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ON THE APPROXIMATION OF SINGULARITY SETS BY ANALYTIC VARIETIES

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We study the problem of approximating the singularity set of an analytic function of two complex variables, lying in a product domain in C^2, by analytic varieties.

Let D denote the open unit disk and let

$$D \times \mathbf{C} = \{(z, w) \mid z \in D, w \in \mathbf{C}\} \subset \mathbf{C}^2.$$

We consider a compact set X contained in the unit bidisk $|z| \leq 1$, $|w| \leq 1$. Let X^0 denote $X \cap (D \times \mathbf{C})$. We assume that there exists a function ϕ which is analytic on $D \times \mathbf{C} \setminus X^0$ and singular at each point of X^0. If there exists such a ϕ we call X a *singularity set*.

For each λ in $\bar{D}$ we put

$$X_\lambda = \{w \in \mathbf{C} \mid (\lambda, w) \in X\}.$$

Each X_λ is then a compact subset of $|w| \leq 1$. We assume $X_\lambda \neq \varnothing$, for each λ.

Singularity sets were first studied by Hartogs, in [3]. Hartogs showed that if for some integer p X_λ contains at most p points for each λ, then X^0 is an analytic subvariety of $D \times \mathbf{C}$. Further results on singularity sets were given by Oka, [5], and Nishino, [4].

Recently one of us in [7] and Slodkowski in [6] studied general singularity sets. In particular, Theorem 1 in [7] gives that the maximum principle holds on X^0 for restrictions to X^0 of polynomials in z and w, in the sense that for each compact subset N of X^0 and each $(z_0, w_0) \in N$,

$$|P(z_0, w_0)| \leq \max_{\partial N} |P|,$$

for each polynomial P.

(See also [6], Theorem II, (vi).) Here ∂N denotes the boundary of N relative to X^0. In particular, fix $R < 1$. Put $N = \{(\lambda, w) \in X \mid |\lambda| \leq R\}$. Then $\partial N = \{(\lambda, w) \in X \mid |\lambda| = R\}$. Hence, for $(z_0, w_0) \in X$, $|z_0| < R$, we have for each polynomial P,

$$|P(z_0, w_0)| \leq \max_{X \cap (|z| = R)} |P|.$$

Letting $R \to 1$, we get

(1) $$|P(z_0, w_0)| \leq \max_{X \cap (|z|=1)} |P|, \qquad (z_0, w_0) \in X.$$

In order to account for the inequality (1), one of us, (Alexander), suggested that it may be possible to approximate an arbitrary singularity set by one-dimensional analytic subvarieties of $D \times \mathbf{C}$.

Question 1. Given a singularity set X and an open neighborhood U of X in $\mathbf{C}^2$. Can we find a one-dimensional analytic subvariety of $D \times \mathbf{C}$ which is contained in U?

In this paper, we look at a special case of this question. For each λ in $|\lambda| < 1$, we fix an open disk D_λ with center $a(\lambda)$ and radius r (r fixed and > 0). Consider the tube

$$T = \{(\lambda, w) \,|\, |\lambda| < 1, w \in D_\lambda\}.$$

Question 2. Suppose X is a singularity set contained in the tube. Then for each λ, $X_\lambda \subset D_\lambda$. Can we find a function f analytic on $|\lambda| < 1$ such that the graph of f lies in T, i.e. such that, for every λ, $f(\lambda) \in D_\lambda$, or, in other words,

$$|f(\lambda) - a(\lambda)| < r?$$

We have the following partial answer.

THEOREM. *Let $\lambda \mapsto a(\lambda)$ be a continuous function defined for $|\lambda| \leq 1$ with $|a(\lambda)| \leq 1$ for all λ. Fix $r > 0$. Suppose that there exists a singularity set X such that for each λ, $|\lambda| \leq 1$, $X_\lambda \subset \{w \,|\, |w - a(\lambda)| < r\}$. Then there exists an analytic function $\lambda \mapsto f(\lambda)$ such that for each λ in the unit disk*

(2) $$|f(\lambda) - a(\lambda)| \leq 4r.$$

Let $C(\Gamma)$ denote the Banach space of functions g which are continuous on $\Gamma = \{|z| = 1\}$, with

$$\|g\| = \max_{\Gamma} |g|.$$

Let $A(\Gamma)$ denote the closed subspace of $C(\Gamma)$ consisting of those functions in $C(\Gamma)$ which have an analytic extension to $|\lambda| < 1$. We write a for the element of $C(\Gamma)$ obtained from our function a by restricting it to Γ. The distance from a to $A(\Gamma)$, in the norm of $C(\Gamma)$, equals

$$\sup |L(a)|$$

taken over all bounded linear functionals L on $C(\Gamma)$ with $\|L\| = 1$ and $L(g) = 0$ for every $g \in A(\Gamma)$. Each such functional L is given, in view of the Theorem of F. and M. Riesz, by a summable function ϕ in the Hardy space H^1 such that

$$L(g) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\phi(e^{i\theta})\, d\theta, \tag{3a}$$

$$\phi(0) = 0, \tag{3b}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{i\theta})|\, d\theta = 1. \tag{3c}$$

LEMMA. *Let* $\phi \in H^1$ *and assume* (3b), (3c). *Then*

$$\left|\frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})\phi(e^{i\theta})\right| \leq r. \tag{4}$$

Proof. Since polynomials in λ are dense in H^1 in L^1-sense, it suffices to prove (4) where ϕ is a polynomial.

