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ON THE TRANSFORMATION OF FOURIER COEFFICIENTS OF CERTAIN CLASSES OF FUNCTIONS II

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If $\{a_\nu\}_1^\infty$ is the sequence of Fourier cosine coefficients of a function in the space L^p, $1 \leq p < \infty$, a Theorem of Hardy states that the sequence of averages $\{(\sum_{j=1}^{\nu} a_j)/\nu\}_1^\infty$ arise as Fourier cosine coefficients of a function also in L^p. Analogous results for the sequence $\{\sum_{j=\nu}^{\infty} a_j/j\}_1^\infty$ were obtained by Bellman. In this paper, sufficient conditions on the non-negative weight function $\omega(x)$ are given in order that the weighted Lebesgue space $L^p(\omega(x)\,dx)$ may replace the spaces L^p in the Theorems of Hardy and Bellman.

1. Introduction and statement of results. Let $\{a_\nu\}_0^\infty$ denote the sequence of Fourier cosine coefficients of the integrable function $f(x)$, that is,

$$a_\nu = \frac{2}{\pi}\int_0^\pi f(x)\cos \nu x\,dx, \qquad \nu = 0, 1, 2, \ldots,$$

and let $A_0 = A'_0 = a_0$,

$$A_\nu = \frac{1}{\nu}\sum_{j=1}^{\nu} a_j, \qquad A'_\nu = \sum_{j=\nu}^{\infty} a_j/j, \qquad \nu = 1, 2, \ldots.$$

G. H. Hardy [4] has shown that if $f(x)$ belongs to $L^p(0, \pi)$ for some p, $1 \leq p < \infty$, then $\{A_\nu\}_0^\infty$ is the sequence of Fourier cosine coefficients of a function $F(x)$, also in $L^p(0, \pi)$; R. Bellman [3] proved the analogous statement for $\{A'_\nu\}_1^\infty$, except that now p satisfies $1 < p \leq \infty$. These results have been generalized by several authors in various directions. In particular, we [1] have recently characterized those function spaces $L^\sigma(0, \pi)$, given by a rearrangement invariant metric σ, that may replace the $L^p(0, \pi)$ spaces in the Theorems of Hardy and Bellman.

In this paper, we consider a generalization in a direction complementary to that of the rearrangement invariant spaces. We shall consider here weighted spaces of functions $L^p(\omega) = \{f\colon \int_0^\pi |f(x)|^p \omega(x)\,dx = \|f\|_{p,\omega}^p < \infty\}$, giving conditions on the non-negative weight function $\omega(x)$ which ensure that $L^p(\omega)$ may replace the (unweighted) spaces L^p in the Theorems of Hardy and Bellman. We suppose throughout, when required, that functions f and weights ω defined initially on $(0, \pi)$ are defined on $(-\infty, \infty)$ by the requirements of evenness on $(-\pi, \pi)$ and 2π-periodicity.

Since we shall be concerned with the Fourier cosine coefficients of an f in $L^p(\omega)$ we shall have to have $L^p(\omega) \subset L^1$. Hölder's inequality

$$\int_0^\pi |f(x)|\,dx \le \begin{cases} \left(\int_0^\pi |f(x)|^p \omega(x)\,dx\right)^{1/p} \left(\int_0^\pi \omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} & \text{if } 1 < p < \infty, \\ \left(\int_0^\pi |f(x)|\,\omega(x)\,dx\right)\left(\operatorname*{ess\cdot sup}_{(0,\pi)} 1/\omega(x)\right) & \text{if } p = 1, \end{cases}$$

and its converse show that this is equivalent to the requirement that

$$\begin{cases} \int_0^\pi \omega(x)^{-1/(p-1)}\,dx < \infty & \text{if } 1 < p < \infty, \\ \operatorname*{ess\cdot sup}_{(0,\pi)} 1/\omega(x) < \infty & \text{if } p = 1. \end{cases}$$

