PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

SOME PROPERTIES OF THE CHARACTERISTIC OF CONVEXITY RELATING TO FIXED POINT THEORY

DAVID J. DOWNING AND BARRY TURETT

Fixed point theorems for uniformly lipschitzian mappings often restrict the characteristic of convexity, $\varepsilon_0(X)$, of the underlying Banach space to be less than one. This condition is discussed; in particular, it is shown that, for Banach spaces, $\varepsilon_0(x) < 1$ is equivalent to a condition imposed by E. A. Lifschitz in arbitrary metric spaces. The stability of this condition with respect to Banach-Mazur distance and Lebesgue-Bochner function spaces is also considered.

Let K be a nonempty, closed, bounded, convex subset of a Banach space X. A mapping T: $K \to K$ is said to be uniformly k-lipschitzian ($k \geq 1$) if, for each x, y in K and each natural number n, $\|T^n x - T^n y\| \leq k\|x - y\|$. Such mappings provide an intermediate class between the class of nonexpansive mappings and the class of lipschitzian mappings with Lipschitz constant greater than one. It is well-known (cf. **[9]**) that mappings in this latter class may fail to have fixed points even if the underlying space is Hilbert space and the Lipschitz constant is arbitrarily near one. However, fixed point theorems for uniformly lipschitzian mappings have been obtained by Goebel and Kirk [7], Goebel, Kirk, and Thele **[8]**, and Lifschitz **[11]**. (See also **[6]**.) In obtaining their results, two formally different geometric conditions are imposed on the space in question. In this paper, the relationship between the two geometric conditions is explored. It is shown that, in Banach spaces, the conditions are qualitatively, although not quantitatively, equivalent. In addition, the stability of these conditions is discussed; in particular, we show that these conditions lift from a Banach space X to the corresponding Lebesgue-Bochner function space $L^p(\mu, X)$ for $1 < p < \infty$ and μ an arbitrary measure.

Uniformly lipschitzian mappings were originally considered by Goebel and Kirk [7] and then by Goebel, Kirk, and Thele **[8]** in a more general semigroup setting. They discovered a relationship between the modulus of convexity of X and fixed points for uniformly lipschitzian mappings. Recall, for a normed linear space X, the modulus of convexity of X is the function δ_X: $[0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}.$$

The characteristic of convexity of X, $\varepsilon_0(X)$, is then defined to be $\sup\{\varepsilon \in [0,2]\colon \delta_X(\varepsilon) = 0\}$. It is well-known that X is uniformly rotund (respectively, uniformly non-square) if and only if $\varepsilon_0(X) = 0$ [**4**, p. 145] (respectively, $\varepsilon_0(X) < 2$ [**4**, p. 146]). The main idea of [7] and [**8**] may be stated as follows:

THEOREM 1 (*Goebel, Kirk, Thele*). *Let X be a Banach space with $\varepsilon_0(X) < 1$ and let $\gamma > 1$ satisfy $\gamma(1 - \delta_X(1/\gamma)) = 1$. If K is a nonempty, closed, bounded, convex subset of X and $T\colon K \to K$ is uniformly k-lipschitzian for $k < \gamma$, then T has a fixed point in K.*

In a subsequent development, Lifschitz [**11**] initiated a more topological approach and considered uniformly lipschitzian mappings in metric spaces. Instead of using the modulus of convexity, Lifschitz associated, with each metric space (M, ρ), a constant $\kappa(M)$ defined as follows:

$$\begin{aligned}\kappa(M) = \sup\{\beta > 0\colon \exists \alpha > 1 \text{ such that } \forall x, y \in M \text{ and } r > 0,\\ \rho(x, y) > r \Rightarrow \exists z \in M \text{ such that } B(x, \beta r) \cap B(y, \alpha r) \subset B(z, r)\},\end{aligned}$$

where $B(x, r)$ denotes the closed ball of radius r centered at x. It is immediate that $\kappa(M) \geq 1$ for any metric space (M, ρ). Lifschitz proved that if (M, ρ) is a bounded, complete metric space and if $T\colon M \to M$ is uniformly k-lipschitzian for $k < \kappa(M)$, then T has a fixed point in M. In order to compare this to the Goebel-Kirk-Thele resulting in the setting of Banach spaces, define $\kappa_0(X)$ to be the infimum of $\kappa(C)$ where C ranges over all nonempty, closed, bounded, convex subsets of the Banach space X. Then Lifschitz's theorem implies:

