PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 2, 1983

p-HENSELIAN FIELDS: K-THEORY, GALOIS COHOMOLOGY, AND GRADED WITT RINGS

ADRIAN R. WADSWORTH

For a field F with a p-Henselian valuation v, direct sum decompositions will be proved for Milnor's K-theory mod n (n a power of the prime p), for the Galois cohomology of F with $\mathbf{Z}_n$-coefficients, and for the graded Witt ring of quadratic forms of F (with $p = 2$). In each case, the summands of the ring associated to F are copies of the corresponding ring associated to the residue field of v, and the number of summands is determined by its value group. The theorems generalize results known for a field with a complete discrete valuation.

The direct decompositions in K-theory, in Galois cohomology, and for the graded Witt ring, for a field with a complete discrete valuation are a familiar part of the "local" machinery of field theory. In view of the increasing importance of Henselian fields, it seems worthwhile to spell out just how these results for complete discrete fields generalize to the Henselian case. While such generalizations are not surprising, and may in certain cases be known to some, they have not appeared in the literature. (The Witt ring of a Henselian field has been described, see [**15**, §12.2], but not the graded Witt ring.)

The basic setting for our results is a field F with a p-Henselian valuation (p a prime number), as described in §1. The p-Henselian property is a weaker relative version of the Henselian condition on a valuation. We work with p-Henselian valuations because they are exactly the ones for which direct sum decompositions exist (at least when F has enough roots of unity) — see (2.3), (3.10), and (4.7). We will consider K-theory, cohomology, and the graded Witt ring in separate and largely independent sections. While the direct sum formulas are strikingly similar in each category, the methods used to obtain them are quite different.

I would like to thank Richard Elman and Bill Jacob for several helpful conversations during the preparation of this paper.

1. p-Henselian fields and a ring construction. We will use the notation (F, v, Γ) for a field F with valuation v: $\dot{F} \to \Gamma$ (where $\dot{F} = F - \{0\}$). The value group Γ will be written additively. The valuation ring, maximal ring, group of units, and residue field associated to v will be denoted respectively V_v, $\mathfrak{m}_v$, U_v, and $\bar{F}$. For $a \in V_v$, $\bar{a}$ will denote its image in $\bar{F}$.

Let p be a prime number. A field extension $F \subseteq K$ is said to be a *p-extension* if K is Galois over F with Galois group a pro-p-group. The *p-closure* of F, which will be denoted $\tilde{F}(p)$, is the unique maximal p-extension of F in some algebraic closure. A valuation v on F is said to be *p-Henselian* if there is only one extension of v to $\tilde{F}(p)$. This is a special case ($\Omega = \tilde{F}(p)$) of the Ω-Henselian valuations introduced in [**5**] and discussed also in [**4**, Ch. II]. Bröcker points out [**5**, §1] that the usual characterizations and properties of Henselian fields have natural analogues for p-Henselian fields.

All Henselian valuations are p-Henselian. Valuations which are 2-Henselian but not in general Henselian have arisen naturally in connection with superpythagorean fields (see [**5**, (3.5)], [**6**, Cor. 8]) and in quadratic form theory — see [**14**] and [**28**]. Notably, Ware has shown [**28**, Th. 4.4] that essentially whenever the Witt ring of F is a group algebra there is a 2-Henselian valuation on F which induces the group algebra structure on WF.

PROPOSITION 1.1 *Let (F, v, Γ) be a field with valuation. Then v is p-Henselian if and only if v extends uniquely to each Galois extension L of F with $[L: F] = p$.*

Proof. Let $\tilde{v}$ be any extension of v to $\tilde{F}(p)$, and let M be the decomposition field of $\tilde{v}/v$ (see, e.g. [**11**, p. 110]). Suppose v is not p-Henselian; then $M \neq F$ by [**11**, (15.7)]. Let L be a minimal proper extension of F lying in M. Then L lies in some p-extension K of F with $[K: F] < \infty$. Since the Galois group $\mathcal{G}(K/L)$ is a maximal proper subgroup of the p-group $\mathcal{G}(K/F)$, L is Galois over F and $[L: F] = p$. Also, because L is normal over F the decomposition field of the restriction of $\tilde{v}$ to L is $M \cap L = M$, by [**11**, (15.6)(c)]. Hence, v does not extend uniquely to L. So, if v extends uniquely to every Galois extension of F of degree p, then v must be p-Henselian. The converse is clear. □

For any integer $n > 1$, μ_n will denote a group of n nth roots of unity in a field. To say that F contains n nth roots of unity (and hence, char $F \nmid n$) we will often write for short, $\mu_n \subseteq F$. Note that if F has a valuation v and char $\bar{F} \nmid n$, then $\mu_n \subseteq F$ implies $\mu_n \subseteq \bar{F}$; when this occurs, the residue map $V_v \to \bar{F}$ sends the nth roots of unity in F bijectively to those of $\bar{F}$.

