}^{P_1^{(\beta)}} < \frac{b - a}{2}$$

and

$$\int_{W_\alpha} P_{1/2}^{(\beta)} < \frac{b-a}{2}. \tag{9}$$

For any $\lambda \in (0,1]$ we have $T_{D,W_\alpha} w_\lambda^{(\beta)} = v$ and hence

$$\begin{aligned} D_{W_\alpha}\left(w_\lambda^{(\beta)}\right) &= D_{W_\lambda}(v) + \left\langle w_\lambda^{(\beta)}, w_\lambda^{(\beta)} \right\rangle_{W_\alpha}^{P^{(\beta)}} \\ &\leq a + \langle 1, 1\rangle_{W_\alpha}^{P^{(\beta)}} < \frac{a+b}{2}, \end{aligned}$$

which shows that (7) holds for this β and any $\lambda \in (0,1]$. By this and (9) we obtain

$$E_{W_\alpha}^{P_{1/2}^{(\beta)}}\left(w_{1/2}^{(\beta)}\right) = D_{W_\alpha}\left(w_{1/2}^{(\beta)}\right) + \int_{W_\alpha} \left(w_{1/2}^{(\beta)}\right)^2 P_{1/2}^{(\beta)} < \frac{a+b}{2} + \frac{b-a}{2} = b.$$

In view of Lemma 2 we can choose $\lambda \in (\frac{1}{2}, 1)$ so that (8) also holds.

4. We use the notation v_α to indicate the dependence of the function $v \in HBD(W_\alpha; \partial W_\alpha)$ with $v(p^*) = 1$ on α. We claim that

$$D_{W_\alpha}(v_\alpha) = \frac{2\pi}{\pi}. \tag{10}$$

In fact, $v_\alpha \mid W_\alpha = 1 - \alpha^{-1} G_T(\cdot, q_0) \mid W_\alpha$ and hence (10) follows from the formula $D_{W_\alpha}(G_T(\cdot, q_0)) = 2\pi\alpha^{-1}$ (cf. [**6**]). Define $\alpha_n = 4^{n+1}\pi$, $n = 1, 2, \ldots$. Then by (10) we have

$$D_{W_{\alpha_n}}(v_{\alpha_n}) = \frac{1}{2 \cdot 4^n}. \tag{11}$$

According to Lemma 3 we may choose λ_n, β_n such that

$$\delta_n = D_{W_{\alpha_n}}\left(w_{\lambda_n}^{(\beta_n)}\right) < \frac{1}{4^n} \tag{12}$$

and

$$\varepsilon_n = E_{W_{\alpha_n}}^{P_{\lambda_n}}\left(w_{\lambda_n}^{(\beta_n)}\right) = \frac{1}{2^n}, \tag{13}$$

for $n = 1, 2, \ldots$. Consider $W_{2\alpha_n} = \{p \in T \mid r(p) > e^{-2\alpha_n}\}$ and $v_{2\alpha_n} \in HBD(W_{2\alpha_n}; \partial W_{2\alpha_n})$ such that $v_{2\alpha_n}(p_n^*) = 1$. It can easily be seen that

$$v_{2\alpha_n} \mid \partial W_{\alpha_n} = \frac{1}{2}. \tag{14}$$

We prepare infinitely many copies T_n of T, $n = 1, 2, \ldots$ and view $W_{2\alpha_n}$ as being a subsurface of T_n. Let $V = \mathbf{C} \setminus \cup_{1 \leq n < \infty}\{|z - 3n| \leq 1\}$. We weld $W_{2\alpha_n}$ to V by identifying $\partial W_{2\alpha_n}$ with $\{|z - 3n| = 1\}$, $n = 1, 2, \ldots$ and let

R be the resulting Riemann surface. We now view W_{α_n}, $W_{2\alpha_n}$ as subsurfaces of R and denote them simply by W_n, U_n. We regard v_{α_n}, $v_{2\alpha_n}$ as being defined on W_n, U_n and denote them by v_n, u_n. Let Δ be the harmonic boundary of R. Since $\dim HBD(W_n; \partial W_n) = 1$, $\overline{W}_n \cap \Delta$ consists of a single point p_n^*. Set $\Delta_1 = \{p_1^*, p_2^*, \ldots\}$. The fact that $u_n \mid \partial W_n = \frac{1}{2}$, $n = 1, 2, \ldots$, i.e. (14), implies $\overline{\Delta}_1 = \Delta$ (cf. [3]). Let $W = \cup_{1 \leq n < \infty} W_n$ and define v on R by $v \mid W_n = v_n$, $n = 1, 2, \ldots$ and $v \mid R \setminus W = 0$. Then by (11) we see that $v \in HBD(W; \partial W)$. Since $v \mid \Delta_1 = 1$, we must have $v \mid \Delta = 1$ and consequently $\overline{W} \setminus \partial \overline{W}$ is a neighborhood of Δ in R^*.

