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PERMUTATIONS AND CUBIC GRAPHS

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In studying maximal nonparabolic subgroups of the modular group, B. H. Neumann and later C. Tretkoff were led to study pairs of permutations A and B of an infinite set Ω such that $A^2 = B^3 = 1$ and that $C = AB$ is transitive on Ω. Here we study such triples (Ω, A, B), but without the requirement that Ω be infinite. Our method is to associate with each such triple a graph $G^*(\Omega, A, B)$. Such graphs have been used before, especially by Stothers and by Cori.

Our central result here is that the graphs under consideration are precisely those that can be obtained by attaching trees, in certain simple specified ways, to finite or infinite graphs equipped with a reduced path that traverses each edge exactly once in each direction.

The classification or enumeration of all such cubic graphs appears to be a difficult question, to which we are able to contribute only a few remarks and examples. We also give a catalog of all triples of the kind described, for Ω of cardinality up to 12. In addition, we catalog all such *Eulerian paths* on cubic graphs that have no more than six vertices. As the cardinality of Ω increases, the complexity of the catalog increases rapidly.

In another paper [**13**] we show that several large classes of infinite planar graphs have Eulerian paths; these include the 1-skeletons of almost all regular tessellations of the plane (with Euclidean or hyperbolic metric). The results of the present paper are used in [**10**] to extend results of Neumann and Tretkoff. The central result here, which has been obtained earlier and independently by Stothers [**39**], is as follows, where we write ∞ for $\aleph_0$. We note that points (1) and (2) of this theorem had been observed by Tretkoff in the examples she constructed.

THEOREM. *It is well known that the modular group $M = PSL(2, Z)$ has a free factorization $M = Z_2 * Z_3$, and that if A is a generator for Z_2 and B is a generator for Z_3, then the subgroup P generated by $C = AB$ is a maximal parabolic subgroup of M. Let N be a complement of P in M, and let Ω be the family of right cosets of N in M. Then A and B act on N by right multiplication in such a way that $C = AB$ is transitive. Let $G^* = G^*(\Omega, A, B)$. Then* (i) *N is the free product of r_2 groups of order 2, r_3 groups of order 3, and r_∞ infinite cyclic groups*; $0 \leq r_2, r_3, r_\infty \leq \infty$. (ii) *$r_2$ is the number of fixed points of A, r_3 is the number of fixed points of B, and r_∞ is the Betti number of G^*.*

(iii) (1) $r_2 + r_3 + r_\infty = \infty$.

(2) *If r_∞ is finite, it is even.*

Conversely, if three numbers r_2, r_3, r_∞ *with* $0 \le r_2, r_3, r_\infty \le \infty$, *satisfy conditions* (1) *and* (2), *then they are realized by some complement* N *to* P *in* M.

In yet another paper **[12]** we have used the methods of the present article to discuss various similar problems. For example, we find all those permutations of a set that are the product of an involution and a transitive permutation. In other work **[14]** we have again used this method to reprove a theorem of G. A. Miller **[31]** that: given integers a, b, c $2 \le a, b \le c$, there exist permutations A, B of a set of cardinality at most $c + 2$, such that A, B, and $C = AB$ have respective periods a, b, and c. A variant of Miller's theorem is established in **[12]**, while in **[14]** we sharpen Miller's result to obtain in certain cases permutations of smallest degree $|\Omega|$.

2. Graphs. We consider directed graphs in which each edge has an inverse. Explicitly a graph G consists of a set V of vertices, a set E of edges, a function $\alpha\colon E \to V$ assigning to each edge its initial point, and a function $\eta\colon E \to E$ assigning to each edge e its inverse edge e^{-1}. It is required that $e^{-1} \neq e$ but $(e^{-1})^{-1} = e$. One says that the edge e runs from $\alpha(e)$ to $\alpha(e^{-1})$.

One may view each pair $\{e, e^{-1}\}$ of opposite edges as constituting an 'undirected edge' or *line* $|e|$, between $\alpha(e)$ and $\alpha(e^{-1})$, with e and e^{-1} its two directions or orientations. The set of vertices and lines then form a 1-complex $|G|$. We shall not always distinguish between G and $|G|$. For example, if we say that G is a tree, we really mean that $|G|$ is a tree. Similarly, the degree of a vertex v of G is really its degree in $|G|$, that is, the number of lines at v.

A path in G is *reduced* if it contains no consecutive pair of inverse edges. We call a tree *simply infinite* if, at any point, there is exactly one (reduced) infinite path beginning at that point. We call a simply infinite path, beginning at some point, a *ray* at that point.

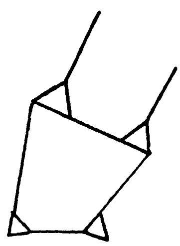
A *cubic* graph is one in which each vertex has degree 3. We call a graph *cuboid* if each vertex has degree at most 3. A *Eulerian path* in a cuboid graph (hence also in a cubic graph) is one that contains each edge exactly once, and which is reduced except at vertices of degree 1.

In speaking of triples (Ω, A, B) it is always understood that A and B are permutations of Ω such that $A^2 = B^3 = 1$. We call the triple *transitive* if $C = AB$ is transitive on Ω.

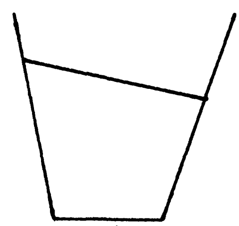
We associate a graph $G = G(\Omega, A, B)$ with each triple (Ω, A, B). The vertex set is Ω, and there is an edge e from p to q if and only if $p \neq q$ and q is one of pA, pB, pB^{-1}. We call e an A-edge, B-edge, or B^{-1}-edge accordingly. The inverse edge e^{-1} is then an A-edge, B^{-1}-edge, or B-edge. (If, for example, $p \neq q$, and both $q = pA$ and $q = pB$, we require both an A-edge and a B-edge from p to q. However, this cannot occur in the cases we consider, where $C = AB$ is transitive, since it would yield $qC = qAB = pB = q \neq p$.)

Evidently an A-edge e meets no A-edge except e and e^{-1}. Similarly, the B-edges fall into triples, forming oriented *triangles*, and any two distinct triangles of this type are disjoint. Conversely, it is easy to see that if the edges of a graph G are divided into three categories: A-edges, B-edges, and B^{-1}-edges, the inverses of which are respectively A^{-1}-edges, B^{-1}-edges, and B-edges; and if these disjointness conditions are met, then there is a unique triple (Ω, X, B) such that $G = G(\Omega, A, B)$.

Let $G^* = G/B$ be the quotient graph of G by B. Explicitly, the vertices of G^* are the B-orbits of Ω, and the edges of G^* are in one-to-one correspondence with the A-edges of G under the natural map from G onto G^*. (We identify each A-edge e with its image e^* in G^*.) Clearly G^* is a cuboid graph.



G



G^*

Let $W = (\ldots, p_{-1}, p_0, p_1, \ldots)$ be a C-orbit in Ω, where (as usual) if W has finite length n, the subscripts are to be taken modulo n. Since p_iA lies in the image p_{i+1}^* of $p_{i+1} = p_iAB$, the relation $p_iC = p_{i+1}$ determines an edge e_i^* from p_i^* to p_{i+1}^*, except that, in case $p_iA = p_i$, we have $p_i^* = p_{i+1}^*$. Thus W determines a path π in G^*, the successive edges of which are these e_i^*; π is either a closed path or a doubly infinite path. Since a given p_i occurs at most once in W, (the image of) a given A-edge e_i occurs at most once in π. If C is transitive on Ω, then every vertex occurs in W, and hence every edge of G^* occurs in π; in this case π is a Eulerian path on G^*.