THEOREM 2 (*Lifschitz*). *Let X be a Banach space with $\kappa_0(X) > 1$. If K is a nonempty, closed, bounded, convex subset of X and $T\colon K \to K$ is uniformly k-lipschitzian for $k < \kappa_0(X)$, then T has a fixed point in K.*

Lifschitz proved that $\kappa_0(\mathcal{H}) \geq \sqrt{2}$ where $\mathcal{H}$ denotes Hilbert space; Goebel, Kirk, and Thele noted that $\gamma = \sqrt{5}/2$ is the solution to $\gamma(1 - \delta_{\mathcal{H}}(1 - \gamma)) = 1$. Thus, for Hilbert spaces, Lifschitz's approach yields a sharper result on how large the Lipschitz constant k may be taken and still guarantee the mappings have fixed points. It should be mentioned that J. Baillion (cf. [**8**]) has found an example of a fixed point-free uniformly $\pi/2$-lipschitzian self-mapping defined on a closed, bounded, convex subset of l^2. Our first result states that the approach of Lifschitz will always provide estimates on the size of k at least as good as those found using the approach of Goebel-Kirk-Thele.

THEOREM 3. *Let X be a Banach space and assume $\gamma > 1$ satisfies $\gamma(1 - \delta_X(1/\gamma)) = 1$. Then $\gamma \le \kappa_0(X)$.*

In order to facilitate the proof of Theorem 3, we state a lemma from **[11]**.

LEMMA 4 (*Lifschitz*). *Let X be a normed linear space, Then $\kappa_0(X) \ge \sup\{\beta > 0$: for some $\alpha > 1$ and all $y \in X$ with $\|y\| > 1$, there exists $t \in [0, 1]$ with $B(0, \beta) \cap B(y, \alpha) \subset B(ty, 1)\}$.*

Proof of Theorem 3. Let $y \in X$ with $\|y\| > 1$ and suppose $x \in B(0, \gamma) \cap B(y, \gamma)$. Then $\|x/\gamma\| \le 1$, $\|(x - y)/\gamma\| \le 1$ and $\|x/\gamma - (x - y)/\gamma\| > 1/\gamma$. Therefore, by the definition of δ_X, $\|1/2(x/\gamma + (x - y)/\gamma)\| \le 1 - \delta_X(1/\gamma)$. Thus, $\|x - y/2\| \le \gamma(1 - \delta_X(1/\gamma)) = 1$; i.e., $x \in B(y/2, 1)$. By Lemma 3, $\kappa_0(X) \ge \gamma$. This completes the proof of Theorem 4.

Although, in a quantitative sense, Lifschitz's result yields sharper estimates on the size of k than does the Goebel-Kirk-Thele result, the next theorem shows that, in the setting of Banach spaces, the results are qualitatively equivalent.

THEOREM 5. *Let X be a Banach space. Then $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.*

Proof. If $\varepsilon_0(X) < 1$, it is immediate that γ satisfying $\gamma(1 - \delta_X(1/\gamma)) = 1$ is greater than 1. So, by Theorem 4, $\kappa_0(X) \ge \gamma > 1$.

