PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 1, 1983

SPACES WITHOUT REMOTE POINTS

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All spaces considered are completely regular and X^* denotes $\beta X - X$. The point $x \in X^*$ is called *a remote point of* X if $x \notin \mathrm{Cl}_{\beta X} A$ for each nowhere dense subset A of X. If $y \in Y$, then the space Y is said to be *extremally disconnected at* y if $y \notin \overline{U} \cap \overline{V}$ whenever U and V are disjoint open sets. In this paper we construct two noncompact σ-compact spaces X, one locally compact and one nowhere locally compact, such that X has no remote points, and in fact such that βX is not extremally disconnected at any point.

Our examples were motivated by the following results from [**6**]:

(1) *X has remote points if X has countable π-weight, in particular if X is separable and first countable, and is not pseudocompact*, [**6**, 1.5]; see also [**7**] for an earlier consistency result, and [**1**] for a more general result.

(2) *βX is extremally disconnected at each remote point of X*, [**6**, 5.2]. Via the observation that

(3) *if Y is dense in Z, and $y \in Y$, then Y is extremally disconnected at y iff Z is extremally disconnected at y*,

these results and the following imply a nonhomogeneity result, which applies for example to the rationals and the Sorgenfrey line

(4) *if X is a nowhere locally compact nonpseudocompact space which has a remote point and if $\{x \in X$: X is not extremally disconnected at $x\}$ is dense in X, e.g. if X is first countable, then X^* is not homogeneous because X^* is extremally disconnected at some but not at all points.*

(This is a special case of Frolík's theorem that X^* is not homogeneous if X is not pseudocompact, [**8**]. The proof of Frolík's theorem does not yield a simple "because" as in (4). X is called nowhere locally compact if no point of X has a compact neighborhood, or, equivalently, if X^* is dense in βX.)

In this paper we produce two closely related examples which show that the condition on the π-weight cannot be omitted altogether in (1), thus answering a question of [**6**].

Our two examples are rather big: they have cellularity at least ω_3. This suggests the question of whether every nonpseudocompact separable space has a remote point. (This would generalize (1).) It follows from a construction in [**7**] that the answer is affirmative under CH.

EXAMPLES. *There are two noncompact σ-compact spaces X, one locally compact and one nowhere locally compact, such that X has no remote points, and in fact such that βX is not extremally disconnected at any point.*

Because of (3) the nowhere locally compact example shows that the condition on the π-weight cannot be omitted altogether in the nonhomogeneity result (4). We will show that an older nonhomogeneity proof, involving far points, still applies.

No remote points.

A subset P of a space X is called a *P-set* if for each F_σ-subset F of X, if $F \cap P = \varnothing$ then $\overline{F} \cap P = \varnothing$. A subset T of a space X is called a *2-set* if there are disjoint open U and V in X with $T \subseteq \overline{U} \cap \overline{V}$.

LEMMA 1. *There is a compact space U such that for each $q \in U$ there is a decreasing ω_1-sequence $\langle P_\xi \colon \xi \in \omega_1\rangle$ of clopen sets such that $\bigcap_{\xi\in\omega_1} P_\xi$ is a nowhere dense set of U which contains q.*

□ Give ω_2 the discrete topology. Identify ω_2^* with the space of free ultrafilters on ω_2. Then

$$U = \left\{q \in \omega_2^* \colon |Q| = \omega_2 \text{ for all } Q \in q\right\},$$

the space of uniform ultrafilters on ω_2, is a closed, hence compact, subspace of ω_2^* of course. We need the following result due to Čudnovskiĭ and Čudnovskiĭ, [**3**] and, independently, to Kuen and Prikry, [**11**], and earlier, but with GCH to Chang [**2**]:

(∗) for each $q \in U$ there is a decreasing ω_1-sequence $\langle Q_\xi \colon \xi \in \omega_1\rangle$ in q such that $\displaystyle\bigcap_{\xi\in\omega_1} Q_\xi = \varnothing$.