Fix w_0 so that $(O, w_0) \in X$. It follows from the inequality (1) that there exists a representing measure μ for (O, w_0) relative to polynomials which is carried on $X \cap \{|\lambda| = 1\}$. (See e.g. [2], pp. 79–81.) μ is a probability measure such that

$$Q(O, w_0) = \int Q\, d\mu \tag{5}$$

for every polynomial Q.

We denote by μ^* the projection of μ under the map λ. Then μ^* is a probability measure on the circle $|\lambda| = 1$ so that

$$\int_{X \cap \{|\lambda| = 1\}} F(\lambda)\, d\mu(\lambda, w) = \int_{|\lambda| = 1} F(\lambda)\, d\mu^*(\lambda) \tag{6}$$

for every F in $C(\Gamma)$.

Then for $n = 1, 2, \ldots$

$$\int_{|\lambda| = 1} \lambda^n\, d\mu^*(\lambda) = \int \lambda^n\, d\mu = \lambda^n(O, w_0) = 0.$$

Conjugating, we get

$$\int_{|\lambda| = 1} \overline{\lambda^n} d\mu^*(\lambda) = 0, \qquad n = 1, 2, \ldots.$$

Finally, μ^* has total mass 1. So $\mu^* = d\theta/2\pi$ on Γ. Thus for all $F \in C(\Gamma)$:

$$\int F(\lambda)\, d\mu(\lambda, w) = \frac{1}{2\pi}\int_{|\lambda|=1} F(\lambda)\, d\theta. \tag{7}$$

Fix now a polynomial ϕ satisfying (3b), (3c). Then

$$\begin{aligned}\frac{1}{2\pi}\int_{|\lambda|=1} a(\lambda)\phi(\lambda)\, d\theta &= \int_{X\cap\{|\lambda|=1\}} a(\lambda)\phi(\lambda)\, d\mu(\lambda, w)\\ &= \int (a(\lambda) - w)\phi(\lambda)\, d\mu(\lambda, w),\end{aligned}$$

since

$$\int w\phi(\lambda)\, d\mu(\lambda, w) = w_0\phi(0) = 0.$$

Hence

$$\left|\frac{1}{2\pi}\int_{|\lambda|=1} a(\lambda)\phi(\lambda)\, d\theta\right| \le \int | a(\lambda) - w\,|\,|\,\phi(\lambda)\,|\, d\mu(\lambda, w).$$

For each $(\lambda, w) \in \operatorname{supp}\mu$, $| a(\lambda) - w | \le r$. Hence the right hand side

$$\le \int r\,|\,\phi(\lambda)\,|\, d\mu(\lambda, w) = \int_{|\lambda|=1} r\,|\,\phi(\lambda)\,|\,\frac{d\theta}{2\pi} = r.$$

Thus (4) holds and we are done.

Proof of Theorem. In view of the Lemma, $\sup | L(a) |$, taken over all functionals L with $\| L \| = 1$ and $L = 0$ on $A(\Gamma)$, $\le r$, and so the distance from a to $A(\Gamma) \le r$. Thus $\exists f_0 \in A(\Gamma)$ with

$$| f_0(\lambda) - a(\lambda) | \le 2r, \qquad | \lambda | = 1. \tag{8}$$

Without loss of generality, we take f_0 to be a polynomial in λ.

In view of (8), for each λ on $| \lambda | = 1$ $| w - f_0(\lambda) | \le 3r$ for every $w \in X_\lambda$.

Denote by Q the polynomial in λ and w,

$$Q(\lambda, w) = w - f_0(\lambda).$$

We just saw that

$$| Q(\lambda, w) | \le 3r$$

for each $(\lambda, w) \in X \cap (| \lambda | = 1)$. Because of (1), it follows that for every (λ_0, w_0) in X,

$$| Q(\lambda_0, w_0) | \le 3r.$$

Thus

$$|f_0(\lambda_0) - w_0| \leq 3r, \qquad (\lambda_0, w_0) \in X.$$

Hence

$$|f_0(\lambda_0) - a(\lambda_0)| \leq 4r, \qquad |\lambda_0| \leq 1.$$

This was to be proved.

Note. A natural extension of the situation treated in the Theorem is the following: let X be a singularity set. Fix $r > 0$.

Assume there exist continuous functions $a_1(\lambda), a_2(\lambda), \ldots, a_n(\lambda)$ defined on $|\lambda| \leq 1$ such that the roots $w_1(\lambda), w_2(\lambda), \ldots, w_n(\lambda)$ of the equation

$$w^n + a_1(\lambda)w^{n-1} + \cdots + a_n(\lambda) = 0 \tag{9}$$

give an approximation to X_λ in the sense that for each λ in $|\lambda| \leq 1$, X_λ is contained in the union of the disks

$$|w - w_j(\lambda)| < r, \qquad j = 1, 2, \ldots, n.$$

Can we find good approximations $a_1^0(\lambda), \ldots, a_n^0(\lambda)$ to the $a_j(\lambda)$, where the a_j^0 are analytic on $|\lambda| < 1$? In this case, the goodness of approximation should be expressed in terms of r and n. The analytic variety defined by

$$w^n + a_1^0(\lambda)w^{n-1} + \cdots + a_n^0(\lambda) = 0, \qquad |\lambda| < 1, \tag{10}$$

will then provide an approximation to X.

Note. Singularity sets occur in the spectral theory of linear operators. See Aupetit, **[1]**, and Slodkowski, **[6]**.

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