Now assume $\varepsilon_0(X) \ge 1$ and let $\beta > 1$ and $\alpha > 1$. Then there exist norm one elements x, y in X such that $\|x - y\| > 1/\gamma$ and $\|(x + y)/2\| > 1/\gamma$ where $\gamma = \min\{\alpha, \beta, 2\} > 1$. Consider $B(0, \beta) \cap B(\gamma(x - y), \alpha)$. Since $\|\gamma x\| = \gamma \le \beta$ and $\|\gamma x - \gamma(x - y)\| = \gamma\|y\| = \gamma \le \alpha$, $\gamma x \in B(0, \beta) \cap B(\gamma(x - y), \alpha)$. Similarly $-\gamma y \in B(0, \beta) \cap B(\gamma(x - y), \alpha)$. But $\|\gamma x - (-\gamma y)\| = \gamma\|x + y\| > 2$ and hence there does not exist z in X with $B(0, \beta) \cap B(\gamma(x - y), \alpha) \subset B(z, 1)$. Since 0 and $\gamma(x - y)$ are in $4B_X$, and $\beta > 1$ and $\alpha > 1$ are arbitrary, $\kappa(4B_X) = 1$; thus $\kappa_0(X) = 1$. This completes the proof of Theorem 5.

The conditions compared in Theorem 5 are, in some senses, stable. Recall that, for isomorphic Banach spaces X and Y, the Banach-Mazur distance coefficient from X to Y, denoted $d(X, Y)$, is defined to be the infimum of $\|U\|\,\|U^{-1}\|$, the infimum being taken over all invertible operators U from X onto Y.

THEOREM 6. *Let X be a Banach space with $\varepsilon_0(X) < 1$ and let $\gamma > 1$ satisfy $\gamma(1 - \delta_X(1/\gamma)) = 1$. If Y is a Banach space isomorphic to X and $d(X, Y) < \gamma$, then $\varepsilon_0(Y) < 1$.*

Proof. Without loss of generality, let U be an isomorphism from X onto Y such that $\|U^{-1}\| = 1$ and $d(X, Y) \leq \|U\| < \gamma$. Choose norm-one elements y_1 and y_2 in Y such that $\|y_1 - y_2\| \geq \|U\|/\gamma$ and define $x_1 = U^{-1}(y_1)$ and $x_2 = U^{-1}(y_2)$. It is immediate that $\|x_1\| \leq 1$, $\|x_2\| \leq 1$ and $\|x_1 - x_2\| \geq 1/\gamma$. Then, by the definition of δ_X, $\|(x_1 + x_2)/2\| \leq 1 - \delta_X(1/\gamma)$. Therefore

$$\begin{aligned} \|(y_1 + y_2)/2\| &\leq \|U\| \, \|(x_1 + x_2)/2\| \\ &\leq \|U\|(1 - \delta_X(1/\gamma)) < \gamma(1 - \delta_X(1/\gamma)) = 1. \end{aligned}$$

This implies that $\delta_Y(\|U\|/\gamma) \geq 1 - \|U\|(1 - \delta_X(1/\gamma)) > 0$. Thus $\varepsilon_0(Y) \leq \|U\|/\gamma < 1$ and the proof is complete.

Theorem 6 allows us to generalize some recent work of Bynum [**2**]. He has shown that if X is uniformly rotund, then there exists $\beta > 1$ so that if Y is a Banach space with $d(X, Y) < \beta$, closed, bounded, convex subsets of Y have the fixed point property for nonexpansive mappings. The next corollary follows immediately from Theorems 1 and 6.

COROLLARY 7. *Let X be a Banach space with $\varepsilon_0(X) < 1$. Then there exist constants $\gamma > 1$ and $\beta > 1$ such that if Y is a Banach space isomorphic to X and $d(X, Y) < \beta$, then closed, bounded, convex subsets of Y have the fixed point property for uniformly k-lipschitzian mappings with $k < \gamma$.*

Finally we show that these conditions are stable in a second sense.