We now suppose, conversely, that a connected cuboid graph H is given, together with a Eulerian path π on H, and we seek to construct a transitive triple (Ω, A, B) such that $H \simeq G^*$, in which π is the path described above. We begin by constructing an undirected graph $\tilde{H}$ as follows. We replace each vertex p of degree 2 or 3 in H, and some arbitrarily chosen set of vertices of degree 1 in H, by an unoriented triangle (p_1, p_2, p_3). Each line l of H, between vertices p and q, is replaced by a line $\tilde{l}$ between one of the p_i and one of the q_j in such a way that the various lines $\tilde{l}$ are disjoint. We next orient the triangles in $\tilde{H}$, that is, we choose one of the two cyclic orders (p_1, p_2, p_3) and (p_3, p_2, p_1) to be the orientations of the B-edges. Suppose first that p has degree 3 in H, and suppose the lines l_1, l_2, l_3 at p have covers $\tilde{l}_1$, $\tilde{l}_2$, $\tilde{l}_3$ in H that contain p_1, p_2, p_3. Let e_1, e_2, e_3 be the (directed) edges of H that enter p along l_1, l_2, l_3. If, Case 1, π contains all three of $e_i e_{i+1}^{-1}$ (subscripts mod 3), we choose the orientation (p_1, p_2, p_3). It is easy to see that in the contrary case, Case 2, π must contain all three of $e_i e_{i-1}^{-1}$, in which case we choose the other orientation (p_3, p_2, p_1). We orient the remaining triangles (which correspond to vertices of degrees 1 and 2 in H) arbitrarily.

The graph $\tilde{H}$, with the given orientations, now satisfies the conditions for it to be $\tilde{H} \simeq G = G(\Omega, A, B)$ for a certain triple (Ω, A, B), whence $H \simeq G^*$. Moreover, we have chosen the orientations in such a way that π is the image of a C-orbit W in Ω. Since π contains all the edges of H, W must contain all the initial points of A-edges in G. If C were not transitive, an orbit W_0 other than W would then contain only points fixed by A, hence would be a B-orbit, mapping into an isolated point of H. This contradicts the assumption that H is connected. We conclude that C is indeed transitive. We have proved the following.

(2.1) THEOREM. *Each transitive triple (Ω, A, B) determines uniquely a connected cuboid graph H together with a Eulerian path π on H. Conversely each pair consisting of a connected cuboid graph H with Eulerian path π arises thus from some transitive triple (Ω, A, B); however, this triple fails of uniqueness in two respects*:

(2.11) *a vertex of degree 1 in H may be the image of either a 1-cycle (fixed point) or of a 3-cycle of B,*

(2.12) *the orientation of any 3-cycle of B mapping to a vertex of degree 2 in H (that is, containing exactly one point fixed by A) may be chosen arbitrarily (that is, the cycle may be replaced by its inverse).*

If Ω contains no fixed point of either A or B, then G^* will be a cubic graph. Conversely, if H is a cubic graph, and π a Eulerian path on H, then the ambiguities of (2.11, 2.12) do not arise, and a transitive triple (Ω, A, B) is determined uniquely; in this triple clearly neither A nor B has a fixed point.

(2.13) COROLLARY. *The transitive triples (Ω, A, B) in which neither A nor B has a fixed point correspond one-to-one with the Eulerian paths of cubic graphs.*

3. Reduction. We shall see that every cuboid graph with Eulerian path can be obtained in a straightforward way from a cubic graph with Eulerian path.

(3.1) LEMMA. *Let a cuboid graph H be obtained from the disjoint union of graphs H_1 and H_2 by first subdividing the lines l_1 and l_2 of H_1 and H_2 by vertices v_1 and v_2, to obtain graphs H_1', H_2' and then adding a line l between v_1 and v_2. Let e be the (directed) edge along l from v_1 to v_2. Let π be a Eulerian path on H. Then after possible interchange of H_1 and H_2, the following holds.*

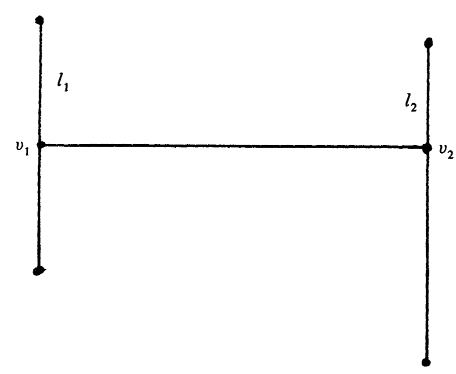


FIGURE 3.10

(3.11) *$\pi = \pi_1 e \pi_2 e^{-1} \pi_1'$ where π_1, π_1' are in H_1 and π_2 in H_2, and (apart from subdivision of edges) $\pi_1 \pi_1'$ is a Eulerian path on H_1 and π_2 is an Eulerian path on H_2;*

(3.12) *H_2 is finite.*

Conversely,

(3.13) *if $\pi_1\pi_1'$ is an Eulerian path for H_1' where π_1 ends at v_1, and π_2 is an Eulerian path on H_2' beginning and ending at v_2, then π, as given in* (3.11), *is an Eulerian path on H.*

These assertions are all clear from an inspection of the graphs, if one notes that, since π_2 is finite and contains all edges of H_2, H_2 must be finite.

(3.2) COROLLARY. *If an infinite tree H has an Eulerian path, then H must be simply infinite.*

The same argument shows that if H is an infinite graph with an Eulerian path and if deletion of some finite set of edges separates H into components $H_1, \ldots, H_k$, then all but one of the H_i must be finite. In particular, if H is a locally finite planar graph with an Eulerian path, then the complement of H in the plane can have at most one infinite component.

Now suppose that H, H_1, H_2, and l are as above, and that H_2 is a tree. Then $T = H_2 \cup l$ is a tree, and we say H is obtained from H_1 by *attaching* the tree T at the point v_1, or that H_1 is obtained from H by *deleting* the tree T. We call T a *branch* of H. It is convenient to incorporate here a degenerate case: if H consist merely of H_1 with the line l_1 subdivided by the vertex v_1, we regard H as obtained from H_1 by attaching the trivial branch consisting of the point v_1 alone, and H_1 as obtained from H by deleting this trivial branch.

From the above it is clear that if H has an Eulerian path and H_1 is obtained from H by deleting a finite branch, then H_1 has an Eulerian path. The same is true if H_1 is obtained by deleting a possibly infinite number of finite branches, attached at different points. We seek to delete all finite branches.

It is easy to see that if H contains an infinite ascending chain of finite trees, then H is in fact the union of this chain, and is an infinite tree. If H has an Eulerian path, then H is a simply infinite tree. Putting aside this case, every finite branch in H is contained in a maximal finite branch, and, after deleting all maximal finite branches, we arrive at a graph H_1 that contains no finite branch. If H has finite branches not contained in any maximal finite branch, then H is a simply infinite tree, and, by deletion of finite branches, H can be reduced to a ray H.

It may be that H_1 contains an infinite branch. By (3.12) we see that H_1 can contain at most one maximal infinite branch, which must then be simply infinite, and that, if it does, H_1 is obtained by attaching this infinite branch to a finite subgraph H_0, which will have an Eulerian path. In this latter case, we say that H_0 is obtained from h_1 by deleting a simply infinite branch.

In all cases, by deleting branches we can pass from H to a subgraph H_0 that contains no branches, and which possesses an Eulerian path. In the corresponding triple (Ω_0, A_0, B_0), there can be no fixed point for B_0, else there would be a vertex of degree 1 in H_0. Suppose that H_0 is not a single point; then no B_0-orbit can contain three fixed points of A_0. Since H_0 contains no vertex of degree 1, no B_0-orbit can contain two fixed points of A_0. Finally, if some B_0-orbit contained a single fixed point of A_0, it would give a vertex of degree 2 in H_0; this vertex can be removed by detaching a trivial branch. We conclude that either H_0 is a single point, or that A_0 and B_0 are without fixed points and H_0 is a cubic graph. Finally, if a single point H_0 is left after detaching branches from a larger graph H, then H must in fact be a tree.

We have proved the following.