PROPOSITION 1.2. *Let p be a prime number and (F, v, Γ) a field with valuation. Suppose $\mu_p \subseteq F$ and* char $\bar{F} \neq p$. *Then, the following are equivalent*:

(i) *v is p-Henselian*;

(ii) $1 + \mathfrak{m}_v \subseteq F^p$;
(iii) $1 + \mathfrak{m}_v \subseteq F^{p^c}$; *for every integer* $c \geq 1$.

Proof. (i) $\Rightarrow$ (iii) Take any $a \in 1 + \mathfrak{m}_v$, and let $f(X) = X^{p^c} - a$. Then f has an unrepeated linear factor mod $\mathfrak{m}_v$ and f splits in $\tilde{F}(p)$. So, the usual argument for Henselian fields applies here (cf. [**5**, (1.2)]), showing that f has a linear factor over F, i.e., $a \in F^{p^c}$.

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Suppose $1 + \mathfrak{m}_v \subset F^p$. Then, for any $s \in U_v$, if $\bar{s} \in \bar{F}^p$, then $s \in F^p$. Let (L, w, Δ) be any extension of (F, v, Γ), such that L is Galois over F and $[L: F] = p$. By Kummer theory, $L = F(d^{1/p})$, for some $d \in F - F^p$. If $v(d) \notin p\Gamma$, then $|\Delta : \Gamma| \geq p$. If $v(d) \in p\Gamma$, we may assume $v(d) = 0$ (replacing d by dt^p for suitable $t \in F$). Then, $\bar{d} \notin \bar{F}^p$, so $g(X) = X^p - \bar{d}$ is irreducible in $\bar{F}[X]$. Since g has a root in $\bar{L}$, $[\bar{L} : \bar{F}] \geq p$. In either case, the fundamental inequality $\sum e_i f_i \leq [L: F] = p$ [**11**, (17.5)] shows that w is the only extension of v to L. Hence, by (1.1), F is p-Henselian. □

For $p = 2$, (1.1) and (1.2) were proved by Dress [**8**, Sätze 2, 3].

REMARKS 1.3. (i) Note that, for any field with valuation (F, v, Γ) the condition $1 + \mathfrak{m}_v \subseteq F^p$ is of interest only when char $\bar{F} \neq p$. For, when char $\bar{F} = p$, we have $1 + \mathfrak{m}_v \subseteq F^p$ iff $F = F^p$ or v is the trivial valuation. (If char $F = 0$, this is deducible from the identity in [**4**, p. 2, bottom line].)

(ii) If char $F = p$, then the analogue of (1.2), at least for a discrete valuation, is: v is p-Henselian iff $\mathfrak{m}_v \subseteq \{a^p - a \mid a \in F\}$. This is a little harder to prove than (1.2).

The next lemma, which is well known, gives a basic property of p-Henselian valuations (when $\mu_p \subseteq F$) which we will use heavily. We write $\mathbf{Z}_n$ for $\mathbf{Z}/n\mathbf{Z}$ (with $\mathbf{Z}$ the integers).

LEMMA 1.4. *Let* $n = p^c$, *p prime. If* (F, v, Γ) *is a field with valuation for which* $1 + \mathfrak{m}_v \subseteq F^n$, *then there is a canonical short exact sequence*

$$(1.5) \qquad 1 \to \dot{\bar{F}}/\dot{\bar{F}}^n \xrightarrow{i} \dot{F}/\dot{F}^n \xrightarrow{\bar{v}} \Gamma/n\Gamma \to 0,$$

which is split exact, not canonically, since $\Gamma/n\Gamma$ *is a free* $\mathbf{Z}_n$*-module.*