THEOREM 8. *If X is a Banach space with $\varepsilon_0(X) < 1$, μ an arbitrary measure, and $1 < p < \infty$, then $\varepsilon_0(L^p(\mu, X)) < 1$.*

Theorem 8 follows immediately from Theorem 9 below. A discussion of Lebesgue-Bochner function spaces may be found in [**1**] or [**5**]. If μ is counting measure over some set, $L^p(\mu, X)$ is the Banach sequence space $l^p(X)$. The next theorem demonstrates the relationship between the characteristic of convexity of a Banach space X and the characteristic of convexity of the corresponding Lebesgue-Bochner function space. The proof is closely modelled on Day's proof [**3**] that $L^p(\mu, X)$ is uniformly rotund if and only if X is uniformly rotund and $1 < p < \infty$.

THEOREM 9. *Let X be a Banach space and μ a measure. Then $\varepsilon_0(L^p(\mu, X)) = \max\{\varepsilon_0(l^p), \varepsilon_0(X)\}$.*

Proof. Since both X and $L^p(\mu)$ are isometric to subspaces of $L^p(\mu, X)$, it is clear that

$$\varepsilon_0(L^p(\mu, X)) \geq \max\{\varepsilon_0(L^p(\mu)), \varepsilon_0(X)\} = \max\{\varepsilon_0(l^p), \varepsilon_0(X)\}.$$

It is then immediate that if $p = 1$, $p = \infty$, or $\varepsilon_0(X) = 2$, the desired equality is obtained with $\varepsilon_0(L^p(\mu, X)) = 2$. Thus, for the remainder of the proof, assume $1 < p < \infty$ and $\varepsilon_0(X) < 2$. Since $\varepsilon_0(l^p) = 0$ for $1 < p < \infty$, it suffices to demonstrate that $\varepsilon_0(L^p(\mu, X)) \leq \varepsilon_0(X)$.

Assume further that μ is counting measure on the set of natural numbers. Although this assumption appears quite restrictive, once the theorem is verified for counting measure, the theorem follows quickly for an arbitrary measure μ by defining an embedding of simple functions in $L^p(\mu, X)$ into the space $l^p(X)$ as done by Day [3, p. 507]. Thus it suffices to prove that $\varepsilon_0(l^p(X)) \leq \varepsilon_0(X)$ for $1 < p < \infty$ and $\varepsilon_0(X) < 2$.

Let $b = (x_i)$ and $b' = (x_i')$ be elements of $l^p(X)$ and let $0 < \eta \leq 2$. First, consider the case $\|b\| = \|b'\| = 1$, $\|b - b'\| \geq \varepsilon_0(X) + \mu$, and $\|x_i\| = \|x_i'\|$ for all $i \in \mathbf{N}$. For convenience, let $\|x_i\| = \beta_i$ and $\|x_i - x_i'\| = \gamma_i$. Then

$$\|b + b'\| = \left(\sum_i \|x_i + x_i'\|^p\right)^{1/p} \leq 2\left\{\sum_i \left[\left(1 - \delta_X\left(\frac{\gamma_i}{\beta_i}\right)\right)\beta_i\right]^p\right\}^{1/p}.$$

Note that $\gamma_i \leq 2\beta_i$ for each $i \in \mathbf{N}$. Define $E = \{i \in \mathbf{N}\colon \gamma_i/\beta_i > 2(\varepsilon_0(X) + \eta)/(2 + \eta)\}$ and $F = \mathbf{N} \setminus E$. Then

$$1 = \left(\sum_i \beta_i^p\right)^{1/p} \geq \left(\sum_{i \in F} \beta_i^p\right)^{1/p} \geq \frac{2 + \eta}{2(\varepsilon_0(X) + \eta)}\left(\sum_{i \in F} \gamma_i^p\right)^{1/p};$$

that is,

$$\left(\sum_{i \in F} \gamma_i^p\right)^{1/p} \leq \frac{2(\varepsilon_0(X) + \eta)}{2 + \eta}$$

or

$$\left(\sum_{i \in E} \gamma_i^p\right)^{1/p} = \left(\sum_i \gamma_i^p - \sum_{i \in F} \gamma_i^p\right)^{1/p} \geq (\varepsilon_0(X) + \eta)\left(1 - \left(\frac{2}{2 + \eta}\right)^p\right)^{1/p}.$$