Further, since we wish $L^p(\omega)$ to contain the constant functions, we assume that $\int_0^\pi \omega(x)\,dx < \infty$. Thus, $\theta(u)$ given by

$$\theta(u) = \theta(\omega, p; u) = \begin{cases} \sup_a \left(\int_a^{a+u} \omega(x)\,dx\right)^{1/p} \left(\int_a^{a+u} \omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} & \text{if } 1 < p < \infty, \\ \sup_a \left(\int_a^{a+u} \omega(x)\,dx\right)\left(\operatorname*{ess\cdot sup}_{(a,a+u)} 1/\omega(x)\right) & \text{if } p = 1, \end{cases}$$

is finite for all $u > 0$, and we assume throughout that ω satisfies the additional mild condition $\int_0^\delta \theta(u)\,du/u < \infty$ for some $\delta > 0$. The restrictions we have placed on ω thus far may then be summarized by the equivalent, single requirement that

$$\int_0^{2\pi} \theta(\omega, p; u)\frac{du}{u} < \infty. \tag{1.1}$$

For example, $\omega(x) = |\sin x|^\alpha$ satisfies (1.1) if and only if

$$\begin{cases} -1 < \alpha < p - 1 & \text{if } 1 < p < \infty \\ -1 < \alpha \le 0 & \text{if } p = 1. \end{cases}$$

More generally, the well known and important Muckenhoupt class A_p of weights, defined by the requirement that $\theta(u) \le Cu$, satisfy (1.1). Of course, a weight satisfying (1.1) need not satisfy the A_p condition; for example, with $p = 2$, $\omega(x) = x^{-1}(\log(\pi/x))^{-2}$ satisfies (1.1) but not the A_2 condition.

We can now state our results.

THEOREM 1. *Let* $p = 1$. *Suppose* ω *satisfies* (1.1) *and there is a constant* C *such that for almost all* t, $0 < t < \pi$,

$$\int_0^t \omega(x)\, dx \le Ct\omega(t). \tag{1.2}$$

If $\{a_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of an* $f \in L^1(\omega)$ *then* $\{A_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of a function* F *also in* $L^1(\omega)$; *moreover, there is a constant* c *independent of* f *such that* $\|F\|_{1,\omega} \le c\|f\|_{1,\omega}$.

Since ω clearly satisfies (1.2) if ω satisfies the A_1 condition, we have the following result.

COROLLARY 1. *If* ω *is an* A_1 *weight then the conclusion of Theorem* 1 *holds.*

THEOREM 2. *Let* $1 < p < \infty$. *Suppose* ω *satisfies* (1.1) *and that there is a constant* C *such that for some* $\varepsilon > 0$

$$\left(\int_0^r \omega(x)\, dx\right)^{1/p} \left(\int_r^\pi \left(\frac{r}{x}\right)^\varepsilon x^{-p'} \omega(x)^{-1/(p-1)}\, dx\right)^{1/p'} \le C \tag{1.3}$$

holds for all $0 < r < \pi$. *If* $\{a_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of an* $f \in L^p(\omega)$ *then* $\{A_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of a function* F *also in* $L^p(\omega)$; *moreover, there is a constant* c *independent of* f *such that* $\|F\|_{p,\omega} \le c\|f\|_{p,\omega}$.

If ω satisfies the A_p condition, then as we shall show, (1.3) is satisfied so that we have the following corollary.

COROLLARY 2. *If* $1 < p < \infty$ *and* ω *satisfies the* A_p *condition then the conclusion of Theorem* 2 *holds.*

Concerning the sequences $\{A'_\nu\}$ we have the following results.

THEOREM 3. *Let* $p = 1$. *Suppose* ω *satisfies* (1.1) *and there is a constant* C *such that for almost all* t, $0 < t < \pi$,

$$\int_t^\pi \omega(x)\, \frac{dx}{x} \le C\omega(t). \tag{1.4}$$

If $\{a_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of an* $f \in L^1(\omega)$ *then* $\{A'_\nu\}_0^\infty$ *is the sequence of Fourier cosine coefficients of a function* F *also in* $L^1(\omega)$; *moreover, there is a constant* c *independent of* f *such that* $\|F\|_{1,\omega} \le c\|f\|_{1,\omega}$.

THEOREM 4. *Let $1 < p < \infty$. Suppose ω satisfies* (1.1) *and that there is a constant C such that for some $\varepsilon > 0$*

$$(1.5)\qquad \left(\int_r^{\pi}\left(\frac{r}{x}\right)^{\varepsilon} x^{-p}\omega(x)\,dx\right)^{1/p}\left(\int_0^r \omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} \le C$$

holds for all $0 < r < \pi$. If $\{a_\nu\}_0^\infty$ is the sequence of Fourier cosine coefficients of an $f \in L^p(\omega)$ then $\{A'_\nu\}_0^\infty$ is the sequence of Fourier cosine coefficients of a function F also in $L^p(\omega)$; moreover, there is a constant c independent of f such that $\|F\|_{p,\omega} \le c\|f\|_{p,\omega}$.

