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ANALYTIC AND ARITHMETIC PROPERTIES OF THIN SETS

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Let $\mathbf{Z}^+$ and $\mathbf{Z}^-$ denote the sets of positive and negative integers respectively. We study relations between various thinness conditions on subsets E of $\mathbf{Z}^+$, with particular emphasis on those conditions that imply $\mathbf{Z}^- \cup E$ is a set of continuity. For instance, if E is a $\Lambda(1)$ set, a p-Sidon set (for some $p < 2$), or a UC-set, then E cannot contain parallelepipeds of arbitrarily large dimension, and it then follows that $\mathbf{Z}^- \cup E$ is a set of continuity; on the other hand there is a set E that is Rosenthal, strong Riesz, and Rajchman, which is not a set of continuity.

1. Introduction. Let $\mathbf{T}$ be the circle group and $\mathbf{Z}$ the integers; denote by $M(\mathbf{T})$ the customary convolution algebra of Borel measures on $\mathbf{T}$, and, given $\mu \in M(\mathbf{T})$ and $n \in \mathbf{Z}$, let

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu(\theta).$$

For a subset S of $\mathbf{Z}$ with infinite complement, call S a *set of continuity* if, for each number $\varepsilon > 0$ there is a $\delta > 0$ such that, if $\mu \in M(\mathbf{T})$, $\|\mu\| \leq 1$, then the condition that

$$\limsup_{n \in \mathbf{Z} \setminus S} |\hat{\mu}(n)| < \delta \text{ implies that } \limsup_{n \in S} |\hat{\mu}(n)| < \varepsilon. \tag{1.1}$$

Less formally, S is a set of continuity if, for measures μ in the unit ball of $m(\mathbf{T})$, the size of $\limsup_{n \in S} |\hat{\mu}(n)|$ can be controlled by the size of $\limsup_{n \in \mathbf{Z} \setminus S} |\hat{\mu}(n)|$.

The definition of a set of continuity [**19**] was inspired by the theorem of K. de Leeuw and Y. Katznelson [**5**] to the effect that $\mathbf{Z}^+$ and $\mathbf{Z}^-$ are sets of continuity. Different proofs of the de Leeuw-Katznelson result were subsequently found by J. A. R. Holbrook [**22**] and L. Pigno [**34**]. Yet another proof will be presented in §2 of the present paper.

In §2, we consider analytic conditions. We first show that the theorem of de Leeuw and Katznelson is a rather direct consequence of Paley's theorem concerning the Fourier coefficients of $H^1(\mathbf{T})$-functions. We next recall the analytic conditions that E be a $\Lambda(1)$ set, a p-Sidon set ($p < 2$), or a UC-set, and we show by the same method that if E satisfies any of these conditions, then $Z^- \cup E$ is a set of continuity. We also exhibit an example of a set E which is a Rosenthal set, a strong Riesz set, and a Rajchman set, but is *not* a set of continuity; see §2 for all definitions.

In [**34**], Pigno went on to show via the construction of Cohen-Davenport that if $E \subset \mathbf{Z}^+$ satisfies lacunarity condition ($\mathfrak{L}$) (defined in §3), then $\mathbf{Z}^- \cup E$ is a set of continuity. In §3 we consider various arithmetic conditions on E and show via refinements in the original method of de Leeuw and Katznelson that these conditions imply that $\mathbf{Z}^- \cup E$ is a set of continuity. Then we show that if E is a $\Lambda(1)$ set, a p-Sidon set ($p < 2$), or a UC-set, then E satisfies such an arithmetic condition, because E cannot contain parallelepipeds of arbitrarily large dimension. As a byproduct of our analysis, we exhibit a $4/3$-Sidon set that is also a $\Lambda(q)$ set for all $q < \infty$ but is *not* a UC-set.

After completing most of our work in this paper we learned of other refinements of the method of de Leeuw and Katznelson due to F. Méla [**29**] and B. Host and F. Parreau [**23; 25**]. In particular Host and Parreau [**25**] have found an arithmetic characterization of sets of continuity. It is easy to see that if E satisfies one of our arithmetic conditions then it must satisfy the arithmetic condition of Host and Parreau. It also follows easily from our work in §2 that every $\Lambda(1)$ set, or p-Sidon set ($p < 2$), or UC-set satisfies the arithmetic condition of Host and Parreau. We have therefore concentrated in our presentation on the relations between various analytic conditions and various arithmetic conditions that are stronger than the one considered by Host and Pareau. This part of our work is essentially disjoint from their work.

In his proof [**22**] of the theorem of de Leeuw and Katznelson, Holbrook actually dealt directly with a generalization concerning contractions on a Hilbert space; he also pointed out that the generalization was really equivalent, via unitary dilations and the spectral theorem for unitary operators, to the original de Leeuw-Katznelson theorem. In §4, we show that the method of de Leeuw and Katznelson can be modified to work in the setting considered by Holbrook.

2. Analytic conditions. Call a sequence $\{h_k\}_1^\infty$ of strictly positive integers a *strong Hadamard sequence* if $h_{k+1} \geq 3h_k$ for all k; let $H^1(\mathbf{T})$ be the classical space of all functions $f \in L^1(\mathbf{T})$ such that $\hat{f}(n) = 0$ for all $n < 0$. Paley's Theorem asserts that

$$\left(\sum_{k=1}^{\infty} |\hat{f}(h_k)|^2 \right)^{1/2} \leq C \|f\|_1 \tag{2.1}$$

for all $f \in H^1(\mathbf{T})$ and all strong Hadamard Sequences $\{h_k\}_1^\infty$; it is known [**15**] that the best value of the constant C is $\sqrt{2}$.

We now derive the de Leeuw-Katznelson result from inequality (2.1).

THEOREM 0. *The sets* $\mathbf{Z}^+$ *and* $\mathbf{Z}^-$ *are both sets of continuity.*

Proof. Since $\mathbf{Z}^-$ is a set of continuity if and only if $\mathbf{Z}^+$ is a set of continuity, it is enough to show $\mathbf{Z}^+$ is a set of continuity. To show $\mathbf{Z}^+$ is a set of continuity we proceed as follows:

If f is a trigonometric polynomial on $\mathbf{T}$, then as a consequence of (2.1) we have

$$(2.2)\qquad \left(\sum_{k=1}^{\infty} |\hat{f}(h_k)|^2\right)^{1/2} \leq C\left[\|f\|_1 + \left(\sum_{n<0} |\hat{f}(n)|^2\right)^{1/2}\right]$$

for all strong Hadamard sequences $\{h_k\}_1^\infty$; to confirm this, just apply Paley's inequality to the function $f(\theta) - \Sigma_{n<0}\hat{f}(n)e^{in\theta}$. Next, fix a strong Hadamard sequence $\{h_k\}_1^\infty$ and a positive integer K, and let $R_K(\theta)$ be the standard Riesz product

$$\prod_{k=1}^{K} (1 + \cos h_k\theta);$$

given $\mu \in M(\mathbf{T})$, put

$$\mathfrak{L}_K(\mu) = \left(\sum_{n<0} |(\mu * R_K)\hat{}(n)|^2\right)^{1/2}.$$

Applying inequality (2.2) to the convolution product $\mu * R_K$ we get that

$$(2.3)\qquad \frac{1}{2}\left(\sum_{k=1}^{K} |\hat{\mu}(n)|^2\right)^{1/2} \leq C[\|\mu\| + \mathfrak{L}_K(\mu)].$$

Suppose now that $\|\mu\| \leq 1$, and that $\limsup_{n\to-\infty} |\hat{\mu}(n)| < \delta$, while $\limsup_{n\to+\infty} |\hat{\mu}(n)| > \varepsilon > 0$, where the relationship of δ to ε will be specified later. By replacing μ by $e^{im\theta}\mu$ for some integer m, we can arrange that $|\hat{\mu}(n)| < \delta$ for all $n < 0$, and we can then choose a strong Hadamard sequence $\{h_k\}_1^\infty$, so that $|\hat{\mu}(h_k)| > \varepsilon$ for all k. For each integer K, the support of $\hat{R}_K$ has fewer than 3^K negative integers; so, by inequality (2.3),

$$(2.4)\qquad \varepsilon\sqrt{K} < 2C[1 + 3^{K/2}\delta].$$

Given ε, let K be the smallest integer for which $\varepsilon \geq 4C/\sqrt{K}$, and put $\delta = 3^{-K/2}$. Then inequality (2.4) does not hold, and we conclude that, if $\mu \in M(\mathbf{T})$, $\|\mu\| \leq 1$, and $\limsup_{n\to-\infty} |\hat{\mu}(n)| < \delta$, then $\limsup_{n\to+\infty} |\hat{\mu}(n)| \leq \varepsilon$. That is, $\mathbf{Z}^+$ is a set of continuity, as in $\mathbf{Z}^-$. Our proof is complete.

We now recall various analytic conditions on subsets of $\mathbf{Z}$. Given $E \subset \mathbf{Z}$, and an integrable function f on $\mathbf{T}$, call f an *E-function* if $\hat{f}$ vanishes off E; call trigonometric polynomials that are E-functions *E-polynomials*.