As usual, let $\hat{A}$ denote $U \cap \overline{A}$ (closure in $\beta\omega_2$), for $A \subseteq \omega_2$. For a given $q \in U$ let $\langle Q_\xi \colon \xi \in \omega_1\rangle$ be as in (∗), and define $\langle P_\xi \colon \xi \in \omega_1\rangle$ by $P_\xi = \hat{Q}_\xi$ for $\xi \in \omega_1$. Clearly $\langle P_\xi \colon \xi \in \omega_1\rangle$ is a decreasing ω_1-sequence of clopen subsets of U such that $P = \bigcap_{\xi\in\omega_1} P_\xi$ contains q. Now recall that $\{\hat{B}\colon B \subseteq \omega_2 \text{ and } |B| = \omega_2\}$, being the collection of all nonempty clopen subsets of U, is a base for U. Consider any $B \subseteq \omega_2$ with $|B| = \omega_2$. There is an $\eta \in \omega_1$ with $|B - Q_\eta| = \omega_2$. Then $\varnothing \neq (B - Q_\eta)\hat{} = \hat{B} - \hat{Q}_\eta \subseteq \hat{B} - P$. It follows that P is nowhere dense. □

REMARK. Instead of ω_1 we can take any regular cardinal κ, and then U will be the space of uniform ultrafilters on κ^+.

Clearly Lemma 1 implies that there is a compact space which is covered by the collection of its nowhere dense closed P-sets. Since evidently each 2-set is nowhere dense the following is a stronger assertion.

LEMMA 2. *There is a compact space H such that for each $q \in H$ there is a closed P in H with $q \in P$ such that P is both a P-set and a 2-set.*

□ Let U be as in Lemma 1, and let $H = U \times U$. Consider any $q_0, q_1 \in U$. For $i \in 2$ choose a decreasing ω_1-sequence $\langle P_{i,\xi} \colon \xi \in \omega_1 \rangle$ of clopen sets in U such that $P_1 = \bigcap_{\xi \in \omega_1} P_{i,\xi}$ is a nowhere dense subset of U which contains q_i. Then $P_0 \times P_1$ is a nowhere P-set in H which contains $\langle q_0, q_1 \rangle$. We show that $P_0 \times P_1$ is also a 2-set

For $i \in 2$ define an open $V_{i,\xi}$ with recursion on $\xi \in \omega_1$ by

$$V_{i,\xi} = (U - P_{i,\xi}) - \Big(\bigcup_{\eta \in \xi} V_{i,\eta} \Big)^{-} \qquad \Big(\bigcup_{\nu \in 0} V_{i,\eta} = \varnothing \text{ of course} \Big).$$

Then evidently $(\bigcup_{\eta \leq \xi} V_{i,\eta})^{-} = U - P_{i,\xi}$ for $i \in 2$ and $\xi \in \omega_1$. Since P_0 and P_1 are nowhere dense it follows that

$$(\dagger) \qquad \Big(\bigcup_{\xi \in \omega_1} V_{i,\xi} \Big)^{-} = (U - P_i)^{-} = U, \qquad \text{for } i \in 2.$$

Define open subsets W_0 and W_1 of H by

$$W_0 = \bigcup_{\xi \in \omega_1} P_{0,\xi} \times V_{1,\xi} \quad \text{and} \quad W_1 = \bigcup_{\xi \in \omega_1} V_{0,\xi} \times P_{1,\xi}.$$

Then $W_0 \cap W_1 = \varnothing$ since if $\xi \leq \eta < \omega_1$ then $V_{i,\xi} \subseteq U - P_{i,\xi} \subseteq U - P_{i,\eta}$, for $i \in 2$ (so that $(P_{0,\xi} \times V_{1,\xi}) \cap (V_{0,\eta} \times P_{1,\xi}) = \varnothing$ for all $\xi, \eta \in \omega_1$). To prove that $P_0 \times P_1 \subseteq \overline{W_0} \cap \overline{W_1}$ we have only to prove that $P_0 \times P_1 \subseteq \overline{W_0}$, because of symmetry. We have

$$W_0 \supseteq \bigcup_{\xi \in \omega_1} \Big(\Big(\bigcap_{\eta \in \omega_1} P_{0,\eta} \Big) \times V_{1,\xi} \Big) = P_0 \times \bigcup_{\xi \in \omega_1} V_{1,\xi},$$

hence $\overline{W_0} \supseteq P_0 \times U \supseteq P_0 \times P_1$ as required. □