Proof. For any field with valuation (F, v, Γ) we have the canonical short exact sequence

$$(1.6) \qquad 1 \to U_v/(U_v)^n \to \dot{F}/\dot{F}^n \xrightarrow{\bar{v}} \Gamma/n\Gamma \to 0,$$

where $\bar{v}$ is induced from v: $\dot{F} \to \Gamma$. If $1 + \mathfrak{m}_v \subseteq \dot{F}^n$, then the canonical surjection $U_v/(U_v)^n \to \dot{\bar{F}}/\dot{\bar{F}}^n$ is an isomorphism, which we substitute into (1.6) to obtain (1.5). To obtain a $\mathbf{Z}_n$-base of $\Gamma/n\Gamma$, take any subset of $\Gamma/n\Gamma$ mapping bijectively to a $\mathbf{Z}_p$-base of $\Gamma/p\Gamma$. □

For a field with p-Henselian valuation, we will compare a ring (in K-theory, Galois cohomology, or a graded Witt ring) for F with the corresponding one for $\bar{F}$. In each case the same kind of ring extension occurs, which we will now describe in general terms. All the rings we consider are assumed to be associative.

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded ring with 1 which is *anticommutative*, i.e., in which

$$(1.7) \qquad a_i a_k = (-1)^{ik} a_k a_i, \quad \text{for all } a_i \in A_i \text{ and } a_k \in A_k.$$

Take any $t \in A$ with $2t = 0$, any index set J, and let $\{x_j\}_{j \in J}$ be a collection of distinct symbols not in A.

DEFINITION 1.8. $A[J;\, t]$ denotes the anticommutative graded ring extension of A by the x_j's (with each x_j given degree 1), subject only to the relations

$$x_j^2 = t x_j, \quad \text{for each } j \in J.$$

More precisely, let $A_0\{X_j\}_{j \in J}$ denote the polynomial ring over A_0 in noncommuting indeterminates X_j. Then $A[J;\, t]$ is the factor ring of the free product $A *_{A_0} A_0\{X_j\}_{j \in J}$ modulo its ideal $\mathcal{I}$ generated by $\{X_j a_i - (-1)^i a_i X_j,\ X_j X_k + X_k X_j,\ X_j^2 - tX_j \mid a_i \in A_i,\ i \geq 0,\ j, k \in J\}$; x_j is the image of X_j. The free product is given a grading extending that on A by assigning degree 1 to each X_j. Since $\mathcal{I}$ is a homogeneous ideal there is an induced grading on $A[J;\, t]$. The first two types of generators of $\mathcal{I}$ (and the anticommutativity of A) assure that $A[J;\, t]$ is anticommutative. To describe $A[J;\, t]$ more fully we need further

NOTATION 1.9. For an index set J, let $\mathcal{J}_m$ denote the collection of all subsets of J with m elements, for $m \in \mathbf{Z}$, $m > 0$. Let $\mathcal{J} = \bigcup_{m=1}^{\infty} \mathcal{J}_m$. A typical member of $\mathcal{J}_m$ will be written as $\vec{j}$, and the elements of the set $\vec{j}$ listed as $j_1, j_2, \ldots, j_m$.

REMARKS 1.10. The ring $A[J;\, t]$ of (1.8) is a free left (and right) A-module, with base $\{1\} \cup \{x_{j_1} x_{j_2} \cdots x_{j_m} \mid \vec{j} = \{j_1, \ldots, j_m\} \in \mathcal{J}\}$. This is easily seen by induction on the cardinality $|J|$ of J if $|J| < \infty$, and by a direct limit argument if $|J| = \infty$. (Note that in the base for $A[J;\, t]$ there is one term $x_{j_1} \cdots x_{j_m}$ for each $\vec{j} \in \mathcal{J}$. That term depends on the order of the j_i (not just on $\vec{j}$); but the dependence is only up to sign, and can be ignored.) Since the terms in the base are homogeneous, we can read off the homogeneous components of $A[J;\,t]$: for $k \geq 0$,

$$A[J;\,t]_k = A_k \oplus \bigoplus_{m=1}^{k} \bigoplus_{\vec{j} \in \mathcal{J}_m} A_{k-m} x_{j_1} \cdots x_{j_m}. \tag{1.11}$$

In this direct decomposition, each summand $A_{k-m} x_{j_1} \cdots x_{j_m}$ is isomorphic to A_{k-m}.