(3.3) THEOREM. *Every cuboid graph H with Eulerian path is either a finite or simply infinite tree, or is obtained, from a cubic graph H_0 with Eulerian path, by attaching finite trees and, in case H_0 is finite, possibly one additional simply infinite tree.*

4. Appendix A. *Enumeration of the permutations A, B of a finite set such that $A^2 = B^3 = 1$ and AB is transitive.*

In this appendix we give some lemmas concerning the enumeration problem in the title. We raise some questions that we cannot solve. Especially, we cannot give a reasonable algorithm for the counting problem. In the last part of this Appendix (Section 5), we do enumerate all solutions of the conditions

$$(4.01) \qquad A^2 = 1, \quad B^3 = 1, \quad C = AB \text{ transitive on } \Omega,$$

for Ω of cardinality $c = |\Omega|$ no greater than 12.

If (A, B) is a solution of (4.01) and $P \in \operatorname{Sym}\Omega$, then $(A', B') = (A^P, B^P)$, where $X^P = P^{-1}XP$, is also a solution, *conjugate* to (A, B). If

C_0 is any cycle transitive on Ω, for example, $C_0 = (1, 2, \ldots, c)$, then clearly every solution for (4.01) is conjugate to a solution of

$$(4.011) \qquad A^2 = 1, \quad B^3 = 1, \quad \textit{and} \quad AB = C_0.$$

Since $\operatorname{Sym}\Omega$ contains exactly $(c-1)!$ cycles C of length c, the number of solutions of (4.01) is $(c-1)!$ times the number of solutions of (4.011).

If (A, B) and (A^P, B^P) are conjugate solutions of (4.011) then $C_0^P = C_0$, that is, P commutes with C_0. Moreover, $(A, B) = (A^P, B^P)$ if and only if P commutes with A and B as well. In this case P effects an automorphism of the graph H, and every automorphism of H is of this kind. Now P commutes with C_0 if and only if P is a power of C_0, whence the group Aut H of all automorphisms of H is a subgroup of the cyclic group, of order c, generated by C_0. If Aut H has order s, then s divides $c = rs$, where C_0^r generates Aut H, and r is the number of solutions of (4.011) that are conjugate to (A, B).

We find that the values $s = 1, 2, 3, 6$ occur as the number of symmetries of finite cubic graphs H with Eulerian paths. We conjectured that no other values could occur, and this has been proved by Bianchi and Cori [2], indeed with the orders 2 and 3 replaced by any pair of distinct primes. Obvious examples (see below) exhibit the values $s = 1, 2, 3$. The case $s = 6$ is illustrated in (4.05) below, with $c = 6$. The value $s = 6$ occurs for infinitely many graphs H; the construction of such graphs is illustrated by Figure 4.051, in which A and B are without fixed points, and $c = 2.3.13 = 78$.

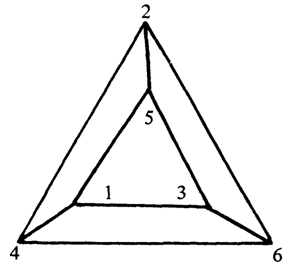


FIGURE 4.05

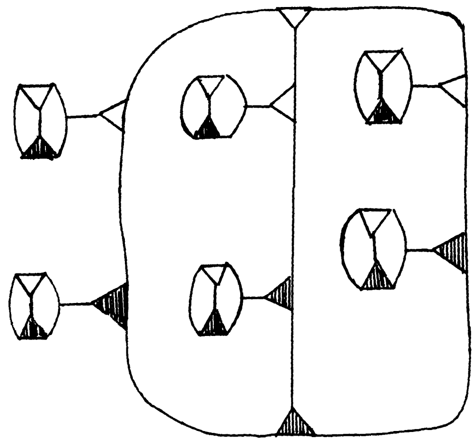


FIGURE 4.051

Unshaded triangles are oriented $+$; shaded $-$.

In this Appendix, the Cauchy-Jordan lemma is used repeatedly. This lemma states that if P, Q are cycles, then the transposition of an element from P with an element from Q will link them into a single cycle.

In the notation of the earlier sections, we are addressing the problem of enumerating all pairs (A, B) of permutations of a finite set Ω such that $A^2 = 1$, $B^3 = 1$, and such that $C = AB$ is transitive on Ω. If Ω has cardinal $|\Omega| = c$, we may suppose that $\Omega = \{1,2,\ldots,c\}$ and that $C = (1,2,\ldots,c)$. To avoid trivialities, we assume that $c \geq 3$.

Two solutions (A, B) and (A', B') are *conjugate*, or lie in the same *conjugacy class*, if $A' = P^{-1}AP$ and $B' = P^{-1}BP$ for some permutation P of Ω. Thus P must be a power of C, and all conjugates of a solution (A, B) are obtained by repeated conjugation by C. This implies the following.

(4.03) *The number of solutions in any conjugacy class is a divisor of c.*

We next examine the possible number r of solutions in a conjugacy class.

(4.04) *A conjugacy class consists of a single solution only in the case $c = 6$ and $A = (14)(25)(36)$, $B = (135)(246)$.*

Proof. We have assumed that $C = AB = (1, 2, \ldots, c)$. Suppose that (A, B) is the sole member of its conjugacy class. Then conjugation by C leaves (A, B) unchanged, that is, C commutes with A and B. Since $C = AB$, this implies that A and B commute. Since $c \geq 3$, the permutation $A \neq 1$ must exchange two elements p and p' of Ω. Now B cannot fix both p and p'. Therefore some element p belongs both to a non-trivial A-orbit (p, p') and to a non-trivial B-orbit (p, q, r). Since A and B commute, $(pA, qA, rA) = (p', q', r')$ is also a B-orbit, whence it follows that (q, q') and (r, r') are distinct A-orbits. But now the set $\{p, q, r, p', q', r'\}$ is invariant under both A and B, hence must be all of Ω. We find that $C = (p, q', r, p', q, r')$ which, up to notation, gives the conclusion. $\square$

The associated graph $G = G(A, B)$ is the 'prismatic' graph shown in Figure 4.05.

(4.06) *If c is a prime, then every conjugacy class contains exactly c solutions.*

We next observe that r is determined by the order of the automorphism group of the graph $G = G(A, B)$.

(4.07) *The automorphism group of $G(A, B)$ is cyclic, of order s, where $c = sr$, and is generated by $S = C^r$. Thus $r = c$ if and only if G admits no non-trivial automorphism. In particular, if c is a prime, then G admits no nontrivial automorphism.*

(4.08) *An automorphism P of H fixes a vertex, a link, or a triangle only if $P = 1$, $P^2 = 1$, or $P^3 = 1$, respectively.*

Proof. Since P is a power of C, a transitive cycle on the vertices, P can fix an element only if $P = 1$. If P fixes a link, then P^2 must fix the end points of the link, whence $P^2 = 1$. Similarly, if P fixes a triangle, then P^3 fixes the vertices of the triangle, whence $P^3 = 1$.

We call a graph G *strongly planar* if it can be embedded in the plane in such a way that all non-trivial B-orbits have counterclockwise orientation. Since C is transitive, G must then be simply connected, and the graph $H = G/B$ must be a tree. (For example, a necklace of triangles does not correspond to a transitive permutation C, since C will have one orbit on the outside of the necklace and another on the inside.) Conversely, if $H = G/B$, as abstract graph, is a tree, then H can be embedded in the plane, and, by arranging the edges at each vertex of H in the proper cyclic order, we derive a strong embedding of G in the plane. We have shown that

(4.10) *G is strongly planar if and only if $H = G/B$ is a tree.*

Every automorphism of a finite tree has either a fixed edge or a fixed vertex. If T is a non-trivial automorphism of $H = G/B$, a tree, then either T fixes an edge, interchanging its end points, or T fixes a vertex. By virtue of (4.08), T has order 2 or 3. This establishes

(4.11) *If G is strongly planar, then $s = 1, 2,$ or 3; thus the number of solutions conjugate to (A, B) is c, $c/2$, or $c/3$.*

It is easy to see that if $s = 2$, then G can be embedded in the plane in such a way that the symmetry S is effected by a rotation through π about the midpoint of an edge; if $s = 3$, G can be embedded in such a way that S is effected by a rotation through $2\pi/3$ about the center of a triangle.