REMARK 2. We do not know if the space U of Lemma 1 can be used for the space H of Lemma 2. We are indebted to the referee for pointing out that the set $P = \bigcap_{\xi \in \omega_1} P_\xi$ obtained in Lemma 1 is not a 2-set: P has character ω_1, but in U the closure of every open F_{ω_1}-set ($\equiv$ union of ω_1 many closed sets) is easily seen to be open, [**CoN**, Thm. 14.9], which implies that no closed set in U of character ω_1 is a 2-set. To see this let F be a closed set in U of character ω_1 and let V and W be disjoint open sets in U such that $F \subseteq \overline{V}$. Since F has character ω_1 there is an open F_{ω_1}-set $T \subseteq V$ such that $\overline{T} \cap \overline{F} \neq \varnothing$. Now $\overline{T} \cap \overline{W} = \varnothing$ since $T \cap W = \varnothing$, and $\overline{T}$ is clopen. It follows that $F \not\subseteq \overline{W}$.

SUBREMARK. It is at least consistent that $U = U(\omega_2)$ has a closed P-set that is a 2-set. There is a closed nowhere dense $P \subseteq U$ which is a P_{ω_2}-set ($\equiv$ for every F_{ω_2}-set F in U, if $F \cap P = \varnothing$ then $\overline{F} \cap P = \varnothing$), namely $\bigcap\{C: C \subseteq \omega_2 \text{ is a cub}\}$, and if $2^{\omega_2} = \omega_3$ then every nowhere dense P_{ω_3}-set in U (or in any space of weight ω_3) is a 2-set. However, if $2^{\omega_2} = \omega_3$ then U is not covered by the collection of its nowhere dense closed P_{ω_3}-sets, by **[10**, 1.1].

REMARK 3. After this paper had been written another proof of Lemma 1 was discovered by Kunen, van Mill and Mills: the space of nondecreasing functions $\omega_2 \to \omega_1 + 1$, **[10**, 3.1]. It is easy to see that the P-sets obtained there are 2-sets. The example of Lemma 2 has the additional feature that each P-set has character ω_1.

REMARK 4. The above remarks suggest the question of whether there is a compact space which is covered by the collection of its closed nowhere dense P-sets but which has no nonempty closed P-set which is also a 2-set. This question can be answered quite easily. Let E be the projective cover of the example of Lemma 1, i.e. E is the unique extremally disconnected compact space that admits an irreducible map, say π, onto U. As is well known, $\pi^{\leftarrow}(D)$ is nowhere dense in E iff D is nowhere dense in E. Since it is easily seen that $\pi^{\leftarrow}(P)$ is a P-set of E iff P is a P-set of U, we conclude that E can be covered by nowhere dense closed P-sets. Since E is extremally disconnected, there are no nonempty 2-sets in E. The following question however remains open:

Question. Is there (in ZFC) a compact space which is covered by the collection of its closed nowhere dense P-sets but which has no nonempty nowhere dense P_{ω_2}-set?

LEMMA 3. *Let K be a compact space, and let P be a P-set in K. Furthermore, let Y be a countable space, let π: $K \times Y \to K$ be the projection, and let $\beta\pi$: $\beta(K \times Y) \to K$ be the Stone extension of π. Then for each $x \in \beta(K \times Y)$, if $\beta\pi(x) \in P$ then $x \in (P \times Y)^{-}$.*

□ Consider any $x \in \beta(K \times Y) - (P \times Y)^{-}$. Let V be a closed neighborhood of x which misses $P \times Y$. Then $x \in ((K \times Y) \cap V)^{-}$, hence

$$\beta\pi(x) \in (\beta\pi^{\to}((K \times Y) \cap V))^{-} = (\pi^{\to}((K \times Y) \cap V))^{-}.$$

Also, $\pi^{\rightarrow}((K \times Y) \cap V)$ is an F_σ (since $(K \times Y) \cap V$ is σ-compact) in K which misses the P-set P, hence $(\pi^{\rightarrow}((K \times Y) \cap V))^{-} \cap P = \varnothing$. Consequently $\beta\pi(x) \notin P$. □

COROLLARY 1. *If K is a compact space which is covered by nowhere dense P-sets, then $K \times Y$ has no remote points, for each countable space Y.* □

COROLLARY 2. *If K is a compact space which is covered by P-sets which are 2-sets, then $\beta(K \times Y)$ is not extremally disconnected at any point, for each countable space K.*

□ The key observation is that if D is dense in a space X, then the closure in X of each 2-set in D is a 2-set in X. □

If H is as in Lemma 2, if ω is the integers and if Q is the rationals, then our examples are $H \times \omega$ and $H \times Q$.

