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VECTOR EXTENSIONS OF OPERATORS IN L^p SPACES

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We study B-valued extensions of operators of weak or strong type (p, q), where B is a p-Banach space of a certain type, and present several applications.

An old well known result of Marcinkiewicz and Zygmund states that every bounded linear operator T from L^p to L^q has a bounded extension $T \otimes 1_B$ from L^p_B to L^q_B, where B is a Hilbert space. The analogous question for certain types of Banach spaces was also considered in [**9**] and [**5**] and, for weak type operators, in [**13**] and [**14**]. Here we obtain a general result on the extension of bounded linear operators of (weak or strong) type (p, q) to B-valued functions, where B belongs to C. Herz's class of r-spaces (see [**5**]). Applications are given to pointwise convergence of vector valued functions and to weighted norm inequalities for a wide class of operators. Finally, we prove a mixed norm estimate for translation invariant operators which is a weak type analogue of the main result in [**6**].

I. B-Valued extensions of operators. Let (X, μ) and (Y, ν) be measure spaces. Given a linear operator T from $L^p(\mu)$ to $L^0(\nu) = \{$all ν-measurable functions with the topology of local convergence in measure$\}$, and a quasi-Banach space B, the operator

$$T^B = T \otimes 1_B \colon \sum b_i f_i(x) \mapsto \sum b_i T f_i(y) \qquad (b_i \in B; f_i \in L^p(\mu))$$

is defined a priori on $L^p(\mu) \otimes B$. If T is of weak or strong type (p, q), or simply continuous in measure, we ask ourselves if the corresponding continuity condition holds for T^B, in which case, it can be uniquely extended to $L^p_B(\mu)$. In this case, and when B is a Banach space, T^B is characterized by the property:

$$(1) \qquad \langle T^B f(y), b' \rangle = T(\langle f, b' \rangle)(y) \quad \nu\text{-a.e.}$$

for every $f \in L^p_B(\mu)$ and $b' \in B'$ (dual of B).

By $\mathfrak{B}_r$, $0 < r < \infty$, we denote the class of all quasi-Banach spaces which are isomorphic to some subspace of a quotient of a space L^r. The following facts are either known or easy to check:

(F.1) When $r \leq 1$, $B \in \mathfrak{B}_r$ if and only if B is r-normed, since every r-Banach space is isomorphic to a quotient of $l^r(I)$ (Shapiro [**15**]).

(F.2) If $B \in \mathfrak{B}_r$, every operator T of strong type (r, r) has a vector extension T^B of strong type (r, r) with $\|T^B\| \leq C_B \|T\|$. For $r \geq 1$, the converse is also true (Kwapien **[8]**).

(F.3) L^p spaces are of class $\mathfrak{B}_r$ if and only if $r \leq p \leq 2$ or $2 \leq p \leq r$ (Marcinkiewicz and Zygmund **[9]**, Herz **[5]**).

(F.4) The Lorentz space $L(p, q)$ is of class:

$\mathfrak{B}_{p-\varepsilon}$ for every $\varepsilon > 0$ if $1 < p < 2$, $1 \leq q \leq \infty$

$\mathfrak{B}_{p+\varepsilon}$ for every $\varepsilon > 0$ if $2 < p < \infty$, $1 \leq q \leq \infty$

$\mathfrak{B}_p$ if $0 < p < 1$, $1 \leq q \leq \infty$ (or $p = 1$, $q < \infty$).

All this is either contained in (F.1) and (F.3) or obtained from (F.2) by interpolation.

To state the first theorem, we define for each pair (p, q) with $0 < p$, $0 \leq q$, the interval $I(p, q)$ in the following way:

if $q < p < 2$: $I(p, q) = (p, 2]$

if $2 < q < p$: $I(p, q) = [2, q)$

in all other cases: $I(p, q) = [2 \wedge p, 2 \vee q]$

(where $\wedge =$ inf, $\vee =$ sup). Then we have the following

THEOREM 1. (a) *Let* $0 < p < \infty$, $0 \leq q < \infty$. *If* $B \in \mathfrak{B}_r$ *for some* $r \in I(p, q)$, *then every bounded linear operator* T *from* $L^p(\mu)$ *to* $L^q(\nu)$ *has a bounded extension* T^B *from* $L^p_B(\mu)$ *to* $L^q_B(\nu)$. *Moreover* $\|T^B\| \leq C_{p,q,B} \|T\|$ (*when* $q > 0$).