REMARK. The 'prismatic' graph of Figure 4.05 is planar, as Figure 4.12 shows, but it is not strongly planar.

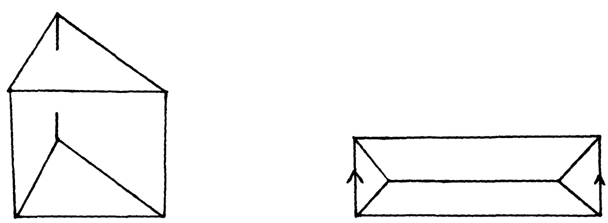


FIGURE 4.12

This graph has $s = 6$, as we have seen, but it is exceptional in that it is the only graph with $s = c$. We note that there exist planar cubic graphs with arbitrarily large symmetry groups. For example, the graph of Figure 4.13, where there are n radial lines, evidently admits (even as orientation preserving maps, if we orient the triangles properly) the dihedral group of order $2n$ as symmetry group. But this graph does not have any Eulerian trail. We are left with the following question.

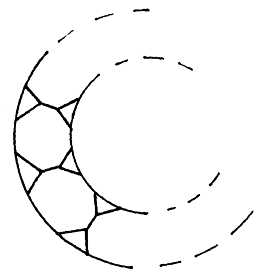


FIGURE 4.13

(4.14) PROBLEM. *With the exception of the 'prismatic graph' does there exist any graph $G = G(A, B)$ with symmetry group of order $s > 3$? Equivalently, does there exist any pair consisting of a cubic graph H on more than 2 vertices and an Eulerian trail τ on H, such that the automorphism group of the pair (H, τ) has order $s > 3$?* (Solved in Fig. 4.051.)

(4.15) THEOREM. *For every value of c, $c \geq 12$, if 2 divides c there is a class of solutions for which $s = 2$, and if 3 divides c there is a class for which $s = 3$. In these cases, $r = c/s < c$.*

Proof. If $c \equiv 0 \pmod 6$, we obtain $s = 2$ with a chain of triangles as shown in Figure 4.16.

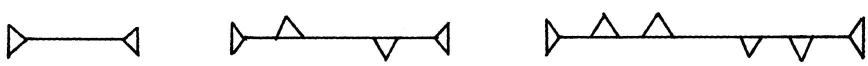


FIGURE 4.16

If $c \equiv 2, 4 \pmod 6$, we modify these figures by attaching spurs. For $s = 3$ we construct graphs with rotational symmetry of order 3, as shown in Figure 4.17, again possibly attaching spurs.

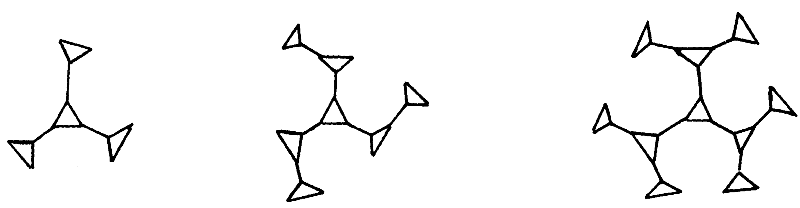


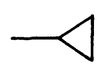
FIGURE 4.17

We turn now to a catalog of all solutions $G = G(A, B)$ of (5.01) for small values of c. We exhibit a representative for each conjugacy class, and also indicate the value of s, that is, the size $r = c/s$ of that conjugacy class. The classes are listed according to the number of nontrivial orbits in B, and, for two classes that agree in this attribute, precedence is given to the classes with the fewest nontrivial orbits in A. (For the graphs, this means, first, according to the number of triangles, and second, according to the number of *spurs*, that is, links attached to only a single triangle.) For each value of c, we explain why the list is exhaustive.

We have carried our catalog only as far as $c = 12$. Already at $c = 6$ we encounter the prismatic graph, which is not strongly planar. However, we encounter no graph that is not planar, in the usual sense, for the reason that the first such graph occurs for $c = 18$. Such a graph G is shown in Figure 5.131; note that the associated quotient graph $H = G/B$ is cubic.

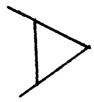
5. Appendix A, continued. Catalog of classes for $1 \leq c \leq 12$. In this section each class, $1 \leq c \leq 12$ is represented by a graph. All these graphs are planar. Nonplanar graphs first arise for $c > 12$, but for $c = 6$, 9, 10, 11, 12 some of the graphs are not strongly planar. The size s of the automorphism group of the graph is always indicated ($s = c/r$ is tabulated). Thus the number of solutions of $A^2 = 1$, $B^3 = 1$, $AB = (1\ 2 \ldots c)$ is obtainable. For example when $c = 12$, the number of solutions is $\Sigma\, r_i = \Sigma\, c/s_i = 232$, and the number of solutions of $A^2 = 1$, $AB =$ some 12-cycle is 232 (11!).

When the orientation of a triangle is not indicated, orientation is counterclockwise; shaded triangles are oriented clockwise.

	c = 1	One class; trivial.		$s = 1$
	c = 2	One class		$s = 2$
	c = 3	One class		$s = 3$
(5.041)	**c = 4**	One class		$s = 1.$

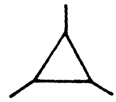
If the triangle is oriented clockwise, turn it over.
Then rotate the entire figure to obtain 5.041.

c $= 5$ One class

(5.051)  $s = 1$.

The argument is the same.

c $= 6$ Three classes. The first class has a single triangle.

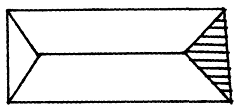


(5.061) $s = 3$

The automorphisms result from plane rotations. The second class has two triangles joined by a single link.

(5.062)  $s = 2$

If either triangle is oriented clockwise, twist it and turn it. The automorphisms are plane rotations.

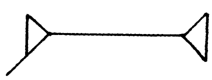


(5.063) $s = 6$.

The graph is not strongly planar. Transitivity requires that one triangle be oriented clockwise, and the other counterclockwise. After a plane rotation, we may suppose that it is the left triangle that is oriented clockwise.

In all the remaining examples, $7 \leq c \leq 12$, all automorphisms are also plane rotations.

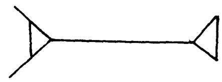
c $= 7$ Two classes. B has two nontrivial orbits, A has two nontrivial orbits. The graphs show that A cannot have four nontrivial orbits.

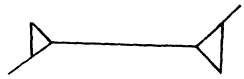
(5.071)  $s = 1$.

(5.072)  $s = 1$.

Any triangle oriented clockwise can be twisted. Then the entire figure can be rotated in the plane so that the triangle to which the first spur is attached is the left triangle.

c = 8 Three classes. Two triangles, two spurs.

(5.081)  $s = 1.$

(5.082)  $s = 2.$

(5.083)  $s = 1.$

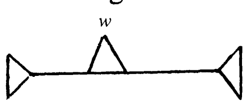
Any other graph with two triangles can be rotated or twisted to have one of these three shapes.

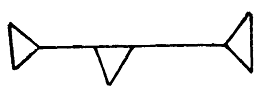
c = 9 Four Classes.

(5.091)  $s = 1.$

(5.092)  $s = 1.$

Any graph with two triangles can be rotated and twisted into one of 5.091, 5.092. But if the right triangle is twisted, its orientation is reversed.

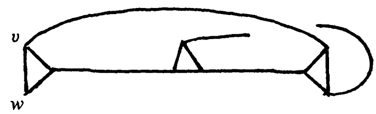
(5.093) 

or  $s = 1.$

These two pictures represent the same class, since one can be rotated into the other.