The ring $A[J;\,t]$ can be described as a tensor product of A with another ring, but this is somewhat cumbersome to do in general (cf. the discussion of products of κ-algebras in [3, p. 366]). We note a few special cases.

EXAMPLES 1.12. (i) If $t = 0$, then $A[J;\,t] \cong A \otimes_{\mathbf{Z}} \Lambda(J)$, where $\Lambda(J)$ denotes the exterior algebra (over $\mathbf{Z}$) of the free $\mathbf{Z}$-module with base $\{x_j\}_{j \in J}$. Note that $t = 0$ whenever A is (additively) an n-torsion group, n odd.

(ii) If A is 2-torsion, then A and $A[J;\,t]$ are commutative rings, and we have the graded ring isomorphism

$$A[J;\,t] \cong A \otimes_{\mathbf{Z}_2[X]} GR, \tag{1.13}$$

where $\mathbf{Z}_2[X]$ is the polynomial ring over $\mathbf{Z}_2$ and $GR = \mathbf{Z}_2[X][J;\,X]$. That is, GR is the factor ring of the (commutative) polynomial ring $\mathbf{Z}_2[X, \{X_j\}_{j \in J}]$ modulo the ideal generated by $\{X_j^2 - XX_j \mid j \in J\}$. ($GR$ can also be described as a graded Witt ring — see §4.) A is made into a $\mathbf{Z}_2[X]$-algebra by mapping X to t. To see that (1.13) is an isomorphism, note that each side is a free A-module, and corresponding basis elements multiply analogously.

2. Milnor's K-theory mod n. Let $(K_*F)_n$ denote Milnor's K-theory for a field F, taken mod n. We now show how $(K_*F)_n$ is related to $(K_*\bar{F})_n$ for a p-Henselian field, n a power of p. The arguments are easy and natural generalizations of those given by Milnor [19, §2] for a complete discrete valuation. The resulting decomposition of $(K_*F)_n$ in terms of $(K_*\bar{F})_n$ provides a prototype of what should be expected in the setting of Galois cohomology and of graded Witt rings.

We recall briefly Milnor's construction in [19] of K_*F. For a field F let $T(\dot{F})$ denote the tensor algebra of $\dot{F}$ as a $\mathbf{Z}$-module. Reducing $T(\dot{F})$ modulo the (homogeneous) ideal generated by $\{a \otimes (1-a) \mid a \in \dot{F},\, a \neq 1\}$ yields the anticommutative [19, (1.1)] graded ring K_*F, whose ith homogeneous component is denoted K_iF. For any integer $n > 1$, let $(K_*F)_n$ denote the graded anticommutative ring obtained by reducing $K_*F \bmod n$; so $(K_iF)_n = K_iF/n(K_iF)$. For $a \in \dot{F}$, let $l(a)$ denote the image of a in $(K_1F)_n$ (which is canonically isomorphic to $\dot{F}/\dot{F}^n$). Since the group operation on $(K_1F)_n$ is written additively, we have $l(ab) = l(a) + l(b)$. This is a slight modification of Milnor's "logarithmic" notation in that our $l(a)$ lies in $(K_1F)_n$, not in K_1F.

PROPOSITION 2.1. (cf. [**19**, (2.1), (2.2)]). *Let (F, v, Γ) be any field with valuation, and let $n = p^c$, where p is prime. Choose any subset $\{\pi_j\}_{j\in J}$ of $\dot{F}$ which maps bijectively under $\bar{v}$ to a $\mathbf{Z}_n$-basis of $\Gamma/n\Gamma$. Then there is a surjective graded ring homomorphism θ: $(K_*F)_n \to (K_*\bar{F})_n[J; l(\overline{-1})]$ such that*

$$l(u) \mapsto l(\bar{u}) \quad \text{for } u \in U_v \quad \text{and} \quad l(\pi_j) \mapsto x_j \quad \text{for } j \in J.$$

Proof. Let $B = (K_*\bar{F})_n[J;\ l(-\bar{1})]$, the extension ring of $(K_*\bar{F})_n$ described in (1.8)–(1.11).