Hence

$$\alpha = \left(\sum_{i \in E} \beta_i^p\right)^{1/p} \geq \frac{1}{2}\left(\sum_{i \in E} \gamma_i^p\right)^{1/p} \geq \frac{\eta}{2}\left(1 - \left(\frac{2}{2 + \eta}\right)^p\right)^{1/p}.$$

Thus,

$$\|b + b'\| \le 2\left\{\left(1 - \delta_X\left(\frac{2(\varepsilon_0(X) + \eta)}{2+\eta}\right)\right)^p \sum_{i\in E} \beta_i^p + \sum_{i\in F} \beta_i^p\right\}^{1/p}$$

$$= 2\left\{\left(1 - \delta_X\left(\frac{2(\varepsilon_0(X) + \eta)}{2+\eta}\right)\right)^p \alpha^p + 1 - \alpha^p\right\}^{1/p}$$

$$= 2\left\{1 - \left[1 - \left(1 - \delta_X\left(\frac{2(\varepsilon_0(X) + \eta)}{2+\eta}\right)\right)^p\right]\alpha^p\right\}^{1/p}$$

$$\le 2\left\{1 - \left[1 - \left(1 - \delta_X\left(\frac{2(\varepsilon_0(X) + \eta)}{2+\eta}\right)\right)^p\right]\left(\frac{\eta}{2}\right)^p\left(1 - \left(\frac{2}{2+\eta}\right)^p\right)\right\}^{1/p}.$$

Since $0 < \eta \le 2$ and $2(\varepsilon_0(X) + \eta)/(2 + \eta) > \varepsilon_0(X)$, if we set the right-hand side equal to $2(1 - \delta_0(\varepsilon_0(X) + \eta))$, then $\delta_0(\varepsilon_0(X) + \eta) > 0$ for all $\eta > 0$. Thus the first case is finished.

Now suppose $\|b\| = \|b'\| = 1$ and assume $\|b + b'\| > 2(1 - \delta_p(\xi))$ where δ_p is the modulus of convexity of l^p and

$$\xi = \min\{\eta/4, \tfrac{1}{2}\delta_0(\varepsilon_0(X) + \eta/2)\} > 0.$$

Then

$$2(1 - \delta_p(\xi)) \le \left(\sum_i \|x_i + x_i'\|^p\right)^{1/p} \le \left(\sum_i (\|x_i\| + \|x_1'\|)^p\right)^{1/p} \le 2.$$

Since $(\|x_i\|)$ and $(\|x_i'\|)$ are norm-one elements in l^p,

$$\left(\sum_i |\, \|x_i\| - \|x_i'\|\, |^p\right)^{1/p} < \xi.$$

Let $b'' = (x_i')$ where

$$x_i'' = \begin{cases} \dfrac{x_i'\|x_i\|}{\|x_i'\|} & \text{if } x_i' \ne 0 \\ x_i & \text{if } x_i' = 0. \end{cases}$$

Note $\|x_i\| = \|x_i''\|$ for all $i \in \mathbf{N}$ and $\|b' - b''\| = (\Sigma\,|\,\|x_i'\| - \|x_i\|\,|^p)^{1/p} < \xi < \eta/2$. Thus,

$$\begin{aligned}\|b + b''\| &\geq \|b + b'\| - \|b' - b''\| \geq 2\big(1 - \delta_p(\xi)\big) - \xi \\ &= 2\left(1 - \delta_p(\xi) - \frac{\xi}{2}\right) \geq 2\left(1 - \frac{\xi}{2} - \frac{\xi}{2}\right), \text{ since } \delta_p(\xi) \leq \frac{\xi}{2}, \\ &= 2(1 - \xi) > 2\left(1 - \delta_0\left(\varepsilon_0(X) + \frac{\eta}{2}\right)\right).\end{aligned}$$