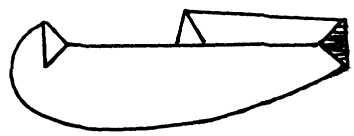
Suppose now that there are four links (rather than two). By Lemma 3.09, if $D = (\alpha\beta)(\gamma\delta)C$ is a 9-cycle, then so also is $(\alpha\beta)(\gamma\delta)D = C$. Thus two of the four links (and the triangles) are disposed as in 5.093.

Suppose w to be the vertex to which none of the four links is attached. Every vertex except w has degree 3. Thus every triangle is connected to every other triangle, and the graph may be redrawn so that w is on the left-hand triangle (with the two extra links not yet attached). It is now clear that v, the vertex before x on the left-hand triangle, must be attached as in 5.094; the other possible attachments result in an ineligibile (intransitive) graph.

(5.094) 

$s = 1.$

We remark that 5.094 is planar (in the usual sense); the embedding is obtained by twisting the right-hand triangle:



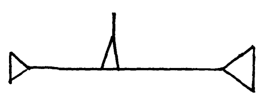
c = 10 Seven classes. If there are only two triangles, there must be four spurs ending at four additional vertices:

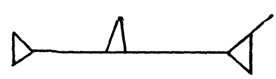
(5.101)  $s = 1.$

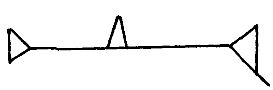
In every other case, there are three triangles and one spur. This spur has a unique end point, so that $s = 1$ for these cases. Deletion of the spur contracts the graph to 5.093 or 5.094. From 5.093, five classes arise. They are

(5.102)  $s = 1.$

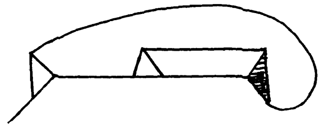
(5.103)  $s = 1.$

(5.104)  $s = 1.$

(5.105)  $s = 1.$

(5.106)  $s = 1.$

From 5.094, only one class arises, since there is only one place to attach the spur.

(5.107)  $s = 1.$

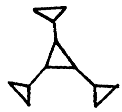
c = 11 Ten classes. There must be three nontrivial B-orbits, hence three triangles, so that two spurs are attached to 5.093.

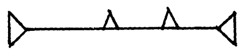
Since there are 5 (free) vertices of degree 2 in 5.093, there are $C_2^5 = 10$ classes, all strongly planar, and all with $s = 1$.

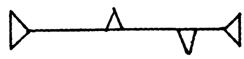
c = 12 Twenty-two classes, of which fourteen are strongly planar.

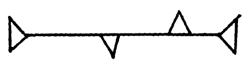
If there are three triangles, there must be three spurs. The number of classes in this case is $C_3^5 = 10$; in each case $s = 1$.

If there are four triangles, there are several classes with three links each; they are shown. Their automorphisms result from plane rotations.

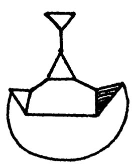
(5.1211)  $s = 3$.

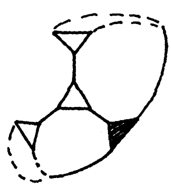
(5.1212)  $s = 1$.

(5.1213)  $s = 2$.

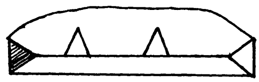
(5.1214)  $s = 2$.

More classes arise if there are four triangles and five links. First consider the number of ways of adding two links to the class (5.1211). After the links are added, either: (i) each of three triangles has three attached links (one class, not strongly planar); or (ii) two triangles have three links, and the other two have two links (four classes, not strongly planar).

(5.1215) 

(5.1216–5.1219) 

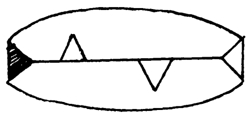
Next, consider the number of ways of attaching more links in Fig. 5.1212. One way to do this is to attach both links at the four vertices of the end triangles.



(5.1220) This gives a new class (with $s = 1$). It is not strongly planar. Any other way of adding two links to 5.1212 either results in an intransitive AB, or else gives a figure that is isomorphic to one of the preceding.

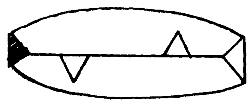
The same remarks apply to Figures 5.1213, 5.1214.

(5.1221)



$s = 2$

(5.1222)



$s = 2.$

The automorphisms of all these classes can be exhibited as plane rotations; the canonical form of each skeleton is obtained by flipping, rotating, and twisting.

The number of classes for the cases $c = 13, 14$ is easily obtained from the pictures for $c = 12$.

We now exhibit some graphs (with C transitive) for $c = 18, 24$.

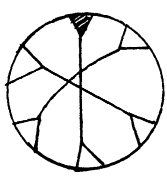


FIGURE 5.131
A nonplanar graph
with $c = 18$, $C = AB$
transitive

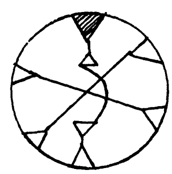


FIGURE 5.132
A nonplanar graph
with $c = 24$, $C = AB$
transitive

FIGURE 5.133
A planar graph with $c = 18$, not strongly planar, $C = AB$ transitive.

To see that 5.131 is nonplanar, note that it can be contracted to $K_{3,3}$, which is known to be nonplanar [**5**, pp. 135–156].

Note further that example 5.131 can be generalized as follows. Let k be odd, and attach $2k$ equally spaced triangles to a circumference, all facing inward. Connect opposite triangles, and orient all triangles (with one exception) clockwise. 5.131 is the case $k = 3$; 5.063 is the case $k = 1$. The cubic graph so obtained has a Eulerian circuit. For $k \geq 3$, the graph is nonplanar. We have proved:

5.14 THEOREM. *There exists a cubic graph with $6k$ vertices that has a Eulerian trail, if k is odd.*

5.15. *If k is even, there cannot be a Eulerian trail.* For C would have to be an odd permutation, that is, a cycle on $6k$ symbols. However, $AB = C$ is impossible, since both A and B are even permutations (A is the product of $3k$ transpositions.)

5.16 THEOREM. *If (and only if) k is odd, $k \geq 1$, there exists a cubic graph with $V = 2k$ vertices.*

Proof of $\Rightarrow$. To construct H, let T be a linear tree with $2k$ vertices, hence $2k - 1$ edges. Form T^* by replacing (in one way or another) each vertex of T by a triangle, positively oriented. Now $C_{T^*} = A_{T^*}B_{T^*}$ is transitive on the vertices of T^*. A *free vertex* is a vertex (of a triangle) not on a link. Let the $2k + 2$ free vertices of C_{T^*} be labeled $v_1, \ldots, v_{2k+2}$, in the order in which they occur in C_{T^*}.

FIGURE 5.161. The construction.

Since k is odd, 4 divides $2k + 2$. Thus the free vertices can be divided into $\frac{1}{2}(k + 1)$ blocks of four vertices each: $v_{4i+1}, \ldots, v_{4i+4}$ $(0 \leq i \leq \frac{1}{2}(k + 1))$. Form H by drawing links (v_{4i+1}, v_{4i+3}), (v_{4i+2}, v_{4i+4}).

The final step is to contract this graph at every triangle (compare Figure 5.133).

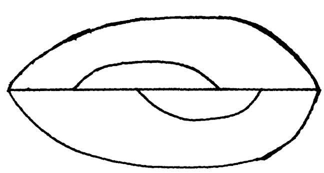


FIGURE 5.162. □

5.17 REMARK. The same idea works if T is any tree (not necessarily a linear tree) with $2k$ vertices.

6. Appendix B. *Cubic Graphs with an Eulerian path.* In §5 we noted that if a cubic graph H possesses an Eulerian path π, then H must have $v = 2k$ vertices, for some odd natural number k. We indicate how to enumerate all such H and π for $v = 2$ and for $v = 6$. We do this partly to indicate the abundance of such pairs (giving a method for finding them) and partly to illustrate the apparent difficulty of obtaining any overall view of the totality of cubic graphs with Eulerian paths. Note that 5.161 is not the only possible cubic graph with $2k$ vertices and with a Eulerian path.