Given an index $q \in (0, \infty)$ we say E is a *$\Lambda(q)$-set* if there is an $r \in (0, q)$, and a constant $C_{q,r}(E)$ so that

(2.5) $$\|f\|_q \leq C_{q,r}\|f\|_r$$

for all E-polynomials f. The smallest value of $C_{q,r}$ for which inequality (2.5) holds for all E-polynomials f is called the *$\Lambda(q, r)$ constant of E*. Given $p \in [1, 2)$, call E a *p-Sidon set* if there is a constant $C_p(E)$ so that

(2.6) $$\|\hat{f}\|_p \leq C_p\|f\|_\infty$$

for all E-polynomials f. Call the smallest such constant $C_p(E)$ the *p-Sidon constant of E*. Finally, call E a *UC-set* if, for every continuous E-function f, the Fourier series of f converges uniformly. It turns out that there is a constant K such that, for every E-polynomial f, the symmetric partial sums $S_N(f)$ of its Fourier series satisfy

(2.7) $$\|S_N(f)\|_\infty \leq K\|f\|_\infty;$$

again there is a smallest such constant K, called the *UC-constant of E*, and denoted by $\kappa(E)$.

We will discuss some examples of sets satisfying (2.5), (2.6) and (2.7) later in this section and the next section. For further information, see **[28]** and the references cited therein.

THEOREM 1. *Let E be a $\Lambda(1)$ set, a p-Sidon set ($p < 2$), or a UC-set. Then $Z^- \cup E$ is a set of continuity.*

Proof. Suppose first that E is a $\Lambda(1)$ set. Then by **[2]** there is an index q in the interval $(1, 2)$ for which E is also a $\Lambda(q)$ set. Without loss of generality assume E is an infinite subset of $\mathbf{Z}^+$. It then follows by a standard argument **[32]** that

(2.8) $$\left(\sum_{n \in E} |\hat{f}(n)|^{q'}\right)^{1/q'} \leq C\|f\|_{1/2}$$

for all $(\mathbf{Z}^- \cup E)$-functions f; here q' denotes the index conjugate to q, and C is a constant depending only on the $\Lambda(q, 1/2)$ constant of E.

Inequality (2.8) will play the same role in the present proof as Paley's theorem did in our proof of Theorem 0; here are the details of the argument which establishes (2.8). First it suffices to confirm (2.8) for all $(\mathbf{Z}^- \cup E)$-polynomials. To this end let f be such a trigonometric polynomial and put $g(\theta) = \sum_{n>0} \hat{f}(n)e^{in\theta}$. By Kolmogoroff's theorem,

$$\|g\|_{1/2} \leq C_1\|f\|_1.$$

On the other hand, the fact that E is a $\Lambda(q)$ set implies [**38**, p. 204] that versions of inequality (2.5) hold for all indices r in the interval $(0, q)$; therefore

$$\|g\|_q \leq C''\|g\|_{1/2},$$

because g is an E-polynomial. Finally,

$$\|\hat{g}\|_{q'} \leq \|g\|_q,$$

by the Hausdorff-Young theorem. As a consequence of the above three inequalities, we obtain inequality (2.8).

Now let $\mu \in M(\mathbf{T})$ with $\|\mu\| \leq 1$. We first show that the size of $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)|$ controls the size of $\limsup_{n\in E}|\hat{\mu}(n)|$. To establish this assertion, suppose that

$$\limsup_{n\in\mathbf{Z}\setminus E}|\hat{\mu}(n)| < \delta \text{ while } \limsup_{n\in E}|\hat{\mu}(n)| > \varepsilon.$$

Without loss of generality assume that $|\hat{\mu}(n)| < \delta$ for all $n \in \mathbf{Z}^+\setminus E$. Choose a strong Hadamard sequence $\{h_k\}_1^\infty$ in E so that $|\hat{\mu}(h_k)| > \varepsilon$ for all k. Given a positive integer K, let

$$\mathcal{R}_K(\mu) = \left(\sum_{n\in\mathbf{Z}^+\setminus E}\left|(\mu * R_K)\hat{\ }(n)\right|^2\right)^{1/2};$$

it now follows from (2.8) that

$$\frac{1}{2}\left(\sum_{k=1}^{K}|\hat{\mu}(h_k)|^{q'}\right)^{1/q'} \leq C\left[\|\mu\| + \mathcal{R}_K(\mu)\right].$$

Inasmuch as $\|\mu\| \leq 1$ we obtain

$$\varepsilon K^{1/q'} \leq 2C\left[1 + 3^{K/2}\delta\right].$$

Well, given ε, choose K and then δ so that the above inequality cannot hold; then the condition that

$$\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)| < \delta \text{ implies that } \limsup_{n\in E}|\hat{\mu}(n)| \leq \varepsilon.$$

In other words, we can make $\limsup_{n\in\mathbf{Z}^+}|\hat{\mu}(n)|$ small by making $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)|$ sufficiently small; but by Theorem 0, we can make $\limsup_{n\in\mathbf{Z}^-}|\hat{\mu}(n)|$ small by making $\limsup_{n\in\mathbf{Z}^+}|\hat{\mu}(n)|$ sufficiently small. Therefore $\mathbf{Z}^-\cup E$ is a set of continuity.

Suppose next that E is a p-Sidon set for some fixed index p in the interval $[1,2)$, and, as before, that E is an infinite subset of $\mathbf{Z}^+$; let

$s = 2p/(2-p)$. Then it turns out that

$$\left(\sum_{n \in E} |\hat{f}(n)|^s\right)^{1/s} \leq C \| f \|_1 \tag{2.9}$$

for all $(E \cup \mathbf{Z}^-)$-functions f, where C is a constant depending only on the p-Sidon constant of E. Once this inequality is established, the rest of the proof that $E \cup \mathbf{Z}^-$ is a set of continuity proceeds as in the case of a $\Lambda(1)$ set E.

To prove inequality (2.9), recall that it is known that such an inequality holds for all E-functions f; what is new here is the fact that the inequality holds for the larger class of $(E \cup \mathbf{Z}^-)$-functions. Now the proof in [**12**, p. 406] of (2.9) for E-functions uses a multiplier theorem of Edwards [**11**, Corollary 2.3]; to deal with $(E \cup \mathbf{Z}^-)$-functions, just use the similar multiplier theorem due to Steckin [**39**, Theorem 1].

Finally, suppose that E is an infinite UC-subset of $\mathbf{Z}^+$. Then [**31**], for each integer $N > 0$ there is a measure μ_N for which $\|\mu_N\| \leq \kappa(E)$, and

$$\hat{\mu}_N(n) = \begin{Bmatrix} 1 & \text{if } n \in E \text{ and } n \leq N \\ 0 & \text{if } n \in E \text{ and } n > N \end{Bmatrix}.$$

It then turns out that there is an integer-valued function $K(\cdot)$ so that for all $\varepsilon > 0$, and all $(E \cup \mathbf{Z}^-)$-functions f,

$$|\{n \in E: |\hat{f}(n)| > \varepsilon\}| < K(\kappa(E)\|f\|_1/\varepsilon); \tag{2.10}$$

here $|B|$ denotes the cardinality of a given set B. Again, once this inequality is established, the rest of the proof proceeds much as in the previous two cases.

To prove inequality (2.10), we use a recent result due to Pigno and Smith [**37**]. By homogeneity, (Replace f by $[f/\kappa(E)\|f\|_1]$, and ε by $\varepsilon/[\kappa(E)\|f\|_1]$) we can reduce matters to the case where $\kappa(E)\|f\|_1 = 1$. For each positive integer N, let $f_N = f * \mu_N$; then $\|f_N\|_1 \leq 1$ for all N, and

$$\hat{f}_N(n) = \begin{Bmatrix} \hat{f}(n) & \text{if } 0 < n \leq N \\ 0 & \text{if } n > N \end{Bmatrix}$$

because f is an $(E \cup \mathbf{Z}^-)$-function. Pigno and Smith exhibit a function $K(\cdot)$ so that, if $\|S_N(f)\|_1 \leq 1$ for all N, then

$$|\{n > 0: |\hat{f}(n)| > \varepsilon\}| < K(1/\varepsilon) \tag{2.11}$$

for all $\varepsilon > 0$. In their argument, however, they only use the existence of a sequence $\{f_N\}_{N=1}^{\infty}$ as above. Since f is an $(E \cup \mathbf{Z}^-)$-function with

$\kappa(E)\|f\|_1 = 1$, and since $E \subset \mathbf{Z}^+$, inequalities (2.10) and (2.11) are equivalent. This completes our proof of the theorem.

The following notion will be convenient in the rest of this section, and in the next section.

DEFINITION. Let $\mathcal{F}$ be a family of subsets of $\mathbf{Z}$. Call $\mathcal{F}$ a *uniform family of sets of continuity* if for each $\varepsilon > 0$ there is a $\delta > 0$ so that for each set S in $\mathcal{F}$ and every measure μ with $\|\mu\| \le 1$, the condition that

$$\limsup_{n \in \mathbf{Z}^+ \setminus S} |\hat{\mu}(n)| < \delta \text{ implies that } \limsup_{n \in S} |\hat{\mu}(n)| < \varepsilon.$$

Thus, to show that a given family $\mathcal{F}$ is a uniform family of sets of continuity, we have to exhibit a uniform relation between ε and δ for all sets E in $\mathcal{F}$. Let $1 < q < \infty$, and let E be a $\Lambda(q)$ set; for each r with $0 < r < q$ denote the $\Lambda(q, r)$ constant of E by $c_{q,r}(E)$, and call $c_{q,1}(E)$ the $\Lambda(q)$ *constant of* E. Recall [**38**, p. 204] that $c_{q,1/2}(E) \le [c_{q,1}(E)]^{(2q-1)/(q-1)}$. Thus the proof of Theorem 1 shows that, if $1 < q < \infty$ and $C > 0$, then the family of all $\Lambda(q)$ sets E with $c_{q,1}(E) \le C$ is a uniform family of sets of continuity, as is the family of all sets $E \cup \mathbf{Z}^-$ with E as above. Similarly, if $1 \le p < 2$ and $C > 0$, then the family of all sets $E \cup \mathbf{Z}^-$, where E is p-Sidon with p-Sidon constant at most C, is a uniform family of sets of continuity. Finally, the same conclusion holds for sets E with UC-constant at most C.