Any $a \in \dot{F}$ can be written $a = b\pi_{j_1}^{m_1} \cdots \pi_{j_k}^{m_k} d^n$ where $b \in U_v, j_1, \ldots, j_k$ are distinct elements of J, and $d \in \dot{F}$. Such a decomposition of a is not unique, but $m_1, \ldots, m_k$ are uniquely determined mod n (by $\bar{v}(a)$ in $\Gamma/n\Gamma$) and $\bar{b}$ in $\dot{\bar{F}}$ is unique mod $\dot{\bar{F}}^n$. Thus, there is a well-defined surjective group homomorphism α: $\dot{F} \to B_1$ given by

$$a \mapsto l(\bar{b}) + \sum_{i=1}^{k} m_{j_i} x_{j_i}.$$

This map extends uniquely to a ring homomorphism β: $T(\dot{F}) \to B$. To see that β induces a map γ: $K_*F \to B$, we show that $\beta(a \otimes (1 - a)) = 0$, for every $a \in \dot{F}$, $a \neq 1$. By definition, $\beta(a \otimes (1 - a)) = \alpha(a) \cdot \alpha(1 - a)$.

Case 1. $a \in \mathfrak{m}_v$. Then $\overline{1 - a} = \bar{1}$ in $\bar{F}$, so $\alpha(1 - a) = l(\bar{1}) = 0$, hence $\beta(a \otimes (1 - a)) = 0$.

Case 2. $a \in U_v$. If $\bar{a} = \bar{1}$ in $\bar{F}$, then $\alpha(a) = 0$, and we are done as in Case 1. If $\bar{a} \neq \bar{1}$, then $\overline{1 - a} = \bar{1} - \bar{a} \in \dot{\bar{F}}$, and $\beta(a \otimes (1 - a)) = l(\bar{a}) \cdot l(\bar{1} - \bar{a}) = 0$.

Case 3. $a \notin V_v$. Then $v(1 - a) = v(a)$ and $\overline{(1 - a)/a} = -\bar{1}$ in $\bar{F}$. Writing $a = b\pi_{j_1}^{m_1} \cdots \pi_{j_k}^{m_k} d^n$ as above we have correspondingly $1 - a = (b(1 - a)/a)\pi_{j_1}^{m_1} \cdots \pi_{j_k}^{m_k} d^n$. Since $l(\overline{b(1 - a)/a}) = l(-\bar{b})$ in $(K_1\bar{F})_n$, and $2l(-\bar{1}) = l(\bar{1}) = 0$, and $m_{j_i}^2 - m_{j_i}$ is even, it follows that

$$\begin{aligned}\beta(a \otimes (1 - a)) &= \Big[l(\bar{b}) + \sum_i m_{j_i} x_{j_i}\Big]\Big[l(\overline{-b}) + \sum_i m_{j_i} x_{j_i}\Big] \\ &= l(\bar{b}) \cdot l(-\bar{b}) + \big(l(\bar{b}) - l(-\bar{b})\big)\sum_i m_{j_i} x_{j_i} + \sum_i m_{j_i}^2 x_{j_i}^2 \\ &= 0 + \sum_i \big(-l(\overline{-1}) m_{j_i} + l(\overline{-1}) m_{j_i}^2\big) x_{j_i} = 0.\end{aligned}$$

Thus, γ exists, and since B is n-torsion, γ induces a homomorphism θ: $(K_*F)_n \to B$. One easily sees that θ has the properties described in the proposition; θ is surjective since α is surjective and the ring B is generated by its terms of degree 1 (since this is true for $(K_*\bar{F})_n$). □

Note that the ring $(K_*\bar{F})_n[J;\ l(-\bar{1})]$ has the form described in (1.12)(i) if n is odd and (1.12)(ii) if $n = 2$. Observe also that the argument of (2.1) shows that if Γ itself is a free abelian group (not just $\Gamma/n\Gamma$), then there is a map $K_*F \to K_*\bar{F}[J;\ l(-\bar{1})]$.