It now follows from the first case that $\|b - b''\| \leq \varepsilon_0(X) + \eta/2$. Finally,

$$\|b - b'\| \leq \|b - b''\| + \|b'' - b'\| < \varepsilon_0(X) + \frac{\eta}{2} + \frac{\eta}{2} = \varepsilon_0(X) + \eta.$$

Therefore, if $\|b\| = \|b'\| = 1$ and $\|b - b'\| \geq \varepsilon_0(X) + \eta$, then $\|(b + b')/2\| \leq 1 - \delta_p(\xi)$. This implies that $\delta_{l^p(X)}(\varepsilon_0(X) + \eta) \geq \delta_p(\xi) > 0$. Thus $\varepsilon_0(l^p(X)) \leq \varepsilon_0(X) + \eta$. Since $\eta > 0$ was arbitrary, $\varepsilon_0(l^p(X)) \leq \varepsilon_0(X)$ and the proof of Theorem 9 is complete.

Theorems 1 and 8 combine to yield the following fixed point theorem.

COROLLARY 10. *Let X be a Banach space with $\varepsilon_0(X) < 1$, μ a measure, and $1 < p < \infty$. Then there exists a constant $\gamma > 1$ so that closed, bounded, convex subsets of the Lebesgue-Bochner function space $L^p(\mu, X)$ have the fixed-point property for uniformly k-lipschitzian mappings with $k < \gamma$.*

Since X is uniformly rotund if and only if $\varepsilon_0(X) = 0$, Day's result follows immediately from Theorem 9.

COROLLARY 11 (*Day* [3]). *Let (Ω, Σ, μ) be a measure space. The Lebesgue-Bochner function space $L^p(\mu, X)$ is uniformly rotund if and only if $1 < p < \infty$ and X is uniformly rotund.*

In a similar vein, Theorem 9 together with the characterization of uniformly non-square spaces in terms of the characteristic of convexity mentioned prior to Theorem 1 combine to prove the next corollary.

COROLLARY 12 (*Smith-Turett* [**12**, *p.* 116]). *Let (Ω, Σ, μ) be a measure space. The Lebesgue-Bochner function space $L^p(\mu, X)$ is uniformly non-square if and only if $1 < p < \infty$ and X is uniformly non-square.*

References

1. S. Bochner and A. E. Taylor, *Linear functionals on certain spaces of abstractly-valued functions*, Ann. of Math., (2) **39** (1938), 913–944.
2. W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math., **86** (1980), 427–436.
3. M. M. Day, *Some more uniformly convex spaces*, Bull. Amer. Math. Soc., **47** (1941), 504–507.
4. _____, *Normed Linear Spaces*, 3rd Ed., Ergebnisse Mathematik und ihrer Grenzgebiete, Band 21, Springer-Verlag, New York, 1973.
5. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R.I., 1977.
6. D. Downing and W. Ray, *Uniformly lipschitzian semigroups in Hilbert space*, Canad. Math. Bull., (to appear).
7. K. Goebel and W. A. Kirk, *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Math., **67** (1973), 135–140.
8. K. Goebel, W. A. Kirk and R. L. Thele, *Uniformly lipschitzian families of transformations in Banach spaces*, Canad. J. Math., **31** (1974), 1245–1256.
9. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.
10. _____, *Fixed point theory for nonexpansive mappings*, Summer Workshop on Fixed Point Theory, Université de Sherbrooke, June 1980.
11. E. A. Lifschitz, *Fixed point theorems for operators in strongly convex spaces*, Voronež Gos. Univ. Trudy Mat. Fak., **16** (1975), 23–28. (Russian)
12. M. A. Smith and B. Turett, *Rotundity in Lebesgue-Bochner function spaces*, Trans. Amer. Math. Soc., **257** (1980), 105–118.

Received April 20, 1981

OAKLAND UNIVERSITY
ROCHESTER, MI 48063