The example below shows that the family of all $\Lambda(q)$ sets *is not* a uniform family of sets of continuity. On the other hand we shall see in the next section, that, for each fixed index p with $1 \le p < 2$, the family of all sets $\mathbf{Z}^- \cup E$, where E is p-Sidon *is* a uniform family of sets of continuity. Finally, we will see in the next section that the class of sets $E \cup Z^-$, where E is a UC-set, *is* a uniform family of sets of continuity.

EXAMPLE 1. Fix a strong Hadamard sequence $\{h_k\}_1^\infty$. For each positive integer N, let E_N consist of all integers n of the form $n = \sum_1^\infty \varepsilon_k h_k$, where $\varepsilon_k \in \{-1, 0, 1\}$ for all k, and $\sum_{k=1}^\infty |\varepsilon_k| \le N$. Then [**28**, p. 65] each set E_N is a $\Lambda(q)$ set for all $q < \infty$; moreover, E_N is $[2N/(N+1)]$-Sidon [**1**]. A standard construction [**28**, p. 20] yields a probability measure μ so that, if $n = \sum_{k=1}^\infty \varepsilon_k h_k$ as above, then $\hat{\mu}(n) = 2^{-\sum|\varepsilon_k|}$, and so that $\hat{\mu}(n) = 0$ otherwise. A simple modification of the construction [**10**, p. 163] yields a measure ν, with $\|\nu\| = 1$, so that $\hat{\nu}(n) = \hat{\mu}(n)$ if n is as above, and $\sum_{k=1}^\infty \varepsilon_k = 1$, while $\hat{\nu}(0) = 0$ otherwise. In particular, $\hat{\mu}$ and $\hat{\nu}$ both vanish off the set $\{0\} \cup \bigcup_{N=1}^\infty E_N$; moreover,

$$\limsup_{n \in E_N} |\hat{\nu}(n)| = \frac{1}{2}, \text{ while } \limsup_{n \in \mathbf{Z}^+ \setminus E_N} \hat{\nu}(n)| \le \left(\frac{1}{2}\right)^{N+1}.$$

Thus the sets E_N do not form a uniform family of sets of continuity, although each E_N is a $\Lambda(q)$ set for all $q < \infty$, and a p-Sidon set for some $p < 2$.

We now consider some other analytic notions of thinness. Call a subset E of $\mathbf{Z}$ a *Rosenthal set* if every E-function in $L^\infty(\mathbf{T})$ is actually continuous. Call a measure μ an *E-measure* if $\hat{\mu}$ vanishes off E. Call E a *Riesz set* if every E-measure is actually absolutely continuous; denote the closure of E in the Bohr compactification of $\mathbf{Z}$ by $\text{cl}(E)$, and call E a *strong Riesz set* if $\text{cl}(E) \cap \mathbf{Z}$ is a Riesz set. Call E a *weak Rajchman set* if every E-measure μ has the property that $\lim_{n \in E} \hat{\mu}(n) = 0$; call E a *Rajchman set* if, for all measures μ, the condition that $\lim_{n \in \mathbf{Z} \setminus E} |\hat{\mu}(n)| = 0$ implies that $\lim_{n \in E} |\hat{\mu}(n)| = 0$.

Concerning these notions see [**28**, pp. 161–163] and [**33**]. Clearly, every set of continuity is a Rajchman set, and Rajchman implies weak Rajchman; also, strong Riesz implies Riesz, and Riesz implies weak Rajchman. It is also known [**9**] that if E is Rosenthal, then $\mathbf{Z}^- \cup E$ is Riesz, as is $F \cup E$ for all Riesz sets F. Finally, Host and Parreau [**24**; **23**] have given two proofs that weak Rajchman actually implies Rajchman.

EXAMPLE 2. Let $h_k = 5^k$ for all k, and form the sets E_N as in the previous example; then let

$$E = \bigcup_{N=1}^{\infty} \left(5^N + 5^{N+1} l E_N\right).$$

Then the set E is Rosenthal, strong Riesz, and Rajchman, but it is *not* a set of continuity.

Indeed, suppose, to force a contradiction, that E is a set of continuity; then its subsets will form a uniform family of sets of continuity. Consider the sets $5^N + 5^{N+1}E_N = F_N$, say; each F_N is a set of continuity with exactly the same relation between ε and δ as E_N, because translation and dilation do not affect this relation. Since the sets E_N do not form a uniform family of sets of continuity, neither do the F_N, and E cannot be a set of continuity.

To see that E is a Rosenthal set, we consider its closure, F say, in the dual of the subgroup of $\mathbf{T}$ given by

$$D = \{\exp(\pi i k/5^n) \colon k \in Z, n \in \mathbf{Z}^+\}.$$

Since D is countable, its dual $\hat{D}$ is metrizable, and any element y of F must be the limit, in $\hat{D}$, of some sequence $\{y_m\}_{m=1}^{\infty}$ taking all its values in E. We suppose first that there is no index N for which $Y_m \in F_N$ for infinitely many values of m; then for each N, we have for all sufficiently large values of m that 5^N divides y_m. We therefore have for each element z of D that $z^{y_m} = 1$ for all sufficiently large values of m, and hence that $y = \lim_{m\to\infty} y_m = 0$ in $\hat{D}$. Next we suppose instead that there is an index N for which $y_m \in F_N$ for infinitely-many values of m; then, by a similar argument, $y = \lim_{m\to\infty} y_m \in F_N$ also. Therefore, the closure F of E coincides with $E \cup \{0\}$, and, in particular, F is countable, and hence residual in $\hat{D}$. By the main theorem of [**35**], the set E must be Rosenthal.

It now follows that E is a Riesz set, because every Rosenthal set is a Riesz set [**9**]. To see that E is a *strong* Riesz set, we show that $\mathrm{cl}(E) \cap Z \subset E \cup \{0\}$: Indeed, if $n \in \mathrm{cl}(E) \cap Z$, then n is the limit, in $b\mathbf{Z}$ the Bohr compactification of $\mathbf{Z}$, of some net (n_α) taking values in E; but then (n_α) also converges to n in $\hat{D}$, whence $n \in F = E \cup \{0\}$. So, $\mathrm{cl}(E) \cap \mathbf{Z}$ is a Riesz set, and E is a strong Riesz set.

Since E is a Riesz set, it is a weak Rajchman set, and by [**23**] also a Rajchman set. In [**25**, Chap. 5] Host and Parreau independently consider a class of examples similar to the one above. and then show that their sets are Riesz and Rajchman, but not sets of continuity. This completes our discussion of this example.

REMARK 1. The proof of Theorem 1 shows that $E \cup \mathbf{Z}^-$ is a set of continuity whenever inequalities like (2.8), (2.9), and (2.10) hold for all $(E \cup \mathbf{Z}^-)$-polynomials. This suggests considering possibly more general classes of sets E for which such inequalities merely hold for all E-polynomials; in the case of inequality (2.9), for instance, such sets have been considered in [**3**] and [**6**], but the only examples known, so far, of such sets are $\Lambda(q)$ sets, with $q > 2$. It is true, however, that if E is such a set, that is if inequality (2.9) holds for some $s < \infty$ and all E-polynomials, then $E \cup \mathbf{Z}^-$ is a set of continuity. The reason is that one can show that inequalities of the form (2.10) then hold for all $(E \cup \mathbf{Z}^-)$-polynomials. In fact, in showing this, one only needs a priori that inequalities of the form (2.10) hold for all E-polynomials; see [**16**, Remark 3] for more details.

3. Arithmetic conditions. In this section, we first modify the method of de Leeuw and Katznelson to show that various arithmetic thinness conditions on a set E imply that $E \cup \mathbf{Z}^-$ is a set of continuity. Later in the section, we shall compare these conditions with each other, with other arithmetic conditions, and with analytic conditions; this will allow us to answer several open questions about UC-sets.

Our first two arithmetic conditions concern the sets

$$\limsup_{j\to\infty} (E - n_j) \equiv \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (E - n_j)$$

for various sequences $(n_j)_{j=1}^{\infty}$. We note that, in the sequel, we could require that $n_j \in E$ for all j, because, if $\limsup_{j\to\infty}(E - n_j)$ is nonempty, with an element m say, then $n_j \in E - m$ for all sufficiently large values of j.

DEFINITION. Call a sequence $(n_j)_{j=1}^{\infty}$ *injective* if $n_i \neq n_j$ whenever $i \neq j$. Say that a subset E of $\mathbf{Z}$ satisfies *lacunarity condition* $(\mathcal{L})$ if the set

$$F = \liminf_{j\to\infty}(E - n_j)$$

is finite for all injective sequences $(n_j)_{j=1}^{\infty}$. Say that E satisfies *lacunarity condition* $(\mathcal{O})$ if, for all such sequences $(n_j)_{j=1}^{\infty}$, the set F has the property that at least one of $F \cap \mathbf{Z}^+$ and $F \cap \mathbf{Z}^-$ is finite.

Our next two conditions concern the set E^a of all accumulation points of E in $b\mathbf{Z}$, the Bohr compactification of $\mathbf{Z}$.

DEFINITIONS. Say that E satisfies *condition* $(\mathcal{M})$ if $E^a \cap \mathbf{Z}^-$ is finite, and that E satisfies *condition* $(\mathcal{Q})$ if $E^a \cap E$ is finite.