As Bellman [3] pointed out, there is a certain 'duality' between the Theorems for $\{A_\nu\}$ and $\{A'_\nu\}$. His duality Theorem may be generalized as follows. For a sequence $\{b_\nu\}_0^\infty$ let $B'_0 = b_0$ and $B'_\nu = \Sigma_{j=\nu}^{\infty} b_j/j$ for $\nu = 1, 2, \ldots$.

THEOREM 5. *Let $1 < p < \infty$. Suppose ω satisfies* (1.1) *and that there is a constant C such that for some $\varepsilon > 0$*

$$(1.6)\qquad \left(\int_0^r \omega(x)\,dx\right)^{1/p}\left(\int_r^{\pi}\left(\frac{r}{x}\right)^{\varepsilon} x^{-p'}\omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} \le C$$

holds for $0 < r < \pi$. If $\{a_\nu\}_0^\infty$ and $\{b_\nu\}_0^\infty$ are the sequences of Fourier cosine coefficients of functions $f \in L^p(\omega)$ and $g \in L^{p'}(\omega^{-1/(p-1)})$ respectively, then $\{A_\nu\}_0^\infty$ and $\{B'_\nu\}_0^\infty$ are the sequences of Fourier cosine coefficients of functions $F \in L^p(\omega)$ and $G \in L^{p'}(\omega^{-1/(p-1)})$ respectively which satisfy the identity

$$(1.7)\qquad \int_0^{\pi}\{f(x)G(x) - F(x)g(x)\}\,dx = 0.$$

The proofs will depend on the following Lemma which is of interest in its own right.

LEMMA. *Let $1 \le p < \infty$ and suppose ω satisfies* (1.1). *If $\{a_\nu\}_0^\infty$ is the sequence of Fourier cosine coefficients of a function $f \in L^p(\omega)$, then $\{c_\nu\}_0^\infty$ given by $c_0 = 0$, $c_\nu = a_\nu/\nu$, $\nu = 1, 2, \ldots$ is the sequence of Fourier cosine coefficients of a function H also in $L^p(\omega)$; moreover, there is a constant c independent of f such that $\|H\|_{p,\omega} \le c\|f\|_{p,\omega}$.*

2. Proof of the lemma. According to [**8**, p. 180] the function H is given by

$$H(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\log\left(\frac{1}{2|\sin(t/2)|}\right)dt, \qquad 0 < x < \pi.$$

We shall carry out the proof assuming that $1 < p < \infty$; the required modifications for the case $p = 1$ will be self-evident.

Consider first $H_1(x)$ where $H(x) = H_1(x) + H_2(x)$ with

$$H_1(x) = \frac{1}{\pi}\int_{\pi \geq |t| \geq \pi/3} f(x+t)\log\left(\frac{1}{2\,|\sin(t/2)|}\right) dt.$$

From Hölder's inequality and the periodic property of f and ω it follows that

$$\begin{aligned}
|H_1(x)| &\leq \left(\frac{\log 2}{\pi}\right)\int_{\pi \geq |t| \geq \pi/3} |f(x+t)|\,dt \\
&\leq \left(\frac{\log 2}{\pi}\right)\left(\int_{\pi \geq |t| \geq \pi/3} |f(x+t)|^p\omega(x+t)\,dt\right)^{1/p} \\
&\quad \times \left(\int_{\pi \geq |t| \geq \pi/3} \omega(x+t)^{-1/(p-1)}\,dt\right)^{1/p'} \\
&\leq \left(\frac{2\log 2}{\pi}\right)\left(\int_0^\pi |f(t)|^p\omega(t)\,dt\right)^{1/p}\left(\int_0^\pi \omega(t)^{-1/(p-1)}\,dt\right)^{1/p'}
\end{aligned}$$

and hence

$$\left(\int_0^\pi |H_1(x)|^p\omega(x)\,dx\right)^{1/p} \leq \frac{2\log 2}{\pi}\theta(\pi)\left(\int_0^\pi |f(t)|^p\omega(t)\,dt\right)^{1/p}. \tag{2.1}$$