REMARK 2.2 (cf. [**19**, (2.1), (2.2)]). We can obtain various "residue maps" from $(K_*F)_n$ to $(K_*\bar{F})_n$ by composing θ of (2.1) with projection onto any of the $(K_*\bar{F})_n$-components of $(K_*\bar{F})_n[J;\ l(-\bar{1})]$ in the direct decomposition given in (1.11). With one exception, these residue maps depend substantially on the choice of $\{\pi_j\}_{j\in J}$. The exception occurs when $\Gamma/n\Gamma$ has finite rank, say r, as a $\mathbf{Z}_n$-module. Then set $J = \{1, 2, \ldots, r\}$. Using projection onto the component of $x_1x_2\cdots x_r$, we have ∂: $(K_*F)_n \to (K_*\bar{F})_n$, of degree $-r$, such that for any $u_1, \ldots, u_k \in U_v$,

$$\partial(l(u_1)\cdots l(u_k)l(\pi_1)\cdots l(\pi_r)) = l(\bar{u}_1)\cdots l(\bar{u}_k),$$
$$\partial\big(l(u_1)\cdots l(u_k)l(\pi_{i_1})\cdots l(\pi_{i_m})\big) = 0 \quad \text{if } 0 \le m < r.$$

If we make a different choice of the π_i, say $\{\pi_1', \ldots, \pi_r'\}$, and form the corresponding map ∂', then $\partial' = d\partial$. The constant d, a unit in $\mathbf{Z}_n$, is the determinant of the change of base matrix from the ordered base $\{\bar{v}(\pi_1'), \ldots, \bar{v}(\pi_r')\}$ of $\Gamma/n\Gamma$ to $\{\bar{v}(\pi_1), \ldots, \bar{v}(\pi_r)\}$. Of course, if $n = 2$ or if there is a preferred basis of $\Gamma/n\Gamma$ (e.g., when $\Gamma \cong \mathbf{Z}$), then ∂ is canonically determined.

PROPOSITION 2.3 (cf. [**19**, (2.6)]). *Let $n = p^c$, where p is prime. For a field with valuation (F, v, Γ) the map θ of* (2.1) *is an isomorphism if and only if $1 + \mathfrak{m}_v \subseteq F^n$. When this occurs, and $(K_*\bar{F})_n$ is identified with its image in $(K_*F)_n$, we have (using notation* (1.9)) *for every $k \ge 0$,*

$$(K_kF)_n = \big(K_k\bar{F}\big)_n \oplus \bigoplus_{m=1}^{k} \bigoplus_{\vec{j}\in\mathcal{J}_m} \big(K_{k-m}\bar{F}\big)_n l(\pi_{j_1})\cdots l(\pi_{j_m}).$$

Furthermore, we have for each summand, $(K_{k-m}\bar{F})_n l(\pi_{j_1})\cdots l(\pi_{j_m}) \cong (K_{k-m}\bar{F})_n$.

Proof. Assume some choice of $\{\pi_j\}_{j\in J}$ has been made as in (2.1). Suppose $1 + \mathfrak{m}_v \subseteq F^n$. It is easy to verify that the canonical injection i: $\dot{\bar{F}}/\dot{\bar{F}}^n \to \dot{F}/\dot{F}^n$ of (1.5) induces a graded ring homomorphism λ: $(K_*\bar{F})_n \to (K_*F)_n$ for which $\lambda(l(\bar{u})) = l(u)$ for any $u \in U_v$. Since

$(K_*\bar{F})_n[J;\ l(-\bar{1})]$ is a free $(K_*\bar{F})_n$-module by (1.10), there is a well-defined map ν: $(K_*\bar{F})_n[J;\ l(-\bar{1})] \to (K_*F)_n$ given by $\nu(\alpha) = \lambda(\alpha)$ and $\nu(\alpha x_{j_1} \cdots x_{j_k}) = \lambda(a)\pi_{j_1} \cdots \pi_{j_k}$ for any $\alpha \in (K_*\bar{F})_n$ and distinct $j_1, \ldots, j_k \in J$. It is easy to check that ν is an inverse for θ; hence θ is an isomorphism. For the converse, note that if $a \in 1 + \mathfrak{m}_v$, then $\theta(l(a)) = 0$. The injectivity of θ implies $l(a) = 0$ in $(K_1F)_n \cong \dot{F}/\dot{F}^n$; hence $1 + \mathfrak{m}_v \subseteq F^n$. The rest of (2.3) is immediate from (1.10). □

COROLLARY 2.4. *Let* $n = p^c$, *p prime, and let* (F, v, Γ) *be a field with valuation such that* char $\bar{F} \neq p$. *Then the map* θ *of* (2.1) *is an isomorphism if F is p-Henselian and* $\mu_p \subseteq F$ *or if F is Henselian.*