Our basic method is that of *adding a line*. A set Σ of (reduced) *paths* on a finite cubic graph H will be called an Eulerian *system* if (i) no path occurs in Σ if its inverse occurs in Σ, (ii) each directed edge of H occurs exactly once in some path π of the set Σ. For our inductive construction, it is technically convenient to count the circle H_0 as a cubic graph with $v = 0$ vertices, and with an Eulerian system consisting of two paths, which describe the circle in its two senses. Now let H be any finite nonempty cubic graph, not necessarily connected. We mark any two distinct points p_1 and p_2 interior to two (not necessarily distinct) lines l_1 and l_2 of H, and form a new graph H' by introducing a new line l joining p_1 and p_2. Then H' is a cubic graph with two more vertices than H. Moreover, every finite nonempty cubic graph can be obtained from a disjoint union of replicas of the circle graph H_0 by iteration of this construction.

If π_1 and π_2 are paths (not necessarily distinct) in an Eulerian system Σ for H, we obtain an Eulerian system Σ^1 for H' by modifying π_1 and π_2 in the manner shown in Figure 6.01.

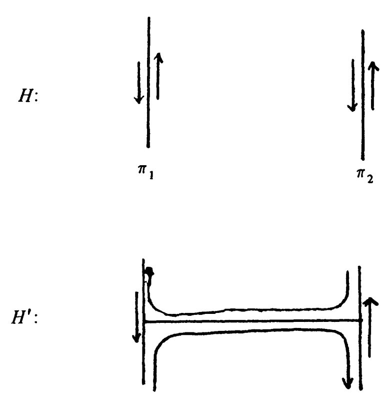


FIGURE 6.01

(We do not spell out a formal description of this construction, since it will be clear what is being done in all the special cases that arise below.) By the familiar argument of Cauchy-Jordan in the theory of permutation cycles, we see that if π_1 and π_2 are distinct, they are replaced by a single path in Σ', while, if $\pi_1 = \pi_2$, then this path is replaced by two paths in Σ'. In any case, Σ' has cardinality $|\Sigma'| = |\Sigma| \pm 1$. This establishes the following.

6.02 PROPOSITION. *Let Σ be an Eulerian system on a cubic graph H with $v = 2k$ vertices, where $k \geq 0$. Then $1 \leq |\Sigma| \leq k+2$, and $|\Sigma| \equiv k$ (modulo 2).*

It is easy to see that every Eulerian system Σ on H' can be obtained from a system Σ on H by the construction suggested by the figure.

We state another consequence of the method of adding a line.

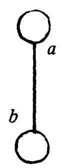
6.03 PROPOSITION. *Let K be any finite cubic graph. Then, by successively adding lines, K can be embedded in a finite cubic graph H that possesses an Eulerian path. In particular, if K has the property that deletion of no pair of lines separates K into two or more components, then H can be chosen with the same property.*

Two Eulerian systems Σ_1 and Σ_2 on the same cubic graph H will be called *equivalent* if some automorphism (preserving incidences) of H carries one into the other. We shall enumerate only *equivalence classes* $[\Sigma]$ of Eulerian systems. With each path π in an Eulerian system we associate the cyclically ordered sequence of the vertices on π, in the cyclic order in which they occur on π. The set of *cycles* σ associated with the paths of a (fully labeled) Eulerian system Σ on H evidently determines the graph H uniquely, and determines the paths π of Σ uniquely except for the choice of edges that connect a pair of vertices whenever there are two or three such edges. In any case, the set of cycles σ determines the equivalence class $[\Sigma]$ uniquely.

Evidently two systems Σ_1 and Σ_2 are equivalent if and only if there exists a permutation of the vertices carrying the set of cycles associated with Σ_1 into the set associated with Σ_2.

If an Eulerian system Σ consists of a single path π, that is, if π is an Eulerian path, we call Σ or π *symmetric* if π is equivalent to its inverse π^{-1}.

We now turn to the cubic graphs D with $v = 2$ vertices, and with no component H_0. There are in fact only two such graphs,

D_1: 

and D_2: 

By inspection we see that D_1 admits only a single class $[\Sigma]$, with $|\Sigma| = 3$, while D_2 admits two classes $[\Sigma_1]$, $[\Sigma_2]$, with $|\Sigma_1| = 3$ and $|\Sigma_2| = 1$. In particular, D_2 is the only cubic graph with two vertices that admits a (reduced) Eulerian path π, and the class of this path is described by the cycle $\sigma = (a\,b\,a\,b\,a\,b)$.

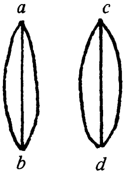
The cubic graphs Q with four vertices and no component H_0 are of three sorts:

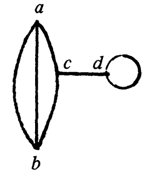
(i) a disjoint union of copies of D_i and D_j, $i, j \in \{1, 2\}$;

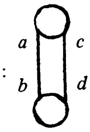
(ii) those obtained by adding a line to the disjoint union of a copy of D_i and a copy of H_0, $i \in \{1, 2\}$;

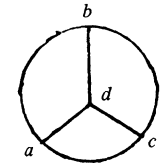
(iii) those obtained by adding a line to a copy of D_i, $i \in \{1, 2\}$.

Since our only interest in graphs with four vertices is to use these for constructing graphs with six vertices that admit an Eulerian path, we confine attention to those 4-vertex graphs Q that possess an Eulerian system Σ with $|\Sigma| = 2$. Inspection shows that these are as follows:

Q_1: 

; Q_2: 

; Q_3: 

; Q_4: 

The lettering of the vertices is for later reference. It is easy to see that each of Q_1, Q_2, and Q_3 admits only a single class of Eulerian systems Σ_i with

$|\Sigma_i| = 2$. These have cycles as follows:

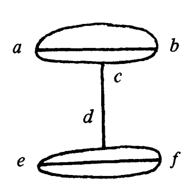
$$\begin{aligned} \Sigma_1&: \quad (a\,b\,a\,b\,a\,b), (c\,d\,c\,d\,c\,d); \\ \Sigma_2&: \quad (c\,d\,d\,c\,a\,b\,a\,c\,b\,a\,b), (d); \\ \Sigma_3&: \quad (a\,c\,d\,c\,a\,b), (b\,d\,c\,d\,b\,a). \end{aligned}$$

For Q_4 there are exactly two classes of Eulerian systems, Σ_4 and Σ_4' with $|\Sigma_4| = |\Sigma_4'| = 2$. To see this observe first that if $\Sigma = \{\pi_1, \pi_2\}$, then $|\sigma_1| + |\sigma_2| = 12$. By symmetry, we may suppose that $|\sigma_1| \leq 6$. Now $|\sigma_1| = 0$, 1, or 2 is clearly impossible. If $|\sigma_1| = 3$, we may suppose that $\sigma_1 = (a\ b\ c)$; this uniquely determines that $\sigma_2 = (a\,c\,d\,b\,a\,d\,c\,b\,d)$. If $|\sigma_1| = 4$, we may suppose that $\sigma_1 = (a\,b\,c\,d)$; this uniquely determines that $\sigma_2 = (a\,d\,b\,a\,c\,b\,d\,c)$. Next, $|\sigma_1| = 5$ is impossible. To see this, we may suppose that $\sigma_1 = (a\,b\,c\,x\,y)$. Since $x \neq a, b, c$, we must have $x = d$ and $\sigma_1 = (a\,b\,c\,d\,y)$. But now y cannot be any one of a, b, c, d. Also, $|\sigma_1| = 6$ is impossible. To see this we may suppose that $\sigma_1 = (a\,b\,c\,x\,y\,z)$. Now $x \neq b, c$, $y \neq a, c$, and $z \neq a, b$. Moreover, x, y, and z must be distinct. This implies that at most one of x, y, z is d, and otherwise that $x = a, y = b$, and $z = c$. But then σ_1 must contain two segments ab, bc, or ca, impossible.