(b) *Let* $0 < p, q < \infty$. *If* $B \in \mathfrak{B}_r$ *for some* $r \in I(p, q - \varepsilon)$ *with* $\varepsilon > 0$, *then every bounded linear operator* T *from* $L^p(\mu)$ *to* $L^{q^*}(\nu) =$ *weak-*$L^q(\nu)$ *has a bounded extension* T^B *from* $L^p_B(\mu)$ *to* $L^{q^*}_B(\nu)$ *with* $\|T^B\| \leq C'_{p,q,B} \|T\|$.

Proof. Let us first prove (a) with $q > 0$. Since the property that T admits a bounded extension T^B is inherited by subspaces and quotient spaces of B, we can assume $B = L^r$. By the Marcinkiewicz-Zygmund theorem [**9**, Theorem 3], T^B is of strong type (p, q) provided that either $r = 2$ or $p \vee q < r < 2$; by duality the same will be true provided $2 < r < p \wedge q$. This establishes (a) for $q > 0$ except when $p \leq r \leq q$. In this case, suppose T is bounded from $L^p(\mu)$ to $L^q(\nu)$.

If $f \in L^p_B(\mu)$, there exist functions: $u \in L^{(r/p)'}(\mu)$ and $v \in L^{(q/r)'}(\nu)$ such that

$$\|f\|_{L^p_B(\mu)} = \|u^{-1/p} \cdot f\|_{L^r_B(\mu)} \text{ with } \|u\|_{(r/p)'} = 1$$

$$\|T^B f\|_{L^q_B(\nu)} = \|v^{1/r} \cdot T^B f\|_{L^r_B(\nu)} \text{ with } \|v\|_{(q/r)'} = 1.$$

Then S: $g \to v^{1/r} \cdot T(g \cdot u^{1/p})$ is a bounded operator from $L^r(\mu)$ to $L^r(\nu)$ with $\|S\| \leq \|T\|$, and by (F.2) it has a bounded extension S^B. Consequently

$$\|T^B f\|_{L^q_B} = \|S^B(u^{-1/p} \cdot f)\|_{L^r_B} \leq C_B \|S\| \cdot \|u^{-1/p} \cdot f\|_{L^r_B} \leq C_B \|T\| \, \|f\|_{L^p_B}.$$

Now we shall see that (b) is actually an easy consequence of (a), because the weak-L^q norm of a function has an equivalent form:

$$(2) \qquad \|f\|_{L^{q^*}(\nu)} \sim \sup\left\{\nu(E)^{-\varepsilon/q(q-\varepsilon)} \|f\chi_E\|_{q-\varepsilon};\ 0 < \nu(E) < \infty\right\}$$

(this is known and was first observed by M. Cotlar; see the proof in [**14**]). Therefore, if $B \in \mathfrak{B}_{p,q-\varepsilon}$ and T is of weak type (p, q), it suffices to apply part (a) to the operators $T_E f = (Tf)\chi_E$ which are of strong type $(p, q - \varepsilon)$, and then use (2).

Finally, if T is bounded from $L^p(\mu)$ to $L^0(\nu)$, it follows from Nikishin's theorem (see Gilbert [**4**]) that T is bounded from $L^p(\mu)$ to $L^{q^*}(h\,d\nu)$, where $h \geq 0$, $h \in L^1(\nu)$ and $q = p \wedge 2$. Applying part (b), T^B is bounded from $L^p_B(\mu)$ to $L^{q^*}_B(h\,d\nu)$ (and a fortiori bounded in measure) provided that $B \in \mathfrak{B}_r$, $r \in I(p, 0) = (p, 2) \cup \{2\}$.

COROLLARY 1. *Let* $0 < p < \infty$. *Every linear operator* T *of strong (resp. weak) type* (p, p) *has a bounded extension* T^B *of strong (resp. weak) type* (p, p) *with* $\|T^B\| \leq C_{p,B}\|T\|$, *provided that* $B \in \mathfrak{B}_r$ *with* $p \leq r \leq 2$ *or* $2 \leq r \leq p$ *(resp.* $p < r < 2$ *or* $2 \leq r < p$ *or* $r = 2$*).*

REMARKS. 1. For strong type operators, the corollary is essentially due to Marcinkiewicz and Zygmund [**9**] (see also [**5**]). For weak type operators, it was proved by Oberlin ([**13**], Lemma 2) when $p < 2$. Our proof covers also the case $p \geq 2$, and is (we feel) simpler than Oberlin's.

2. The results of Theorem 1 are best possible in the sense that, if $r \notin I(p, q)$, it is not true that every operator T of strong type (p, q) has a bounded extension T^B whenever $B \in \mathfrak{B}_r$, and the same can be said for weak type operators if $r \notin \bigcup_{\varepsilon>0} I(p, q - \varepsilon)$. This can be seen in all cases by composing appropriate translation invariant operators with multiplication operators.