Now to treat $H_2(x)$, observe first that for fixed u, $0 < u < 1$, we have

$$\begin{aligned}
&\int_0^\pi \omega(x)\left[\int_{|t| \leq \pi u/2} |f(x+t)|\,dt\right]^p dx \\
&\qquad \leq 4\left[\theta\left(\frac{3\pi u}{2}\right)\right]^p \int_0^\pi |f(t)|^p\omega(t)\,dt.
\end{aligned} \tag{2.2}$$

To see this, choose the integer N so that $Nu \geq 2$, and let, for convenience, $a = \pi u/2$. Then the left side of (2.2) is bounded above by

$$\sum_{k=1}^{N} \int_{(k-1)a}^{ka} \omega(x)\left[\int_{|t| \leq a} |f(x+t)|\,dt\right]^p dx.$$

Enlarging the inner integral and applying Hölder's inequality shows this is further bounded by

$$\sum_{k=1}^{N} \int_{(k-1)a}^{ka} \omega(x)\left[\int_{(k-2)a}^{(k+1)a} |f(t)|\,dt\right]^p dx$$
$$\leq \sum_{k=1}^{N} \int_{(k-1)a}^{ka} \omega(x)\,dx\left[\int_{(k-2)a}^{(k+1)a} \omega(t)^{-1/(p-1)}\,dt\right]^{p-1}$$
$$\times\left[\int_{(k-2)a}^{(k+1)a} |f(t)|^p\omega(t)\,dt\right]$$
$$\leq [\theta(3a)]^p \sum_{k=1}^{N} \int_{(k-2)a}^{(k+1)a} |f(t)|^p\omega(t)\,dt$$

where to obtain the last inequality we have used the definition of θ. Finally, since the intervals $((k-2)a, (k+1)a)$ have limited overlap, we obtain (2.2).

Returning to $H_2(x)$ we have

$$|H_2(x)| \leq \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} |f(x+t)|\left(\int_{2|\sin(t/2)|}^{1} \frac{du}{u}\right) dt$$
$$\leq \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} |f(x+t)|\left(\int_{2|t|/\pi}^{1} \frac{du}{u}\right) dt$$
$$\leq \frac{1}{\pi}\int_{0}^{1} \frac{du}{u}\int_{|t|\leq \pi u/2} |f(x+t)|\,dt$$

by an appeal to Fubini's Theorem. Minkowski's inequality for integrals followed by (2.2) then yields

$$\left(\int_0^{\pi} |H_2(x)|^p\omega(x)\,dx\right)^{1/p}$$
$$\leq \frac{1}{\pi}\int_0^1 \frac{du}{u}\left\{\int_0^{\pi}\omega(x)\left[\int_{|t|\leq\pi u/2} |f(x+t)|\,dt\right]^p dx\right\}^{1/p}$$
$$\leq \frac{4^{1/p}}{\pi}\left(\int_0^1 \theta\left(\frac{3\pi u}{2}\right)\frac{du}{u}\right)\left(\int_0^{\pi} |f(t)|^p\omega(t)\,dt\right)^{1/p}.$$

A change of variable in the first integral on the right shows, in view of (1.1), that

$$\left(\int_0^{\pi} |H_2(x)|^p\omega(x)\,dx\right)^{1/p} \leq c\left(\int_0^{\pi} |f(t)|^p\omega(t)\,dt\right)^{1/p}.$$

This, together with (2.1), completes the proof of the lemma.

3. Proof of Theorem 1. Assume first that $a_0 = 0$. Then as Hardy [4] has shown, F is given by $F(x) = [F_1(x) + H(x)]/2$ where H is the function of the Lemma and $F_1(x) = \int_x^\pi f(t)\cot(t/2)\,dt$. Thus it suffices to prove that $\|F_1\|_{1,\omega} \le c\|f\|_{1,\omega}$. To see this, observe that $\cot(t/2) \le 2/t$ so that

$$\begin{aligned}\int_0^\pi |F_1(x)|\,\omega(x)\,dx &\le 2\int_0^\pi \omega(x)\left(\int_x^\pi |f(t)|\,\frac{dt}{t}\right)dx\\ &= 2\int_0^\pi |f(t)|\left(\frac{1}{t}\int_0^t \omega(x)\,dx\right)dt\\ &\le 2C\int_0^\pi |f(t)|\,\omega(t)\,dt\end{aligned}$$

by Fubini's Theorem and the hypothesis (1.2).