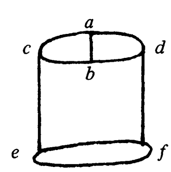
In summary, we have two classes $[\Sigma_1]$ and $[\Sigma_4']$, as follows:

$$\begin{aligned} \Sigma_4&: \quad (a\,b\,c), (a\,c\,d\,b\,a\,d\,c\,b\,d); \\ \Sigma_4'&: \quad (a\,b\,c\,d), (a\,d\,b\,a\,c\,b\,d\,c). \end{aligned}$$

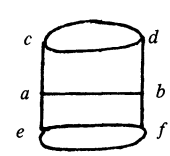
Now every cubic graph H with six vertices and an Eulerian path π must be obtainable from one of Q_1, Q_2, Q_3, Q_4 by adding a line. There is a fairly large number of ways of adding a line to one of these Q_i but, with some effort, one can see that those of the resulting graphs that admit an Eulerian path are of only five types, as follows:



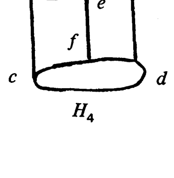
H_1



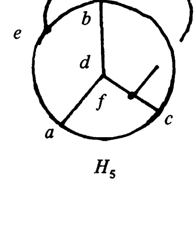
H_2



H_3



H_4



H_5

These five graphs are nonisomorphic. H_1 is the only one that can be disconnected by deleting a single edge. H_5 is the only one that cannot be embedded in the plane. Of the remaining three, H_2 contains exactly one pair of lines with the same end points, while H_3 contains two such pairs and H_4 contains no such pair.

It is easy to see that H_1 admits only one class of Eulerian paths π, with cycle

$$\sigma = (c\,a\,b\,a\,c\,b\,a\,b\,c\,d\,e\,f\,e\,d\,f\,e\,f\,d).$$

This cycle is necessarily equivalent to its inverse: that is, the class $[\pi]$ is symmetric.

In H_2 it is easy to see that if an Eulerian path contained a segment *cefd*, it could not be completed to contain an additional segment *ef* and two segments *fe*. We conclude that π must contain segments *cefec* and, similarly, *dfefd*. Now deletion of the loops *efe* and *fef* from these segments reduces π to π' with cycle $\sigma' = (a\,d\,b\,c\,a\,b\,d\,a\,c\,b)$, or an equivalent cycle. To regain π from π', either occurrence of c may be replaced by *cefec*, and either occurrence of d may be replaced by *dfefd*. These replacements may be made either at occurrences of c and d that are separated in π' by a single letter, or at occurrences that are separated by two letters. We thus have, within equivalence, two cycles:

$$\sigma_1 = (a\,d\,f\,e\,f\,d\,b\,c\,e\,f\,e\,c\,a\,b\,d\,a\,c\,b);$$
$$\sigma_2 = (a\,d\,f\,e\,f\,d\,b\,c\,a\,b\,d\,a\,c\,e\,f\,e\,c\,b),$$

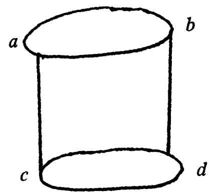
where σ_1 is not equivalent to σ_2 or to σ_2^{-1}. On the other hand, the permutation $(cd)(ef)$ carries σ_1 to σ_1^{-1}, and the permutation $(ab)(cd)(ef)$ carries σ_2 to σ_2^{-1}. In summary, we have two symmetric classes on H_2.

In H_3, by the argument above, an Eulerian path π must contain segments *aefea*, *bfefb*, *acdca*, and *bdcdb*. Now one of the displayed a's must be followed by one of the displayed b's, whence we must have either *aefeabfefb*, *aefeabdcdb*, or a segment equivalent under the permutation $(ce)(df)$. Each of these two segments admits a unique completion, as follows:

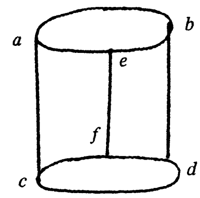
$$\sigma_1 = (a\,e\,f\,e\,a\,b\,f\,e\,f\,b\,a\,c\,d\,c\,a\,b\,d\,c\,d\,b);$$
$$\sigma_2 = (a\,e\,f\,e\,a\,b\,d\,c\,d\,b\,a\,c\,d\,c\,a\,b\,f\,e\,f\,b).$$

In σ_2 the occurrences of the parts *cdc*, *dcd*, are separated by those of *efe* and *fef*, while in σ_1 they are not. Thus σ_1 is not equivalent to σ_2 or σ_2^{-1}. Here $(ab)(cf)(de)$ carries σ_1 to σ_1^{-1}, and $(ab)(cd)(ef)$ carries σ_2 to σ_2^{-1}. As with H_2, we have two symmetric classes on H_3.

We now look for Eulerian paths π on H_4. Observe that H_4 can be obtained from Q_3 by adding a line l between vertices e and f.

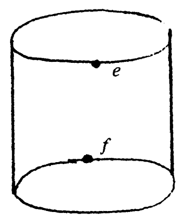


Q_3



H_4

We denote by Q_3' the graph:



Q_3'

If π is an Eulerian path on H_4 it must contain edges ef and fe, hence have the form (1) $\pi = (\alpha_1 ef \alpha_2 fe) = (e\alpha_1 ef\alpha_2 f)$ where (α_1) and $(\alpha_2 e)$ constitute an Eulerian system on Q_3'. The corresponding paths (β_1) and (β_2) will then constitute an Eulerian system B on Q_3. Since $|B| = 2$, B is equivalent to Σ_3 under some automorphism of Q_3. This implies that under some element of the fours group $V = \{1, (ab), (cd), (ab)(cd)\}$, the cycles π_1, π_2 associated with (β_1), (β_2) go into the cycles α_1, α_2 of Σ_3, and, evidently, with $\tau_1 \mapsto \sigma_1$, $\tau_2 \mapsto \sigma_2$. Since every automorphism in V can be lifted to H_4, after replacing π by an equivalent path we may assume that $\tau_1 = \sigma_1$, $\tau_2 = \sigma_2$, except that now, in (1), e and f may be exchanged.

We have then

$$(\beta_1) = (a\,c\,d\,e\,a\,b), \qquad (\beta_2) = (b\,d\,e\,d\,b\,a).$$

The corresponding paths (β_1'), (β_2') in Q_3' will be obtained by insertion of an e and an f in each of (β_1), (β_2), in such a way that (β_1'), (β_2') constitute an Eulerian system on Q_3'. There are four possibilities:

$$\begin{aligned}
EE&: \quad (a\,c\,f\,d\,c\,a\,e\,b)\,(b\,d\,f\,c\,d\,b\,e\,a)\\
EL&: \quad (a\,c\,d\,f\,c\,a\,e\,b)\,(b\,d\,c\,f\,d\,b\,e\,a)\\
LE&: \quad (a\,c\,f\,d\,c\,a\,b\,e)\,(b\,d\,f\,c\,d\,b\,a\,e)\\
LL&: \quad (a\,c\,d\,f\,c\,a\,b\,e)\,(b\,d\,c\,f\,d\,b\,a\,e)
\end{aligned}$$

The notation EE means, for example, that, in (β_1'), e is introduced *early* into the part aba, and f is introduced early into the part cdc. Replacing π by π^{-1} interchanges EE with LL and EL with LE. Thus it suffices to consider cases EE and EL (with the reservation that all inverses of resulting π must be considered).

We consider the case EE. Here π must contain a segment $ebacfdcae$ or $fdcaebaef$; the complementary segment is then fully determined. We obtain

$$\pi_1 = \left(eb\underline{acfdca}e \cdot f\underline{cdbeabd}f \right)$$
$$\pi_2 = \left(fd\underline{caebae}f \cdot eab\underline{dfcdb}e \right).$$

From EL we obtain similarly

$$\pi_3 = \left(eb\underline{acdfca}e \cdot f\underline{dbeabd}cf \right)$$
$$\pi_4 = \left(f\underline{caebac}df \cdot eab\underline{dcfdb}e \right).$$

The underlined *blocks* are the only parts of the form $xyuvyx$. In π_1 and π_2 the blocks are separated by two parts of length 3, while in π_3 and π_4 by parts of lengths 2 and 4. Therefore $\{\pi_1^{\pm 1}, \pi_2^{\pm 1}\}$ is disjoint from $\{\pi_3^{\pm 1}, \pi_4^{\pm 1}\}$.