THEOREM 2. *Let E be a subset of $\mathbf{Z}$ that satisfies one of the conditions $(\mathcal{O})$, $(\mathcal{M})$, or $(\mathcal{Q})$. Then $E \cup \mathbf{Z}^-$ is a set of continuity.*

Proof. We need two lemmas. The first one is due to de Leeuw and Katznelson [5].

LEMMA 1. *For each $\varepsilon > 0$ there is a $\delta > 0$ with the following property. Let X be a set, let μ be a measure on some sigma-algebra of subsets of X, with $\|\mu\| \leq 1$, and let ϕ be a function in $L^\infty(d|\mu|)$ with $\|\phi\|_\infty \leq 1$; then the condition that*

$$\left|\int_X |\phi|^{2m}\phi \, d\mu\right| \leq \delta \text{ for all integers } m \geq 1$$

implies that $|\int_X \phi \, d\mu| \leq \varepsilon$.

For proofs of this lemma, see [5] and [29]. We shall prove a generalization of the lemma in §4, and comment there on various methods of proof, and on the relation of δ to ε.

The other part of the method of de Leeuw and Katznelson is a limiting argument, which we reformulate as Lemma 2 below. Given an odd integer $J > 1$, we define a *standard multi-index of length J* to be a sequence $(\alpha_j)_{j=1}^{J}$ such that $\alpha_j \in \{-1, 1\}$ for all j, and such that $\sum_{j=1}^{J}\alpha_j = 1$. For typographical convenience, we shall sometimes write integer-valued sequences in the form $h = (h(k))_{k=1}^{\infty}$. Given such a standard multi-index

α, such a sequence h, and a nonnegative function Φ on $\mathbf{Z}$, we denote the iterated limit

$$\limsup_{k_1\to\infty}\left\{\limsup_{k_2\to\infty}\left\{\cdots\left\{\limsup_{k_J\to\infty}\Phi\left(\sum_{j=1}^{J}\alpha_j h(k_j)\right)\right\}\cdots\right\}\right\}$$

by $\limsup_{(\alpha,h)}\Phi$.

LEMMA 2. *For each* $\varepsilon>0$ *there is a* $\delta>0$ *with the following property. Let* $\mu\in M(\mathbf{T})$, *with* $\|\mu\|\le 1$, *and let* $h=(h(k))_{k=1}^{\infty}$ *be an integer-valued sequence. Suppose that for each odd integer* $J>1$ *there is a standard multi-index* α *of length* J *for which* $\limsup_{(\alpha,h)}|\hat{\mu}|\le\delta$; *then* $\limsup_{k\to\infty}|\hat{\mu}(h(k))|\le\varepsilon$.

We will prove this lemma, and discuss it further, after the proof of Theorem 2. In proving the theorem, we may assume that E is an infinite subset of $\mathbf{Z}^+$. As in the proof of Theorem 1, we need only show that $\limsup_{n\in E}|\hat{\mu}(n)|$ can be controlled, uniformly for all μ in $M(\mathbf{T})$ with $\|\mu\|\le 1$, by $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)|$, and this is what we do in the cases where E satisfies one of the conditions $(\mathcal{O})$ or $(\mathcal{Q})$. Fix an infinite subset E of $\mathbf{Z}^+$ that satisfies condition $(\mathcal{O})$, and a measure μ in $M(\mathbf{T})$ with $\|\mu\|\le 1$. Given $\varepsilon>0$, choose $\delta>0$ as in Lemma 2, and suppose that $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)|\le\delta$. Choose a strong Hadamard sequence $h=(h(k))_{k=1}^{\infty}$ in E so that

$$|\hat{\mu}(h(k))|\to\limsup_{n\in E}|\hat{\mu}(n)| \text{ as } k\to\infty. \tag{3.1}$$

By passing to a subsequence of h we can demand that

$$F=\lim_{k\to\infty}(E-h(k))$$

exist. By hypothesis, at least one of the sets $F\cap\mathbf{Z}^-$ and $F\cap\mathbf{Z}^+$ is finite.

Suppose first that $F\cap\mathbf{Z}^-$ is finite. By replacing the sequence h by one of its tails, we can arrange that $-h(1)<m$ for all m in F. Now $h(1)-h(2)<-h(1)$, because h is strong Hadamard; so $h(1)-h(2)\notin F$. By passing to a subsequence of h, we can require that $h(1)-h(2)\notin E-h(3)$, that is that $h(1)-h(2)+h(3)\notin E$. We continue to refine the sequence h in this fashion. Specifically, given $(h(k))_{k=1}^{m}$, let L_m be the set of all integers having the term

$$h(k_1)-h(k_2)+h(k_3)-\cdots+h(k_{j-1})-h(k_j),$$

where j is even, the signs alternate in the pattern $+,-,\ldots,+,-$, and $1\le k_1<k_2<\cdots<k_j\le m$. Since the sequence h is strong Hadamard, the set L_m is disjoint from F, and we can then choose $h(m+1)$ so that the

set $h(m+1)+L_m$ is disjoint from E; on the other hand, $h(m+1)+L_m$ is a subset of $\mathbf{Z}^+$.

In other words, if α is a standard multi-index of the form $(1,-1,\ldots,-1,1)$, and if $k_1<k_2<\cdots<k_J$, then

$$\alpha_1 h(k_1)+\alpha q_2 h(k_2)+\cdots+\alpha_J h(k_J)\in\mathbf{Z}^+\setminus E.$$

If $k_1<k_2<\cdots<k_{J-1}$, then

$$\limsup_{k_J\to\infty}\left|\hat{\mu}\left(\sum_{j=1}^{J}\alpha_j h(k_j)\right)\right|\le\delta.$$

Therefore, $\limsup_{(\alpha,h)}|\hat{\mu}|\le\varepsilon$, and, by Lemma 2, $\limsup_{k\to\infty}|\hat{\mu}(h(k))|\le\varepsilon$, as required.

The remaining possibility is that the set $F\cap\mathbf{Z}^+$ is finite. By replacing the sequence h by one of its tails, we can demand in this case that $h(1)>n$ for all n in F. Thus $h(2)-h(1)>n$, and we can pass to a subsequence of h for which $-h(1)+h(2)+h(3)\notin E$. We continue to refine the sequence h in this fashion. Specifically, having chosen $h(1)$, $h(2),\ldots,h(m)$, we will find that the set $-L_m$ is disjoint from F, and we can then choose $h(m+1)$ so that $h(m+1)-L_m$ is disjoint from E; moreover $h(m+1)-L_m$ will be a subset of $\mathbf{Z}^+$. It will then follow, as before, that if α is a standard multi-index of the form $(-1,1,-1,1,\ldots,-1,1,1)$, then $\limsup_{(\alpha,h)}|\hat{\mu}|\le\varepsilon$, and, by Lemma 2, that $\limsup_{k\to\infty}|\hat{\mu}(h(k))|\le\varepsilon$.

As noted above, we follow a similar procedure if E satisfies condition $(\mathcal{Q})$. That is, we choose a strong Hadamard sequence, h with values in E, so that relation (3.1) holds; we then refine h so that $h(m+1)+L_m$ is disjoint from E for all m, and we apply Lemma 2.

In the remaining case, when E satisfies condition $(\mathcal{M})$, we show that $\limsup_{n\in\mathbf{Z}^-}|\hat{\mu}(n)|$, rather than $\limsup_{n\in E}|\hat{\mu}(n)|$, can be controlled by $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)|$. To this end, we choose a sequence h, with values in $\mathbf{Z}^-$, so that $-h$ is strongly Hadamard, and so that

$$|\hat{\mu}(h(k))|\to\limsup_{n\in\mathbf{Z}^-}|\hat{\mu}(n)| \text{ as } k\to\infty.$$

We can then refine h so that the positive integers of the form

$$h(k_1)+h(k_2)-h(k_3)+\cdots+h(k_{J-1})-h(k_J),$$

where $k_1<k_2<\cdots<k_J$, all lie outside the set E. Finally, we apply Lemma 2 with multi indices α of the form $(1,1,-1,\ldots,1,-1)$. This completes our proof of the theorem.

Proof of Lemma 2. Given ε, choose δ as in Lemma 1. Let $\beta=\limsup_{k\to\infty}|\hat{\mu}(h(k))|$; by passing to a subsequence of h, we can require that $|\hat{\mu}(h(k))|\to\beta$ as $k\to\infty$. Let $\phi_k(t)=\exp(-ih(k)t)$ for all real t; by passing to a further subsequence of h, we can make the sequence $(\phi_k)_{k=1}^{\infty}$ converge weakly in $L^2(d|\mu|)$, to a function ϕ say. Then $|\int \phi d\mu| = \beta$, and the desired estimate that $\beta \leq \varepsilon$ will follow from Lemma 1 provided we can confirm, for all integers $m \geq 1$, that $|\int |\phi|^{2m}\phi \, d\mu| \leq \delta$.

To do this, we use the fact that there must be a sequence $(\psi_k)_{k=1}^{\infty}$ so that each ψ_k lies in the convex hull of $\{\phi_j\}_{j\geq k}$, and so that $(\psi_k)_{k=1}^{\infty}$ converges strongly in $L^2(d|\mu|)$, to ϕ. Given an integer m as above, let α be a standard index of length $J = 2m + 1$ for which $\limsup_{(\alpha, h)} |\mu| \leq \delta$. Since $\|\psi_k\|_\infty \leq 1$ for all k, and $\psi_k \to \phi$ in $|\mu|$-measure, we have that $\int |\phi|^{2m}\phi \, d\mu$ is equal to

(3.2)

$$\lim_{k_1\to\infty}\left\{\lim_{k_2\to\infty}\cdots\left\{\lim_{k_J\to\infty}\int|\psi_{k_1}|\cdots|\psi_{k_J}|(\operatorname{sgn}\psi_{k_1})^{\alpha_1}\cdots(\operatorname{sgn}\psi_{k_J})^{\alpha_J}d\mu\right\}\right\}.$$

In fact, the value of this iterated limit is independent of the order in which the limits are computed, and different orders of iteration lead to different estimates for the quantity (3.2), in terms of $\limsup_{n\in S}|\hat{\mu}(n)|$ for various sets S; this is how it is possible to relate quantities like $\limsup_{n\to+\infty}|\hat{\mu}(n)|$ and $\limsup_{n\to-\infty}|\hat{\mu}(n)|$.