II. Some applications. Our first application of Theorem 1 is a general principle of extension of pointwise convergence for scalar valued functions to the case of vector valued functions.

COROLLARY 2. *Let T_n: $L^p(\mu) \to L^0(\nu)$, $0 < p < \infty$, be linear operators continuous in measure and such that $Tf(y) = \lim_n T_n f(y)$ exists* ν-a.e. *for every $f \in L^p(\mu)$. Then*

$$\lim_n \sum_{j=1}^{\infty} \left|T_n f_j(y) - Tf_j(y)\right|^q = 0 \quad \nu\text{-a.e.}$$

provided that $(\Sigma |f_j|^q)^{1/q} \in L^p$ and $p < q < 2$ or $q = 2$.

Proof. Since $T^*f = \sup_n |T_n f|$ is continuous in measure (Banach's principle), Nikishin's theorem implies (see [**4**]) that T^* is bounded from $L^p(\mu)$ to weak-$L^{p\wedge 2}(\nu^*)$ for some measure ν^* equivalent to ν, and this is equivalent to the uniform boundedness of the linear operators

$$T_{s(\cdot)}f(y) = T_{s(y)}f(y)$$

where $s(\cdot)$ are simple functions in Y whose values are natural numbers. Therefore, $\{T^B_{s(\cdot)}\}_s$ are uniformly bounded from $L^p_B(\mu)$ to weak-$L_B^{p\wedge 2}(\nu^*)$ (in particular uniformly bounded in measure) when $B \in \mathfrak{B}_q$, $q \in I(p,0)$, and this easily gives the pointwise convergence of $T^B_n f$ to T^B_f, for every $f \in L^p_B(\mu)$.

The preceding proof is similar to that of [**14**, Theorem 2], where only the result for $q = 2$ was obtained. Corollary 2 can be applied in many different settings. As an illustration, if the functions $(g_n)_1^\infty$ in [0, 1] form a system of convergence for l^p, it follows that $F(x, t) = \Sigma f_k(x)g_k(t)$ can be defined so that

$$\lim_n \left\| \sum_{k=1}^{n} f_k(\cdot)g_k(t) - F(\cdot, t)\right\|_{L^q(\mu)} = 0 \quad (\text{a.e. } t \in [0,1])$$

provided that $f_k \in L^q(\mu)$, $\Sigma_k \|f_k\|_q^p < \infty$ and $p < q < 2$ or $q = 2$. Here we have replaced $L^p(\mu)$ and l^q of the corollary by l^p and $L^q(\mu)$ respectively. Other applications can be given to obtain vector valued analogues of results on permutation of functional series (like Th. 21 and Corol. 1 in [**12**]).

The next application is concerned with factorization theorems (or weighted norm inequalities). The result stated is not the more general possible, but it is indicative of the way in which Theorem 1 can be used in this direction.

COROLLARY 3. *Let T be a linear operator of weak type (p, p) in the probability space (X, μ), $0 < p < 2$. If $p < q \le 2$, and we denote $q = (1 + \alpha)p$, then for every measurable function $u(x) > 0$ such that $u^{-1} \in L^{1/\alpha}(\mu)$, we can find $v(x) > 0$* a.e. *such that T is bounded from $L^q(ud\mu)$ to $L^q(vd\mu)$.*

Proof. By Hölder's inequality $L^q(ud\mu) \subset L^p(\mu)$, and more precisely

$$\left(\int |f|^p d\mu\right)^{1/p} \le \left(\int |f|^q \, ud\mu\right)^{1/q} \left(\int u^{-1/\alpha} d\mu\right)^{\alpha/q}.$$

On the other hand, the l^q-valued extension of T is also of weak type (p, p) by Theorem 1, and we obtain

$$\left\| \left(\sum_j |Tf_j|^q \right)^{1/q} \right\|_{p-\varepsilon} \le \left(\frac{p}{\varepsilon}\right)^{1/p-\varepsilon} \left\| \left(\sum_j |Tf_j|^q \right)^{1/q} \right\|_{p^*}$$

$$\le C \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_p \le C' \left(\sum_j \int |f_j|^q ud\mu \right)^{1/q}.$$

Therefore, Maurey's factorization theorem can be used (e.g. [**11**], Th. 1) to conclude that there exists $g \in L^r(\mu)$, $1/(p - \varepsilon) = 1/q + 1/r$ such that

$$\int \left| \frac{Tf}{g} \right|^q d\mu \le C \int |f|^q ud\mu \qquad (f \in L^q(ud\mu))$$

and we only have to take $v = | g |^{-q}$.