Define a 2-form P on R by

$$P \mid W_n = P_{\lambda_n}^{(\beta_n)},\ n = 1, 2, \ldots \text{ and } P \mid R \setminus W = 0.$$

We view $w_{\lambda_n}^{(\beta_n)}$ as a function on W_n and use the simplified notation w_n for it. In this notation (12) and (13) are written as

$$\delta_n = D_{W_n}(w_n) < \frac{1}{4^n}, \tag{15}$$

$$\varepsilon_n = E_{W_n}^P(w_n) = \frac{1}{2^n}, \tag{16}$$

$n = 1, 2, \ldots$. For $X = D, E$ define measures m^{PX} on Δ by setting $m^{PX}(\Delta \setminus \Delta_1) = 0$ and

$$m^{PD}(p_n^*) = \delta_n,\ m^{PE}(p_n^*) = \varepsilon_n,$$

$n = 1, 2, \ldots$. We denote the bounded continuous functions on Δ by $B(\Delta)$.

LEMMA. *For $X = D$ or E*
(i) $PBX(W; \partial W) \mid \Delta = B(\Delta)$,
(ii) $PX(W; \partial W) \mid \Delta = L^2(\Delta, m^{PX})$.

Since (i) is an easy consequence of (ii) we proceed directly to the proof of (ii). We consider only the case $X = E$ as $X = D$ is analogous. Let $s \in PE(W; \partial W)$. Then $+\infty > E_W^P(s) = \sum_1^\infty E_{W_n}^P(s)$. Recall that $PE(W_n; \partial W_n)$ is spanned by w_n. Thus $s \mid W_n = a_n w_n$ with $a_n = s(p_n^*)$. We see by (16) that $E_{W_n}^P(s) = a_n^2 \varepsilon_n$ and hence $\{a_n\} \in L^2(\Delta, m^{PE})$. Conversely, if $\{a_n\} \in L^2(\Delta, m^{PE})$, then by (16) the function $s = \sum_1^\infty a_n w_n$ is in $PE(W; \partial W)$ with $s \mid \Delta = a_n$, m^{PE}-a.e.

5. We arrive at our main result.

THEOREM. *The 2-form P and the Riemann surface R have the property that*

$$PBE(R) = PBD(R) \text{ and } PE(R) \neq PD(R).$$

Since $\overline{W} \setminus \partial \overline{W}$ is a neighborhood of Δ, we see that μ_{BD} is surjective (cf. [**8**]). By Theorem 1 we see that μ^P_{BD} and μ^P_{BE} are surjective as well. From Lemma 4(i) we deduce that $PBD(W; \partial W) = PBE(W; \partial W)$. Thus the mapping $\mu^P_{BD} \circ (\mu^P_{BE})^{-1}$: $PBE(R) \to PBD(R)$ is a bijection and the first part of the assertion follows.

Let f be defined on Δ_1 by $f(p^*_n) = 2^{n/2}$, $n = 1, 2, \ldots$. By (15) and (16) we see that $f \in L^2(\Delta, m^{PD})$ but $f \notin L^2(\Delta, m^{PE})$. According to Lemma 4(ii) there is a function $s \in PD(W; \partial W)$ such that $s \mid \Delta = f$, m^{PD}-a.e. Set $u = \mu^P_D s \in PD(R)$ and $h = T_D u$. By Theorem 1 we have $h \in \mu_D(HD(W; \partial W))$. If u were in $PE(R)$, then in view of $h = T_E u$ Theorem 1 would imply that $u \in \mu^P_E(PE(W; \partial W))$. But since $u \mid \Delta \notin L^2(\Delta, m^{PE})$, Lemma 4(ii) rules out the possibility of u being in $\mu^P_E(PE(W; \partial W))$ and the assertion $u \notin PE(R)$ follows.

It is clear that there is a neighborhood V^* of Δ with $\int_{V^* \cap R} P < +\infty$ but we have not been able to determine whether $\int_R P < +\infty$. Thus the relation between $\int_R P < +\infty$ and $PE(R) = PD(R)$ remains open.

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