The integral inside the iterated limit (3.2) is a convex combination of terms of the form

$$\int \phi_{j_1}^{\alpha_1}\phi_{j_2}^{\alpha_2}\cdots\phi_{j_J}^{\alpha_J}d\mu,$$

that is, of the terms $\hat{\mu}(\alpha_1 h(j_1) + \alpha_2 h(j_2) + \cdots + \alpha_J h(j_J))$, where $j_1 \geq k_1$, $j_2 \geq k_2, \ldots$, and $j_J \geq k_J$. It follows from this fact, and our hypothesis that $\limsup_{(\alpha, h)}|\hat{\mu}| \leq \delta$ that $|\int |\phi|^{2m}\phi \, d\mu| \leq \delta$. This completes our proof of the lemma.

Before introducing more arithmetic conditions, we compare the ones we have mentioned so far with each other and with the analytic conditions of §2. In **[34]**, Pigno pointed out that all 1-Sidon sets satisfy condition ($\mathfrak{L}$), and he used the Cohen-Davenport Procedure to show that, if E satisfies condition ($\mathfrak{L}$), then $E \cup \mathbf{Z}^-$ is a set of continuity; it had been shown earlier **[19]** that the union of a Sidon set and a set of continuity is always a set of continuity. Condition ($\mathcal{O}$) seems worth mentioning in Theorem 2, because it is easy to devise examples of sets satisfying condition ($\mathcal{O}$) that do not satisfy the stronger condition ($\mathfrak{L}$). On the other hand, the positive part of the set E_2 considered in Example 1 does not satisfy condition ($\mathcal{O}$), although it is a *UC*-set **[41]**, a (4/3)-Sidon set, and a $\Lambda(q)$ set for all $q < \infty$.

Condition ($\mathfrak{M}$) arises in the study **[30]** of strong Riesz sets; many interesting sets, such as the set of all prime powers are known **[8]** to satisfy this condition. Condition $(\mathcal{Q})$ arises in the study of certain sets of interpolation; indeed **[36]**, if $L^1(\mathbf{R})\hat{}|_E \subset l^1(\mathbf{T})\hat{}|\, E$, then $E^a \cap E$ is empty. It is known **[43]**, **[27]** that there are UC-sets and sets that are $\Lambda(q)$ for all $q < \infty$, that are dense in $b\mathbf{Z}$. It does not seem to be known whether a set that is p-Sidon for some $p < 2$ can be dense in $b\mathbf{Z}$.

None of the conditions $(\mathcal{O})$, $(\mathcal{M})$ and $(\mathcal{Q})$ implies any of the others. First, the sets constructed in [7, Lemma 2] have no integer accumulation points, and hence satisfy $(\mathcal{M})$ and $(\mathcal{Q})$, but it is easy to see that they do not satisfy $(\mathcal{O})$. In Example 4, at the end of this section, we will exhibit sets that satisfy $(\mathcal{M})$ but not $(\mathcal{Q})$, other sets that satisfy $(\mathcal{Q})$ but not $(\mathcal{M})$, and a set satisfying the classical Faber gap condition, and hence condition $(\mathcal{L})$, but which is dense in $b\mathbf{Z}$.

The sets E_N of Example 2 suggest the following weakening of condition $(\mathcal{L})$.

DEFINITIONS. Say that a subset E of $\mathbf{Z}$ satisfies *lacunarity condition* $(\mathcal{L}_1)$ if it satisfies condition $(\mathcal{L})$. Define a sequence of *lacunarity conditions* $(\mathcal{L}_N)_{N=2}^{\infty}$ by saying that E satisfies condition $(\mathcal{L}_N)$ if the set $\limsup_{j\to\infty}(E - n_j)$ satisfies condition $(\mathcal{L}_{N-1})$ for all injective sequences $(n_j)_{j=1}^{\infty}$.

THEOREM 3. *Let N be a positive integer. Then the sets $E \cup \mathbf{Z}^-$, where E satisfies condition $(\mathcal{L}_N)$, form a uniform family of sets of continuity.*

Proof. We can proceed much as in the proof of Theorem 2. For instance, given an infinite subset E of $\mathbf{Z}^+$ that satisfies conditions $(\mathcal{L}_N)$, and a measure μ with $\|\mu\| \le 1$, we choose a strong Hadamard sequence h in E along which $|\hat{\mu}|$ converges to $\limsup_{n\in E}|\hat{\mu}(n)|$. We suppose that $\limsup_{n\in\mathbf{Z}^+\setminus E}|\hat{\mu}(n)| \le \delta$, and we show that, if α is a standard multi-index of length $J > N$, and having the form $(1, -1, 1, \ldots)$, then $\limsup_{(\alpha, h)}|\hat{\mu}| \le \delta$. We then use a version of Lemma 2 in which the existence of suitable multi-indices α is only assumed when $J > N$; these versions of Lemma 2, with a relation between ε and δ determined by N, follow from corresponding versions of Lemma 1, which we will discuss in §4. We can deal with $\limsup_{n\in\mathbf{Z}^-}|\hat{\mu}(n)|$ via Theorem 0, or obtain direct estimates for this quantity as in the last part of the proof of Theorem 2.

Finally, we remark that another way to prove the present theorem is to imitate a procedure that is standard **[30]** in the analysis of Riesz sets. Specifically, in order that $E \cup \mathbf{Z}^-$ be a set of continuity it suffices that $E^a \cap \mathbf{Z}$ be a set of continuity, or that the various sets $\limsup_{j\to\infty}(E - n_j)$, for all injective sequences $(n_j)_{j=1}^{\infty}$, form a uniform family of sets of continuity. We omit the details of this second proof of the theorem.

We obtained Theorems 2 and 3 before we learned of the recent work by Host and Parreau [**23**; **25**]. Their arithmetic characterization of sets of continuity holds in all discrete abelian groups. Here is a description of it in the context of **Z**. Given any injective sequence $h = (h_k)_{k=1}^{\infty}$, define corresponding sets $E_N(h)$ as in Example 1.

DEFINITION. Say that a subset S of **Z** satisfies *condition HP* if there is an integer N for which, for every injective sequence h, every translate of $E_N(h)$ intersects $\mathbf{Z} \setminus S$.

In checking to see whether a given set satisfies this condition, it is enough to look at translates of the sets $E_N(h)$ when h is strong Hadamard. It is easy to see that if S is a set of continuity, then S satisfies condition HP; Host and Parreau proved the converse.

Theorems 2 and 3 follow from this result, because it is easy to verify that if E satisfies the hypotheses of Theorem 2 or 3, then $E \cup \mathbf{Z}^-$ satisfies condition HP; in fact, Host and Parreau carry out such a verification in the case where $E^a \cap \mathbf{Z}$ is finite [**25**, Chap. 5]. Theorem 1 also follows from Host and Parreau's characterization, but this is less obvious; one way to carry out the deduction runs via inequalities (2.8), (2.9), and (2.10), and another is to use Theorem 4 below.

The main result of Host and Parreau does not seem accessible by our methods, but they mention something similar that does follow from Lemma 2. Let us say that a subset S of **Z** satisfies *condition FP_N* if, for every injective sequence h, every translate of $E_N(h) \setminus E_{N-1}(h)$ intersects $\mathbf{Z} \setminus S$. Then using versions of Lemma 2 of the present paper, or, for that matter, Lemma 2 of [**5**], one can show that, for each fixed positive integer N, the sets that satisfy condition FP_N form a uniform family of sets of continuity. Note, however, that there are sets [**25**, Chap. 5] that satisfy condition HP, but satisfy condition FP_N for no value of N.

We now consider some arithmetic conditions that turn out to follow from the analytic conditions used in Theorem 1.

DEFINITIONS. Let N be a positive integer. Call a subset P of **Z** a *parallelepiped of dimension N* if P has exactly 2^N elements and can be represented as a sum $P_1 + P_2 + \cdots + P_N$ of N two-element sets. Call a pair of subsets Q and R of **Z** *an alternating pair of size N* if they both have exactly N elements, and if, when they are enumerated in increasing order as $\{q_n\}_{n=1}^{N}$ and $\{r_n\}_{n=1}^{N}$, respectively, it is the case that

$$q_2 - q_1 \leq r_2 - r_1 < q_3 - q_1 \leq r_3 - r_1 < \cdots .$$

THEOREM 4. *Let E be a $\Lambda(1)$ set, a p-Sidon for some $p < 2$, or a UC-set. Then there is an integer N for which E contains no parallelepiped of dimension N; moreover, E satisfies condition $(\mathfrak{L}_N)$ for this value of N. If E is a p-Sidon set, with $p < 2$, then there is an integer I determined by p, and independent of E, so that E satisfies condition $(\mathfrak{L}_I)$. Finally, if E is a UC-subset of $\mathbf{Z}^+$ or $\mathbf{Z}^-$, then there is an integer M for which E contains no difference $Q - R$ arising from an alternating pair (Q, R) of size M.*

COROLLARY 1. *The sets $E \cup \mathbf{Z}^-$, where E is a UC-set, form a uniform family of sets of continuity; so do the sets $E \cup \mathbf{Z}^-$, where E is a p-Sidon set for some fixed value of $p < 2$.*

Proofs. Suppose first that E is a p-Sidon set with $1 \le p < 2$; let I be the smallest integer such that $I > (2 - p)$. Then it is known **[26]** that there is an integer M, depending on E, so that E contains no sum $S_1 + S_2 + \cdots + S_I$ of M-element sets S_i. It follows easily that E satisfies condition $(\mathfrak{L}_I)$; the part of Corollary 1 concerning p-Sidon sets then follows by Theorem 3. Finally, let L be the smallest integer for which $2^L \ge M$; then E cannot contain a parallelepiped of dimension $N = IL$.