Remark. It may be seen from the last proof that, given $\beta > \alpha$, $v(x)$ can be chosen so that $v^{-1} \in L^{1/\beta}(\mu)$ and $\| v^{-1} \|_{1/\beta} \le C_{\alpha,\beta} \| u^{-1} \|_{1/\alpha}$. When this is applied to a self-adjoint operator T on $L^2(\mu)$ which is of weak type $(1, 1)$, we obtain:

(a) If $u \subseteq L^{1/p-1}(\mu)$, $1 < p \le 2$, there exists $v \in L^r(\mu)$ (with $r < 1/(p - 1)$) such that

$$\int |Tf(x)|^p \frac{d\mu(x)}{v(x)} \le C \int |f(x)|^p \frac{d\mu(x)}{u(x)}.$$

(b) If $u \in L^1(\mu)$ and $2 \le p < \infty$, there exists $v \in L^s(\mu)$ (with $s < 1$) such that

$$\int |Tf(x)|^p u(x) d\mu(x) \le C \int |f(x)|^p v(x) d\mu(x)$$

(part (b) follows by duality). We can take as T a multiplier operator on the n-dimensional torus satisfying Marcinkiewicz' hypothesis, or the conjugate function operator in a compact connected Abelian group, extending a previous result of P. Koosis [7].

III. Translation invariant operators. Here we shall use Corollary 1 to obtain some mixed norm estimates for translation invariant operators. First of all, given measure spaces (X, μ) and (Y, ν), let us recall that, when $B = L^q(\nu)$, we can identify $L^p_B(\mu)$ with the mixed norm space $L^{q,p}(Y \times X, \nu \otimes \mu)$ (see Benedek and Panzone [**2**]), and in the same way, $L^{p^*}_B(\mu) = \text{weak-}L^p_B(\mu)$ is identified with the space $L^{q,p^*}(Y \times X, \nu \otimes \mu)$ consisting of all measurable functions $f(y, x)$ such that

$$\sup_{t>0} t^{-1}\mu\big(\{x \in X: \|f(\cdot, x)\|_q > t\}\big)^{1/p} = \|f\|_{q,p^*} < \infty.$$

LEMMA. *Let $B = L^q(\nu)$, $1 < q < \infty$. If T is a linear operator on $L^p(\mu)$ with an extension T^B on $L^p_B(\mu)$ of weak type (p, p), then, the action of T^B on $L^{q,p}(\nu \otimes \mu)$ under the above identifications is as follows*:

$$T^B f(y, x) = T(f(y, \cdot))(x) \tag{3}$$

at least when $f \in L^\infty$ with $\operatorname{supp}(f) \subset Y_0 \times X_0$, $\nu(Y_0) < \infty$, $\mu(X_0) < \infty$.

Proof. Let us assume first that $1 < p < \infty$, so that T has an adjoint T^* defined on the Lorentz space $L(p', 1)(\mu)$. If $g \in L^{q'}(\nu)$, $h \in L(p', 1)(\mu)$, we have by using (1)

$$\begin{aligned} &\int h(x)\,d\mu(x) \int T^B f(y, x) g(y)\,d\nu(y) \\ &\quad = \int h(x) T\Big(\int f(y, \cdot) g(y)\,d\nu(y)\Big)(x)\,d\mu(x) \\ &\quad = \int T^* h(x)\,d\mu(x) \int f(y, x) g(y)\,d\nu(y) \\ &\quad = \int g(y)\,d\nu(y) \int h(x) T(f(y, \cdot))(x)\,d\mu(x) \end{aligned}$$

and (3) follows. Now, if $p \le 1$, take $1 < r < q$ and change μ by a finite measure $\mu_1 \ll \mu$ supported in X_0 with $L^r(\mu_1) \subset L^p(X_0, \mu)$. Then $\|f(\cdot, x)\|_B$ belongs to $L^r(\mu_1)$, T maps $L^r(\mu_1)$ into $L^{r^*}(\mu_2)$ for another measure $\mu_2 \ll \mu$ (by Nikishin's theorem) and the preceding proof applies.