Far points.

A point p of X^* is called a *far* (or *ω-far*) *point* of X if $p \notin \mathrm{Cl}_{\beta X} D$ for each (countable) closed discrete subset D of X. Clearly, if X has no isolated points then each remote point of X is a far point; the converse of this is generally false, [**6**, 4.8]. There is a nonhomogeneity result involving far points, or ω-far points, similar to (4) of the introduction, but less attractive since it involves $X^{**} = (X^*)^*$: If X is nowhere locally compact, and is not countably compact, and has a far (ω-far) point, then X^* is not homogeneous because for some but not for all $x \in X^*$ there is a (countable) closed discrete D in the space X^{**} such that $x \in \mathrm{Cl}_{\beta X^*} D$, [**5**, 2,4.3].

One might hope that our examples can be used to answer the question of [**5**] of whether every noncompact Lindelöf space has an ω-far point (which would be a far point). (It is easy to see that every normal nonLindelöf space has an ω-far point, [**5**, 4.3].) This is not the case: both our examples have far points. This follows from the following result.

THEOREM. *If X has a countably infinite discrete collection K of compact subspaces without isolated points, and if X is normal, or, more generally, if K can be separated by a discrete open family, then X has a far point.*

Before we proceed to the proof we point out an attractive corollary:

COROLLARY. *Every locally compact (or, more generally, Čech-complete) nonpseudocompact space has a far-point.*

□ If X is nonpseudocompact it has a countably infinite family $\mathcal{U}$ consisting of nonempty open sets. By a well-known tree argument one finds for each $U \in \mathcal{U}$ a compact $K_U \subseteq U$ that admits a continuous map f_U onto the Cantor discontinuum ${}^{\omega}2$. For $U \in \mathcal{U}$ choose a compact $L_U \subseteq K_U$ such that $f_U \restriction L_U$ is an irreducible map onto ${}^{\omega}2$, then L_U has no isolated points. □

Proof of Theorem. First recall that **R** has a far point, by an elegant argument due to Eberlein [7, Thm. 1.3]. It follows that $Y = \bigcup\mathcal{K}$ has a far point. As in the proof of the Corollary, each member of $\mathcal{K}$ admits a (necessarily closed) map onto the Cantor discontinuum, hence on the closed unit interval. Since $\mathcal{K}$ is countably infinite it follows that Y admits a closed map onto **R**. The Stone extension βf of f maps ϕY onto $\beta\mathbf{R}$, hence there is $y \in Y^*$ such that $\beta f(y)$ is a far point of R. Since $f^{\rightarrow} D$ is closed discrete in **R** for each closed discrete D in Y this y is a far point of Y, cf. [5, §2, Fact 3].

We now point out that

$$(*) \qquad \text{For any two disjoint closed } F \text{ and } G \text{ in } X, \text{ if } F \subseteq Y \text{ then } \mathrm{Cl}_{\beta X} F \cap \mathrm{Cl}_{\beta X} G = \emptyset .$$

The proof is similar to the known case, [9, 3L], that $\mathcal{K}$ consists of singletons. From $(*)$ we see that $\mathrm{Cl}_{\beta X} Y = \beta Y$. Since Y is closed in X it follows that X contains a far point of Y. This point is a far point of X since, by $(*)$, for each closed discrete subset D of Y we have $\mathrm{Cl}_{\beta X}(D - Y) \cap \mathrm{Cl}_{\beta X} Y = \emptyset$. □

REMARK 5. Dow [4] has shown that every separable nonpseudocompact space has a remote point under MA.

REMARK 6. After this paper was written there has been much progress on the question of whether every Lindelöf space has a far point: It is known that the answer is affirmative under MA, [12, 9.1].

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Received April 11, 1978 and in revised form March 10, 1982. Research of the first author was supported by an NSF grant, and the second author's research was supported by the Netherlands Organization for the Advancement of Pure Research Z.W.O.): Juliana van Stolberglaan 148, 's-Gravenhage, the Netherlands.

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