Suppose next that E is a $\Lambda(1)$-set. Fix an index q in the interval $(1, 2]$ for which E is also a $\Lambda(q)$ set. Recall that

$$\left(\sum_{n\in E} |\hat{f}(n)|^{q'}\right)^{1/q'} \le c_{q,r}(E)\|f\|_r \tag{3.3}$$

for all E-polynomials f, and all indices r satisfying $0 < r < q$. Let $g(t) = 1 + e^{it}$; we shall see below that $\|g\|_r \to 1$ as $r \to 0$. Assuming this to be so, we can choose an index r in the interval $(0, q)$ for which $\|g\|_r < 2^{1/q'}$. Then

$$2^{N/q'} > c_{q,r}(E)(\|g\|_r)^N \tag{3.4}$$

for all sufficiently large values of N. We claim that if inequality (3.4) holds, then E contains no parallelepiped of dimension N.

To verify this claim, suppose to the contrary that E does contain such a parallelepiped $P = P_1 + P_2 + \cdots + P_N$ of dimension N; denote the two elements of each set P_n by k_n and m_n. Given real numbers t_n and θ, let

$$F_n(t_n, \theta) = \exp(ik_n\theta) + \exp[i(m_n\theta + t_n)];$$

let $t = (t_1, t_2, \ldots, t_N)$, and

$$f_t(\theta) = \prod_{n=1}^{N} F_n(t_n, \theta).$$

Then for each fixed value of t, the function $\theta \mapsto f_t(\theta)$ is a P-polynomial, and hence an E-polynomial; also $|\hat{f}_n(m)| = 1$ for all m in P, whence

$$(3.5) \qquad \left(\sum_{m \in E} |\hat{f}_t(m)|^{q'} \right)^{1/q'} = 2^{N/q'}.$$

On the other hand, by Fubini,

$$\frac{1}{2\pi}\int_0^{2\pi} \cdots \left(\frac{1}{2\pi}\int_0^{2\pi} |f_t(\theta)|^r \, d\theta \right) dt_1 dt_2 \cdots dt_N$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\cdots \frac{1}{2\pi}\int_0^{2\pi} |f_t(\theta)|^r \, dt_1 \cdots dt_N \right) d\theta = \left[(\| g \|_r)^r \right]^N.$$

Therefore, there is a choice of t for which $(\| f_t \|_r)^r \le (\| g \|_r)^{rN}$; for any such t, relations (3.3) and (3.5) yield that

$$2^{N/q'} \le c_{q,r}(E)(\| g \|_r)^N,$$

contrary to inequality (3.4). Hence E contains no parallelepiped of dimension N; it follows easily that E satisfies condition $(\mathfrak{L}_N)$.

We still have to verify that $\| g \|_r \to 1$ as $r \to 0$. Now it is known [**20**, Theorem 187] that, if $\| h \|_r < \infty$ for some $r > 0$, then

$$\lim_{r \to 0} \| h \|_r = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log | h(\theta) | \, d\theta \right),$$

the so-called *geometric mean* of h. On the other hand, the function g is the restriction to $\mathbf{T}$ of the analytic function $z \mapsto 1 + z$, which has no zeros in the open unit disc U; therefore the function $z \mapsto \log | 1 + z |$ is harmonic in U, and

$$\frac{1}{2\pi}\int_0^{2\pi} \log | g(\theta) | \, d\theta = \log | 1 + 0 | = 0.$$

Hence the geometric mean of g is equal to 1, as required.

Suppose next that E is a UC-subset of $\mathbf{Z}^+$, with UC-constant $\kappa(E)$. Let M be the smallest integer for which $\log(M/2) \ge \pi\kappa(E)$. We claim that E contains no difference arising from an alternating pair of size M.

To verify the claim, suppose to the contrary that E contains such a difference $Q - R$, with Q and R enumerated as in the definition of alternating pair. Consider the Hilbert matrix $(A_{m,n})_{m,n=1}^M$ given by

$$A_{m,n} = \begin{cases} 0 & \text{if } m = n \\ \dfrac{1}{m - n} & \text{if } m \ne n \end{cases}.$$

Given θ in the interval $[0, 2\pi]$ let $v(\theta)$ and $w(\theta)$ be the vectors in $\mathbf{C}^M$ with components $v_n(\theta) = \exp(iq_n\theta)$, and $w_n(\theta) = \exp(ir_n\theta)$, respectively, and let

$$f(\theta) = (v, Aw) = \sum_{m,n=1}^{M} A_{m,n} v_m(\theta)\overline{w_n(\theta)}.$$

Then f is a $(Q - R)$-polynomial, and hence an E-polynomial; moreover,

$$\| f \|_\infty \leq \pi(\| v \|_2)(\| w \|_2) = \pi M,$$

because the operator on l^2 with matrix A has norm at most π [**20**, Theorem 294].

Let $N = q_1 - r_1$, and observe that

$$q_m - r_n = (q_m - q_1) - (r_n - r_1) + N.$$

Therefore, since the pair (Q, R) alternates, $q_m - r_n > N$ if $m > n$, while $q_m - r_n < N$ if $m < n$; moreover $q_m - r_n > 0$ even in the second case, because $E \subset \mathbf{Z}^+$. Hence

$$S_N(f)(\theta) = \sum_{1 \leq m < n \leq M} \left(\frac{1}{m - n}\right) \exp[i(q_m - r_n)\theta];$$

in particular, the nonzero Fourier coefficients of $S_N(f)$ are all negative, and

$$\| S_N(f) \|_\infty = \left| \sum_{1 \leq m < n \leq M} \frac{1}{m - n} \right| = M\left(\sum_{k=2}^{M-1} \frac{1}{k} \right) + 1 > M \log(M/2).$$

Since $\| f \|_\infty \leq \pi M$, it follows that

$$\| S_N(f) \|_\infty > \frac{1}{\pi}[\log(M/2)] \| f \|_\infty \geq \kappa(E) \| f \|_\infty,$$

contrary to the definition of $\kappa(E)$. Hence E contains no such difference of an alternating pair of size M.

It follows easily from this property of E that $E \cup \mathbf{Z}^-$ satisfies condition FP_3. This accounts for the part of Corollary 1 concerning UC-sets, because, in discussing properties of $E \cup \mathbf{Z}^-$ one may assume that $E \subset \mathbf{Z}^+$.

Finally, to see that a UC-set E cannot contain parallelepipeds of arbitrarily large dimension, observe first that every parallelepiped of dimension 2 either contains at least 2 elements of $\mathbf{Z}^+$ or contains at least 2 elements of $\mathbf{Z}^-$; it follows that if E contains a parallelepiped of dimension $4N$, then either its positive part or its negative part contains a parallelepiped of dimension N. It thus suffices to deal with the case where $E \subset \mathbf{Z}^+$, and in that case E cannot contain differences of arbitrarily large alternating pairs. Any such set E, however, cannot contain parallelepipeds of arbitrarily large dimensions. Indeed every parallelepiped of dimension $2M - 2$ contains a difference of alternating pairs of size M; to verify this, let $P = P_1 + P_2 + \cdots + P_{2M-2}$ be such a parallelepiped, where for each index i, $P_i = \{k_i, m_i\}$, with $k_i < m_i$. Let $P_i' = P_i - k_i$, and let $P' = \sum_{i=1}^{2M-2} P_i'$. Then P' is a translate of P, and it suffices to show that P' contains a difference of an alternating pair of size M. In other words, it suffices to deal with the case where $k_i = 0$ for all i. In that case, assume without loss of generality that $m_i \leq m_{i+1}$ for all i. Then, for each index $n \leq M$, let $q_n = \sum_{i<n} m_{2i-1}$, and $r_n = \sum_{i<n} m_{2i}$, with the usual convention that sums over empty sets are 0, so that $q_1 = r_1 = 0$. Since $0 < m_i \leq m_{i+1}$ for all i, the sets $Q = \{q_n\}_{n=1}^M$ and $R = \{r_n\}_{n=1}^M$ form an alternating pair. Hence, so do the sets $Q + \sum_{n=1}^M r_n$ and R, and, moreover $(Q + \sum_{n=1}^M r_n) - R \subset P'$. This completes the proof of the theorem.

The conclusions above about UC-sets deserve further comment. Fix a set H in $\mathbf{Z}^+$ that is the range of a strong Hadamard sequence. Then $H - H$ is a UC-set [**17**]; this shows that general UC-sets can contain differences of arbitrarily large alternating pairs, although, by Theorem 4, UC-subsets of $\mathbf{Z}^+$ or $\mathbf{Z}^-$ cannot. Also, for each positive integer N, the sum $H + H + \cdots + H$ of N copies of H is a UC-set [**41**]; this shows that, although the class of sets of the form $E \cup \mathbf{Z}^-$, where E is a UC-set, is a uniform family of sets of continuity, there is no integer N such that all UC-sets satisfy condition $(\mathfrak{L}_N)$. Hilbert matrices can be used, as above, to show that the union of two UC-sets need not be a UC-set [**17**]; this answers, in the negative, the first four questions at the end of [**41**]. The fifth question therein also has a negative answer, in view of the example below.