Now we consider an amenable locally compact group G with left Haar measure m, and an arbitrary measure space (Ω, μ). For functions f in $G \times \Omega$, left and right translations are defined acting only on the group variable, i.e.

$$f_y(x, \omega) = f(yx, \omega); \quad f^y(x, \omega) = f(xy, \omega) \quad (y \in G).$$

THEOREM 2. *Let T be a linear operator on $L^p(G \times \Omega, m \otimes \mu)$, $1 \le p < \infty$, invariant under right translations*: $T(f^y) = (Tf)^y$. *If T is of weak type (p, p), then it is also a bounded operator from $L^{q,p}(G \times \Omega)$ to $L^{q,p^*}(G \times \Omega)$ provided that $p < q \le 2$ or $2 \le q < p$.*

Proof. The idea is basically as in **[6]** and **[13]**. First of all, if $B = L^q(G)$, then T^B is bounded from $L^p_B(G \times \Omega)$ to $L^{p*}_B(G \times \Omega)$ (by Corollary 1). Given a function $\varphi \in L^\infty(G \times \Omega)$ with $\operatorname{supp}(\varphi) \subset K_0 \times H$, $\mu(H) < \infty$ and K_0 being a compact subset of G, it suffices to prove the estimate

$$(4) \qquad \mu\left(\left\{\omega \in \Omega : \|\chi_K T\varphi(\cdot, \omega)\|_q \geq t\right\}\right) \leq Ct^{-p} \int \|\varphi(\cdot, \omega)\|_q^p \, d\mu(\omega)$$

for arbitrary large compact subsets $K \supset K_0$. Given such a set K, take a relatively compact open set U such that

$$m(E) \leq 2m(U) \quad \text{with } E = KK^{-1}U$$

(such U exists because G is amenable; see [3]), and define $f \in L^\infty_B(G \times \Omega)$ by

$$f(x, \omega) = \chi_E(x)\varphi_x(\cdot, \omega)$$

or, with the usual identifications:

$$f(y; x, \omega) = f(x, \omega)(y) = \varphi(xy, \omega)\chi_E(x).$$

The preceding lemma can be applied to f, and yields

$$T^B f(y; x, \omega) = T(f(y; \cdot))(x, \omega) = T(\varphi\chi_{Ey \times \Omega})(xy, \omega)$$

by the translation invariance of T. Since $K_0 \subset K \subset Ey$ whenever $y \in U^{-1}K$, it follows for every $x \in U$ that

$$\|T^B f(x, \omega)\|_B \geq \left(\int_{U^{-1}K} |T\varphi(xy, \omega)|^q \, dm(y)\right)^{1/q}$$

$$\geq \|T\varphi(\cdot, \omega)\chi_K\|_q$$

Therefore, the weak type (p, p) of T^B implies

$$m(U) \cdot \mu\left(\left\{\omega \in \Omega : \|\chi_K T\varphi(\cdot, \omega)\|_q \geq t\right\}\right)$$

$$\leq m \otimes \mu\left(\left\{(x, \omega) \in G \times \Omega : \|T^B f(x, \omega)\|_B \geq t\right\}\right)$$

$$\leq Ct^{-p} \int \|f(x, \omega)\|_B^p \, dm(x) \, d\mu(\omega)$$

$$= Ct^{-p} m(E) \int \|\varphi(\cdot, \omega)\|_q^p \, d\mu(\omega)$$

and since $m(E) \leq 2m(U)$, (4) follows.

Corollary 4. *Every linear translation invariant operator of weak type (p, p) in an amenable locally compact group is of strong type (q, q) for $p < q \leq 2$ or $2 \leq q < p$.*

It suffices to take Ω reduced to one point. For $p < 2$, this was proved by Oberlin [**13**]. On the other hand, if Ω is another locally compact group, we can restate Theorem 2 in a form which is a weak type analogue of the main result in [**6**].

COROLLARY 5. *Let G be a locally compact group which is the direct product of two closed subgroups: $G = \Gamma H$, $\Gamma \cap H = \{e\}$, and assume that H is amenable. If T is a bounded linear operator from $L^p(G)$ to $L^{p^*}(G)$ commuting with right translations, and if $1 \leq p < q \leq 2$ or $2 \leq q < p < \infty$, then*

$$\left|\left\{\sigma \in \Gamma: \int_H |Tf(\sigma x)|^q dx \geq t^q\right\}\right| \leq Ct^{-p} \int_\Gamma \left(\int_H |f(\sigma x)|^q dx\right)^{p/q} d\sigma$$

where $d\sigma$ (or $|\cdot|$) and dx denote left Haar measures on Γ and H respectively).

When $G = R^{n+m} = R^n \cdot R^m$, T is a singular integral operator and $p = 1$, this result is contained in the work of Benedek, Calderon and Panzone [**1**]. Corollary 5 can also be applied with $p = 1$ when Γ and H are compact connected Abelian groups and T is the conjugate function with respect to any order defined in the dual of G.

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