If now $a_0 \ne 0$, the above argument shows that there is a function $F(x)$ with Fourier cosine coefficients $\{A_\nu\}_0^\infty$ and which satisfies $\|F - a_0/2\|_{1,\omega} \le c\|f - a_0/2\|_{1,\omega}$. Now the triangle inequality and the observation

$$\begin{aligned}\|a_0/2\|_{1,\omega} &= \left(\int_0^\pi \omega(x)\,dx\right)\left|\frac{1}{\pi}\int_0^\pi f(t)\,dt\right|\\ &\le \frac{1}{\pi}\left(\int_0^\pi \omega(x)\,dx\right)\left(\int_0^\pi |f(t)|\,\omega(t)\,dt\right)\left(\operatorname*{ess\cdot sup}_{(0,\pi)} 1/\omega(t)\right)\\ &\le \frac{1}{\pi}\theta(\pi)\|f\|_{1,\omega}\end{aligned}$$

shows that $\|F\|_{1,\omega} \le c\|f\|_{1,\omega}$ for some constant c. This completes the proof of Theorem 1.

4. Proof of Theorem 2 and Corollary 2. We prove the Theorem first. Just as in the proof of Theorem 1, we may assume that $a_0 = 0$ for the general case follows easily from this, and it therefore suffices to prove that

$$\int_0^\pi \omega(x)\left|\int_x^\pi f(t)\,\frac{dt}{t}\right|^p dx \le c\int_0^\pi |f(t)|^p\omega(t)\,dt \tag{4.1}$$

for some constant c. According to [6] (or [2]) a sufficient (and necessary) condition for (4.1) to hold is that

$$\left(\int_0^r \omega(x)\,dx\right)^{1/p}\left(\int_r^\pi x^{-p'}\omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} \le C \tag{4.2}$$

for all $0 < r < \pi$. Since Lemma 2 of [2] shows that (4.2) and (1.3) are equivalent, the proof is complete.

To prove the Corollary, we shall show that (1.3) holds if $\theta(u) \leq Cu$. To see this, note that the definition of $\theta(t)$ yields

$$\left(\int_0^r \omega(x)\,dx\right)^{p'/p}\left(\int_r^t \omega(x)^{-1/(p-1)}\,dx\right) \leq C^{p'}t^{p'}$$

for $0 < r < t < \pi$. Multiplying this by $(r/t)^{\varepsilon}t^{-p'-1}$ and integrating the result over $r < t < \pi$ leads by Fubini's Theorem to

$$\left(\int_0^r \omega(x)\,dx\right)^{p'/p}\int_r^\pi \omega(x)^{-1/(p-1)}\left\{\left(\frac{r}{x}\right)^{\varepsilon}x^{-p'} - \left(\frac{r}{\pi}\right)^{\varepsilon}\pi^{-p'}\right\}dx$$

$$\leq (p'+\varepsilon)C^{p'}\int_r^\pi \left(\frac{r}{t}\right)^{\varepsilon}\frac{dt}{t} \leq (p'+\varepsilon)\varepsilon^{-1}C^{p'}.$$

Transposing the negative term on the left side and dominating it in terms of $\theta(\pi)$ shows that (1.3) holds. This proves the Corollary.

5. Proof of Theorem 3. Observe first that (1.4) and Fubini's Theorem shows that

$$\int_0^\pi \omega(x)\left|\frac{1}{x}\int_0^x f(t)\,dt\right|dx \leq \int_0^\pi |f(t)|\,dt\int_t^\pi \frac{\omega(x)}{x}\,dx \tag{5.1}$$

$$\leq C\int_0^\pi |f(t)|\,\omega(t)\,dt.$$

Hence, if $f \in L^1(\omega)$,

$$\begin{aligned} \int_0^\pi |f(t)|\log(\pi/t)\,dt &= \int_0^\pi |f(t)|\left(\int_t^\pi \frac{dx}{x}\right)dt \\ &= \int_0^\pi \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)dx \\ &\leq \left(\int_0^\pi \omega(x)\left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)dx\right)\left(\operatorname*{ess\cdot sup}_{(0,\pi)} 1/\omega(x)\right) \\ &\leq C\left(\int_0^\pi |f(t)|\,\omega(t)\,dt\right)\left(\operatorname*{ess\cdot sup}_{(0,\pi)} 1/\omega(x)\right) \\ &< \infty. \end{aligned} \tag{5.2}$$

Now, if $a_0 = 0$, Loo [**5**, pp. 272–274] has shown that (5.2) ensures that $F(x)$ is given by

$$F(x) = \left(\cot(x/2)\int_0^x f(t)\,dt + H(x)\right)\Big/2 \tag{5.3}$$

where $H(x)$ is given by the Lemma. Hence (5.1) and the inequality $\cot(x/2) \leq \pi/x$ show that $F \in L^1(\omega)$. This proves the Theorem for the case $a_0 = 0$, and as before, the general case follows easily from this.