EXAMPLE 3. Let H be an infinite subset of $\mathbf{Z}$, enumerated as $\{h_k\}_{k=1}^\infty$, say. As in Example 1, we consider sums

$$n = \sum_{k=1}^{\infty} \varepsilon_k h_k, \tag{3.6}$$

where $\varepsilon_k \in \{-1, 0, 1\}$ for all k, but we do not require that h be a Hadamard sequence. Recall [**28**, p. 19] that the set H is called *dissociate* if two integers of the form (3.6) are equal only if their coefficients ε_k match. Travaglini [**41**] has asked whether the sum $H + H$ of two copies of a dissociate subset H of $\mathbf{Z}^+$ must be a UC-set. We show that the answer is "no".

Indeed, if n is odd and $n \geq 3$, let $E_n = \{4^{n!-k}\}_{k \text{ odd},\, 1 \leq k < n}$, while if n is even and $n \geq 4$, let $E_n = \{4^{(n-1)!-k}\}_{k \text{ even},\, 2 \leq k < n}$. Let

$$H = \left[\bigcup_{n \text{ odd}} (4^{n!} + E_n)\right] \cup \left[\bigcup_{n \text{ even}} (4^{n!} - E_n)\right].$$

Then H is dissociate [**28**, p. 25], but $H + H$ is not a UC-set, because it contains translates of the differences $E_n - E_{n+1}$ for all odd $n \geq 3$. Moreover [**1**], $H + H$ is a (4/3)-Sidon set, and a $\Lambda(q)$ set for all $q < \infty$.

It is surprising that the fact that $\Lambda(1)$ sets cannot contain parallelepipeds of arbitrarily large dimensions was not discovered much earlier. For instance, Rudin asked [**38**, p. 216] whether every $\Lambda(1)$ set must have upper density 0. The answer is now known to be "yes", because of Szemeredi's result [**40**,] that subsets of **Z** with positive upper density must contain arbitrarily long arithmetic progressions. The latter fact is not easy to prove; in contrast, it is easy, by the method of [**42**], to prove that every set with positive upper density contains parallelepipeds of arbitrarily large dimensions, and this is enough to answer Rudin's question.

Finally, as promised, we give examples of sets satisfying conditions ($\mathfrak{M}$) but not ($\mathfrak{Q}$), etc.

EXAMPLE 4. For each positive integer N, let

$$P_N = (N!)10^N\{1, 2, \ldots, N\}.$$

Then let

$$F_0 = \bigcup_{n \text{ odd}} P_n,$$

let

$$F_1 = \bigcup_{n \text{ odd}} (1 + P_{2n}),$$

let

$$F_{-1} = \bigcup_{n \text{ odd}} (-1 + P_{4n}),$$

let

$$F_2 = \bigcup_{n \text{ odd}} (2 + P_{8n}), \text{ etc.},$$

finally, let $F = \bigcup_{N \in \mathbf{Z}} F_N$. Then F satisfies the Faber gap condition, and hence condition ($\mathfrak{L}$), but F is dense in $b\mathbf{Z}$.

To see this, first note that the sets P_N occupy widely separated intervals in $\mathbf{Z}^+$, and that the increments within each arithmetic progression P_N grow rapidly as $N \to \infty$; it follows that $\bigcup_{N=1}^{\infty} P_N$ satisfies the Faber gap condition. Now the given set F is obtained from $\bigcup_{N=1}^{\infty} P_N$ by such small perturbations that F also satisfies the Faber gap condition.

The fact that F is dense in $b\mathbf{Z}$ follows from the fact that n is a Bohr accumulation point of F_n for each n. To show that 0 is an accumulation point of F_0, for instance, one must show, for each $\varepsilon > 0$ and each finite subset S of $[0, 2\pi)$, that there is an integer m in F_0 so that $|e^{im\theta} - 1| < \varepsilon$ for all θ in S. Adopt temporarily the notation, from Diophantine approximation, that $\|x\|$ denotes the distance from a given real number x to the set $\mathbf{Z}$; then it suffices to show, for each such pair (ε, S), that there is an integer m in F_0 for which $\|m\theta/2\pi\| < \varepsilon$ for all θ in S. To verify this property of F_0 use the fact that F_0 contains arbitrarily long arithmetic progressions of the form $K\{1, 2, \ldots, k\}$. Let J be the number of elements in S; choose an integer $Q > 1/\varepsilon$, and an arithmetic progressions in F_0, as above, so that $k > Q^J$. Then, by the Dirichlet box principle [**21**, p. 170], there is a positive integer $j \leq k$ for which $\| j(K\theta/2\pi)\| < 1/Q$ for all θ in S; in other words, the integer $m = jK$ has the desired properties.

On the other hand, let

$$G_1 = F_0 \cup \left(\bigcup_{n \in F_0} F_n \right), \text{ and } G_2 = \bigcup_{n \in -F_0} F_n.$$

Then G_1 satisfies condition ($\mathfrak{M}$) but not condition ($\mathfrak{Q}$), while G_2 satisfies condition ($\mathfrak{Q}$), but not condition ($\mathfrak{M}$). To verify this, show that $(G_1)^a \cap \mathbf{Z} = \{0\} \cup F_0$, and $(G_2)^a \cap \mathbf{Z} = -F_0$; the details are omitted.

Finally, note that the set F_0 satisfies all the arithmetic conditions ($\mathfrak{L}$), ($\mathfrak{M}$), and ($\mathfrak{Q}$), but it is *not* a $\Lambda(1)$-set, a p-Sidon set for some $p < 2$, or a UC-set, because it contains arbitrary long arithmetic progressions.

4. The iterates of a contraction and its adjoint. Shortly after the paper by de Leeuw and Katznelson appeared, J. A. R. Holbrook [**22**] pointed out that their main result is equivalent to a seemingly more general assertion about powers of contractions on a Hilbert space, and he gave a direct proof of the latter assertion. Similarly, Theorems 1 and 2 are equivalent to certain statements about contractions. In this section, we show that these assertions have direct proofs because versions of Lemma 1 and 2 can be proved directly in the Hilbert space context.

We first state Holbrook's theorem. Call a bounded operator on a Banach space a *contraction* if its norm is at most 1.

THEOREM 0′. *For each $\varepsilon > 0$ there is a $\delta > 0$ with the following property. Let S be a contraction on a Hilbert space $\mathcal{H}$, and let g and f be elements of the unit ball of $\mathcal{H}$; then the condition that*

$$\limsup_{n\to\infty} \left|((S^*)^n g, f)\right| < \delta \text{ implies that } \limsup_{n\to\infty} |(S^n g, f)| < \varepsilon.$$

To obtain Theorem 0 as a special case of this result, factor the given measure μ as $(\operatorname{sgn}\mu)\,|\mu|$, let $\mathcal{H} = L^2(d|\mu|)$, let $g = \operatorname{sgn}\mu$, let $f \equiv 1$, and let S be the unitary operator on $\mathcal{H}$ that multiplies each function by $e^{-i\theta}$. Conversely, to derive Theorem 0′ from Theorem 0, apply the spectral theorem to a strong unitary dilation of the given contraction S; see [**22**] for more details. As noted above, these observations of Holbrook can also be used to transfer Theorems 1 and 2 to this Hilbert space setting. We will state a transferred version of these theorems at the end of this section; here is the transferred version of Lemma 1.

LEMMA 1′. *For each $\varepsilon > 0$ there is a $\delta > 0$ with the following property. Let T be a contraction on a Hilbert space $\mathcal{H}$, and let g and f be elements of the unit ball $\mathcal{H}$. If*

$$|(T|T|^{2m}g, f)| \leq \delta \quad \text{for } m = 1, 2, \ldots,$$

then

$$|(Tg, f)| \leq \varepsilon.$$

Proof. We show, by an elementary method, that the lemma holds with $\delta = (2/e)\varepsilon e^{-1/\varepsilon}$. Using the spectral theorem and the method of [**29**] one can prove this result with $\delta = \varepsilon(1 + \sqrt{2})^{-1-1/\varepsilon}$; it is pointed out in [**29**] that this form of the relation between ε and δ is optimal in the sense that the conclusions of Lemma 1 and Theorem 0 are false when $\delta = e^{2-1/2\varepsilon}$.

Let $a > 0$, and consider the sequences $(R_k)_{k=1}^K$ and $(S_k)_{k=1}^K$ given by letting $R_0 = aT$ and $S_0 = I$, and

$$R_{k+1} = R_k + aTS_k \quad \text{and} \quad S_{k+1} = S_k - aT^*R_k.$$

This variant of the familiar construction due to Shapiro and Rudin was introduced, in the scalar case, by Clunie [**4**]; it is shown in [**14**, pp. 60–61] that the key estimates also hold for bounded operators on a Hilbert space. First, $\|R_K\| \leq (1 + a^2)^{K/2}$. Next,

$$R_K = \sum_{m=0}^{M} c_m a^{2m+1} T|T|^{2m},$$

where M is the largest integer for which $2M + 1 \leq K$, and c_m is the coefficient of a^{2m+1} in the expansion of $[(1 + a)^K - (1 - a)^K]/2$; in particular $c_0 = K$.