Evidently the permutation $(ac)(bd)(ef)$ carries π_1 to π_2. Now π_1 is not equivalent to π_1^{-1}. To see this, note that any permutation carrying π_1 to π_1^{-1} would have to carry the block $acfdca$ of π_1 to one of the two blocks in π_1^{-1}. In each case, this is incompatible with its action on the three letters e, f, and c.

The permutation $(ab)(cd)$ carries π_3 to π_4. Moreover, the permutation $(ad)(bd)(cf)$ carries π_3 to π_3^{-1}.

In summary, on H_4 there exist three classes of Eulerian paths:

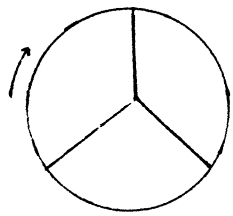
$$[\pi_1],\ [\pi_1^{-1}],\ [\pi_3] = [\pi_3^{-1}].$$

We now look at the Eulerian paths on H_5. The explanations are analogous to those for H_4, and we present only an outline of the calculations.

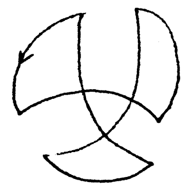
Case I. π is obtained by linking some Σ automorphic in Q_4 to $\Sigma_4 = \{\rho_1, \rho_2\}$ by a line l between points e and f on disjoint lines of Q. We obtain equivalent π' by taking Σ_4 in place of Σ, and l joining some other points e' and f' on disjoint lines in Q. We may suppose e' is on an arc-line in the figure, and f' on the opposite spoke. We now drop primes.

We have $\Sigma_4 = \{\rho_1, \rho_2\}$ where

$$\rho_1 = (abc), \qquad \rho_2 = (acdbadcbd)$$



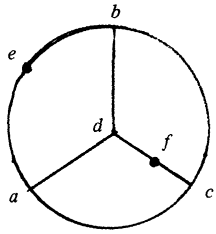
ρ_1



ρ_2

Since Σ_4 has rotational symmetry, we may suppose that e is on ab and f on cd, yielding

Q':



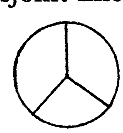
Passing to Q', ρ_1 and ρ_2 become

$$\rho_1' = (aebc), \qquad \rho_2' = (acdfbeadfcbd).$$

Now π must contain a part $ebcae$, and there are 2 ways to complete it. These are

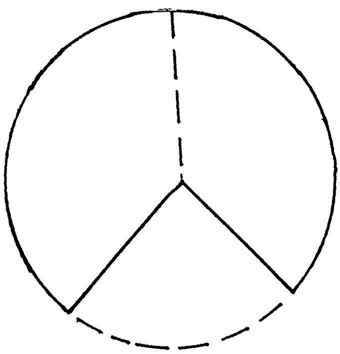
$$\pi_1 = (ebcae\,fcbdacfdbeadf)$$
$$\pi_2 = (ebcae\,fdbeadfcbdacf).$$

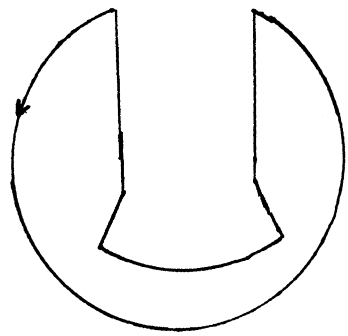
But the permutation (ab) carries π_1 to π_2^{-1}, while $(abf)(cd)$ carries π_1 to π_2. Thus there is exactly one (symmetric) class.

Case II. π is obtained by adding a line l between e and f from some system $\Sigma = \{\rho_1, \rho_2\}$ on Q. Then some automorphism of Q carries Σ to Σ_4', e and f to points e', f', and π to some π' equivalent to π. Changing notation, and replacing π by an equivalent path, we may suppose that π is obtained from Σ_4' by linking two points e and f on disjoint lines of Q. We may suppose e is on an arc-line in the figure  and f on the opposite spoke.

Now $\Sigma_4' = \{\pi_1, \pi_2\}$ where

$$\rho_1 = (abcd), \qquad \rho_2 = (abdacbdc).$$

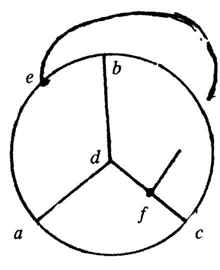




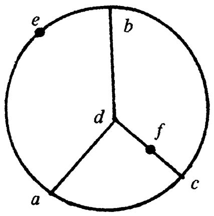
It is not possible that e is on the bottom arc, since in this case neither e nor f would be on ρ_1, and l would not link ρ_1 and ρ_2.

If e is on the second of the consecutive arcs of ρ_1, we replace π, ρ_1, ρ_2 by π^{-1}, ρ_1^{-1}, ρ_2^{-1}. Now e is on the first arc of ρ_1^{-1}. Thus we may assume that e is on the arc ab. Now H_5 is as previously shown, with ρ_1 and ρ_2 as above.

H_5:



In Q':



, ρ_1 and ρ_2 become

$$\rho_1' = (aebcfd), \qquad \rho_2' = (adbeacbdfc).$$

As in Case I, we obtain 2 possibilities:

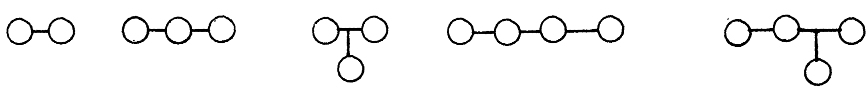
$$\pi_1 = (ebcfdae\,fcadbeacbdf)$$
$$\pi_2 = (bdaebcf\,eacbdfcadbe).$$

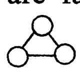
Here we use the fact that each of π_1 and π_2 has exactly 3 *blocks* of the form $x^{***}x^{***}x$, with successive blocks overlapping in a word of length 3. This severely limits permutations θ carrying π_1 to π_2.

We find that the permutation $(ac)(bd)(ef)$ carries π_1 to π_2. We find similarly that $(bf)(cd)$ carries π_1 to π_1^{-1}.

Again we have a single symmetric class. But $(ae)(bdfc)$ carries this class into the class π_1 of Case I. Thus there is just one class of Eulerian paths in H_5. In all of $H_1, \ldots, H_5$ we have a total of 9 classes, consisting of 7 symmetric classes and one pair of inverse classes.

We conclude with two observations. First, there are many finite cubic graphs that admit no Eulerian path. For example any graph with a spur of the form —o fails to have a (reduced) Eulerian path. Examples are



(Chemists are familiar with these patterns.) Another example, with 6 vertices, is . This graph is 2-arc-connected. A 3-arc-connected cubic graph with no (reduced) Eulerian path can be constructed by adding, to H_4, an extra line from the upper ellipse to the lower ellipse. Clearly, there is no need to look for a 4-arc-connected cubic graph of any sort: it does not exist.

Second, any classification of Eulerian paths π on a cubic graph H with v vertices amounts to a classification of the (isomorphism types of) triples (Ω, A, B) where $A^2 = B^3 = 1$, A and B have no fixed points, AB is transitive, and $|\Omega| = v$. For $|\Omega| = 6$, there are nine isomorphism classes of such triples. For 7 of these classes, $(\Omega, A, B) \simeq (\Omega, A, B^{-1})$, while the remaining 2 are nonisomorphic, but of the form (Ω, A_1, B_1) and (Ω, A_1, B_1^{-1}) for certain A_1, B_1.

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