6. Proof of Theorem 4. According to [**6**] and Lemma 2 of [**2**], the hypothesis (1.5) ensures that

$$\int_0^{\pi} \omega(x)\left[\frac{1}{x}\int_0^x |f(t)|\,dt\right]^p dx \le c\int_0^{\pi} |f(t)|^p \omega(t)\,dt. \tag{6.1}$$

Hence, Fubini's Theorem and Hölder's inequality shows that

$$\begin{aligned}\int_0^{\pi} |f(t)|\log(\pi/t)\,dt &= \int_0^{\pi}\left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)dx \\ &\le \left(\int_0^{\pi}\omega(x)\left[\frac{1}{x}\int_0^x |f(t)|\,dt\right]^p dx\right)^{1/p}\left(\int_0^{\pi}\omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} \\ &\le c\left(\int_0^{\pi}|f(t)|^p\omega(t)\,dt\right)^{1/p}\left(\int_0^{\pi}\omega(x)^{-1/(p-1)}\,dx\right)^{1/p'} < \infty\end{aligned}$$

whenever $f \in L^p(\omega)$. Hence, if $a_0 = 0$, $F(x)$ is again given by (5.3) and (6.1) shows that $F \in L^p(\omega)$. The general case follows easily from this.

7. Proof of Theorem 5. The hypothesis and Theorem 2 show that there is $F \in L^p(\omega)$ with Fourier cosine coefficients $\{A_\nu\}_0^\infty$ satisfying

$$\int_0^{\pi} |F(x)|^p\omega(x)\,dx \le c\int_0^{\pi} |f(x)|^p\omega(x)\,dx. \tag{7.1}$$

Further, since $\theta(\omega^{-1/(p-1)}, p'; u) = \theta(\omega, p; u)$, the hypothesis and Theorem 4 yields a $G \in L^{p'}(\omega^{-1/(p-1)})$ with Fourier cosine coefficients $\{B_\nu'\}_0^\infty$ satisfying

$$\int_0^{\pi} |G(x)|^{p'}\omega(x)^{-1/(p-1)}\,dx \le c\int_0^{\pi} |g(x)|^{p'}\omega(x)^{-1/(p-1)}\,dx. \tag{7.2}$$

If the left side of (1.7) is denoted by $L(f, g)$, Hölder's inequality followed by (7.1) and (7.2) then shows that L is a bilinear functional on $L^p(\omega) \times L^{p'}(\omega^{-1/(p-1)})$ satisfying

$$|L(f, g)| \le c\|f\|_{p,\omega}\|g\|_{p',\omega^{-1/(p-1)}}. \tag{7.3}$$

A direct computation (or an appeal to Bellman's Theorem [**3**]) shows that $L(f, g) = 0$ whenever f and g belong to the class $\mathcal{P}$ of finite linear combinations of $\{\cos \nu x\}_0^\infty$. Choose $f_n, g_n \in \mathcal{P}$ with $\|f_n - f\|_{p,\omega} \to 0$ and $\|g_n - g\|_{p',\omega^{-1/(p-1)}} \to 0$ as $n \to \infty$ (see [7, p. 89]). Then

$$\begin{aligned}L(f, g) &= [L(f, g) - L(f, g_n)] + [L(f, g_n) - L(f_n, g_n)] \\ &= L(f, g - g_n) + L(f - f_n, g_n)\end{aligned}$$

so that (7.3) yields

$$|L(f, g)| \leq c\{\|f\|_{p,\omega}\|g - g_n\|_{p',\omega^{-1/(p-1)}} + \|f - f_n\|_{p,\omega}\|g_n\|_{p',\omega^{-1/(p-1)}}\}$$

and since the right side tends to zero as $n \to \infty$, it follows that $L(f, g) = 0$ and the Theorem is proved.

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