Suppose that $|(Tg, f)| < \varepsilon$ and that $|(T|T|^{2m}g, f)| \leq \delta$ for all integers $m \geq 1$. Then, on the one hand,

$$|(R_K g, f)| \leq (1 + a^2)^{K/2} \leq e^{a^2K/2},$$

while, on the other hand,

$$|(R_K g, f)| > aK\varepsilon - \sum_{m=1}^{M} \delta |c_m| a^{2m+1}.$$

Combine these inequalities to get that

$$aK\varepsilon < e^{a^2K/2} + \delta \sum_{m=1}^{M} |c_m| a^{2m+1}$$
$$\leq e^{a^2K/2} + \frac{\delta}{2}(1 + a)^K \leq e^{K/2} + \frac{\delta}{2} e^{aK}.$$

Now let $K \to \infty$, choosing a so that $aK = 1 + 1/\varepsilon$; the outcome is that

$$\varepsilon + 1 \le 1 + \frac{\delta}{2} e^{1+1/\varepsilon},$$

that is that $\delta \ge (2/e)\varepsilon e^{-1/\varepsilon}$, as required.

In Theorems 3 and 4, we also need versions of Lemma 1 in which it is only assumed that $|(T|T|^{2m}g, f)| \le \delta$ for all $m > N$, where N is a fixed positive integer. The easiest way to prove such results is to iterate Lemma 1; we illustrate the idea in the case where $N = 1$. Given $\varepsilon > 0$, let $st(\varepsilon) = (2/e)\varepsilon e^{-1/\varepsilon}$, and let $\delta = st(st(\varepsilon))$. Suppose that $|(T|T|^{2m}g, f)| \le \delta$ for all $m \ge 2$. Then, by Lemma 1, applied with the operator $T|T|^2$ in place of T, we have that $|(T|T|^2 g, f)| \le st(\varepsilon)$; whence, by Lemma 1 again, $|(Tg, f)| \le \varepsilon$.

This method leads, however, to unduly small values for δ; one way to get better values for δ here is to proceed as in [**25**, Chap. 5]. Specifically [**29**], for each integer $N \ge 1$, there is a measure σ_N on the interval $(0, 1]$, with $\|\sigma_N\| < 1$, and so that

$$\int_0^1 t\, d\sigma_N(t) = \frac{1}{2N+1} \text{ and } \int_0^1 t^{2m+1} d\sigma_N(t) = 0 \text{ when } 1 \le m \le N.$$

Denote the operator R_K used above by $R_K(a)$, and let

$$S_K(a) = \int_0^1 R_K(at)\, d\mu_N(t);$$

then proceed as above with S_K in place of R_K. The outcome is that if $|(T|T|^{2m}g, f)| \le \delta$ for all $m > N$, where $\delta = 2\varepsilon e^{-(2N+1)/(1+1/\varepsilon)}$, then $|(Tg, f)| \le \varepsilon$.

Before stating the analogue of Lemma 2, we transfer some of the notation of §3 to the present context. Let $\{h(k)\}_{k=1}^{\infty}$ be an injective sequence of *positive* integers, and let α be a standard multi-index of length $J > 1$; let S be a bounded operator on a Hilbert space $\mathcal{H}$, and let g and f be elements of $\mathcal{H}$. If $\alpha_J = +1$, define $\limsup_{(\alpha, h)} |(S^{\alpha \cdot h}g, f)|$ to be the iterated limit

$$\limsup_{k_1 \to \infty} \left\{ \limsup_{k_2 \to \infty} \left\{ \cdots \left\{ \limsup_{k_J \to \infty} \left| \left(S^{\alpha_1 h(k_1) + \alpha_2 h(k_2) + \cdots + \alpha_J h(k_J)} g, f \right) \right| \right\} \cdots \right\} \right\}$$

notice that for all sufficiently large values of k_J we have a positive power of S above, because $\alpha_J = +1$. If $\alpha_J = -1$, define $\limsup_{(\alpha, h)} |(S^{\alpha \cdot h}g, f)|$ to be the iterated limit obtained, as above, when the power of S is replaced by

$$(S^*)^{-(\alpha h(k_1) + \cdots + \alpha_J h(k_J))}.$$

LEMMA 2′. *For each* $\varepsilon > 0$ *there is a* $\delta > 0$ *with the following property. Let* S *be a contraction on a Hilbert space* $\mathcal{H}$, *and let* h *be an injective sequence of positive integers. Suppose that for each integer* $J > 1$ *there is a standard multi-index* α *of length* J *for which* $\limsup_{(\alpha, h)} |(S^{\alpha \cdot h}g, f)| \leq \delta$; *then* $\limsup_{k \to \infty} |(S^{h(k)}g, f)| \leq \varepsilon$.

We remark first that Theorem 0′ follows from this lemma as in §3, that is, via multi-indices α with $\alpha_J = -1$. We now show how the lemma above follows from Lemma 1′. Suppose to this end, that the hypotheses of Lemma 2′ hold. Let $\beta = \limsup_{k \to \infty} |(S^{h(k)}g, f)|$, and pass to a subsequence of h for which the sequence $\{|(S^{h(k)}g, f)|\}_{k=1}^{\infty}$ converges to β. By restricting our attention to the closed subspace of $\mathcal{H}$ generated by g and f and their images under monomials in S and S^*, we can assume that $\mathcal{H}$ is separable. This done, we can pass to a further subsequence of h for which the sequence of operations $S^{h(k)}$ converges in the weak-operator topology to some contraction T.

Then $|(Tg, f)| = \beta$, and we just have to verify that $|(T|T|^{2m}g, f)| \leq \delta$ for all integers $m \geq 1$. For simplicity, we only do this in the special, but typical, case where $m = 1$. Since T is the weak-operator limit of the sequence $\{S^{h(k)}\}_{k=1}^{\infty}$, there is a sequence of operators T_k, each in the convex hull of $\{S^{h(j)}\}_{j \geq k}$ so that $T_k \to T$ in the strong operator topology. Then

$$\left(T|T|^2 g, f\right) = \lim_{k_1 \to \infty} \lim_{k_2 \to \infty} \lim_{k_3 \to \infty} \left(T_{k_1} T^*_{k_2} T_{k_3} g, f\right),$$

and the value of this iterated limit is in fact independent of the order in which the $k_i \to \infty$, provided that the order of the possibly noncommuting operators T_{k_1}, $T^*_{k_2}$, and T_{k_3} is not changed.

Let α be the standard multi-index of length 3 given by hypothesis. Exactly one of the indices α_i is equal to -1; we suppose that $\alpha_3 = -1$, the proof in the other cases being similar. It is enough to show that

$$(4.1) \qquad \lim_{k_1 \to \infty} \lim_{k_3 \to \infty} \lim_{k_2 \to \infty} \left|\left(T_{k_1} T^*_{k_2} T_{k_3} g, f\right)\right| \leq \limsup_{(\alpha, h)} \left|\left(S^{\alpha \cdot h} g, f\right)\right|;$$

notice the change in the order of iteration of the left-hand limit. By convexity, inequality (4.1) follows from the inequality

$$(4.2) \qquad \limsup_{k_1 \to \infty} \limsup_{k_3 \to \infty} \limsup_{k_2 \to \infty} \left|\left(S^{h(k_1)} (S^*)^{h(k_2)} S^{h(k_3)} g, f\right)\right| \leq \limsup_{\alpha} \left|\left(S^{\alpha \cdot h} g, f\right)\right|.$$

In fact, by Lemma 1 of **[22]**, this relation holds with equality. For completeness we include the simple proof of this assertion. First note **[13]** that for all elements a of $\mathcal{H}$, and all positive integers k,

$$\left\|(S^*)^{h(k)-1}a - S(S^*)^{h(k)}a\right\|^2 \leq \left\|(S^*)^{h(k)-1}a\right\|^2 - \left\|(S^*)^{h(k)}a\right\|^2,$$

because S is a contraction. Now the sequence $\{\|(S^*)^n a\|^2\}_n^\infty$ is nonnegative and decreasing; hence the left side of the inequality above converges to 0 as $k \to \infty$. It follows that

$$\limsup_{k\to\infty}\left|\left(S(S^*)^{h(k)}a,\, b\right)\right| = \limsup_{k\to\infty}\left|\left((S^*)^{h(k)-1}a,\, b\right)\right|$$

for all elements a and b of $\mathcal{H}$, and, by iteration and transposition, that

$$\limsup_{k_2\to\infty}\left|\left(S^{h(k_1)}(S^*)^{h(k_2)}S^{h(k_3)}g,\, f\right)\right|$$

$$= \limsup_{k_2\to\infty}\left|\left((S^*)^{h(k_2)-h(k_1)-h(k_3)}g,\, f\right)\right|.$$

Therefore, relation (4.2) holds with equality.

The following theorem includes Theorems 1 and 2 as special cases. It follows easily from the latter theorems by Holbrook's transference argument, but it can also be deduced directly from variants of Lemma 2′.

THEOREM 2′. *Let E be a subset of $\mathbf{Z}$ that satisfies condition $(\mathcal{L}_N)$ for some N, or one of the conditions $(\mathcal{O})$, $(\mathcal{M})$, or $(\mathcal{Q})$. Then for each $\varepsilon > 0$ there is a $\delta > 0$ with the property that, if S is a contraction on a Hilbert space $\mathcal{H}$, and if g and f are elements of the unit ball of $\mathcal{H}$, then the condition that*

$$\limsup_{n\in\mathbf{Z}^+\setminus E}\left|\left((S^*)^n g,\, f\right)\right| < \delta$$

implies that $\limsup_{n\in E}|((S^*)^n g, f)| < \varepsilon$, *and* $\limsup_{n\to\infty}|(S^n g, f)| < \varepsilon$.

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