Proof. The p-Henselian case is immediate from (1.2) and (2.3). The argument of (1.2)(i) $\Rightarrow$ (iii) showing that $1 + \mathfrak{m}_v \subseteq F^n$ is actually valid if v is Ω-Henselian (cf. [**5**, §1]) for some normal extension Ω of F such that $F \subseteq \Omega^n$. If F is Henselian, we may take Ω to be the separable closure of F. □

REMARK 2.5. The number n was restricted to be a prime power in (2.1)–(2.4) to assure that $\Gamma/n\Gamma$ be a free $\mathbf{Z}_n$-module. For more general values of n one can always reduce to the prime power case: if $n = n_1 \cdots n_k$, with $n_i = p_i^{c_i}$ and the p_i distinct primes, we have the primary decomposition $(K_*F)_n = \oplus_{i=1}^{k} (K_*F)_{n_i}$. This is also a direct sum decomposition of rings. Note that the rank of $\Gamma/n_i\Gamma$ may vary with i.

3. Galois cohomology. We now prove the analogues to (2.1) and (2.3) for Galois cohomology with $\mathbf{Z}_n$ coefficients. The case $F = \bar{F}((t))$ and $n = 2$ was proved by Milnor [**19**, p. 341]. Arason [**1**, pp. 475–477] obtained the corresponding direct decomposition for v a Henselian discrete valuation on F and $n = 2$, and Elman [**9**, Th. 2.6] generalized Arason's argument to arbitrary n. (Of course the analogous direct decomposition for the Brauer group, with F a field with complete discrete valuation is much older — tracing back to Witt [**29**].) Our approach here applies to Henselian valuations with arbitrary value group, using an induction argument with an abstracted version of Arason's argument for the induction step.

For a profinite group G and a discrete G-module M, let $H^i(G, M)$, $i \geq 0$, denote the ith (continuous) cohomology group of G on M. (See [**7**, Ch. IV, V] or [**23**] for definitions and general background.) For any integer $n \geq 2$, we write $H^i(G, n)$ for $H^i(G, \mathbf{Z}_n)$ with G acting trivially on $\mathbf{Z}_n := \mathbf{Z}/n\mathbf{Z}$. Identifying $\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_n$ with $\mathbf{Z}_n$ via $1 \otimes 1 \to 1$, $H^*(G, n) := \oplus_{i=0}^{\infty} H^i(G, n)$ becomes an anticommutative (see (1.7)) graded ring with multiplication given by the cup product (cf. [**7**, pp. 107–108]).

THEOREM 3.1. *Let G be a profinite group, N a closed normal subgroup of G, and $n \geq 2$ an integer. Suppose,*

(i) *there is an isomorphism φ: $H^1(N, n) \cong \mathbf{Z}_n$;*

(ii) *G/N acts trivially on $H^1(N, n)$;*

(iii) *the restriction map $\operatorname{res}_{G\to N}$: $H^1(G, n) \to H^1(N, n)$ is surjective;*

(iv) *$H^i(N, n) = 0$ for every $i > 1$.*

Choose any $\beta \in H^1(G, n)$ such that $\varphi(\operatorname{res}_{G\to N}(\beta)) = 1$ in $\mathbf{Z}_n$. Then, for every $k \geq 1$, there is a short exact sequence

$$0 \to H^k(G/N, n) \xrightarrow{\inf} H^k(G, n) \xrightarrow{\rho} H^{k-1}(G/N, n) \to 0,$$

which is split by the map

$$\psi: H^{k-1}(G/N, n) \to H^k(G, n) \quad \textit{given by } \alpha \mapsto \beta \cup \inf_{G/N\to G}(\alpha).$$

Proof. For any $k \geq 1$, consider the following diagram:

(3.2)

$$\cdots \xrightarrow{d} H^k(G/N, n) \xrightarrow{\inf} H^k(G, n) \xrightarrow{r} H^{k-1}(G/N, H^1(N, n)) \xrightarrow{d} H^{k+1}(G/N, n) \xrightarrow{\inf} \cdots$$
$$\psi \nwarrow \qquad \downarrow \varphi^*$$
$$H^{k-1}(G/N, n)$$

Because $H^i(N, n) = 0$ for $i > 1$, the top line of (3.2) is part of a long exact sequence obtained by Hochschild and Serre [**13**, p. 132, Th. 3] using their spectral sequence for group extensions. The map inf is inflation from G/N to G, and d is the map d_2 of the Hochschild-Serre spectral sequence, $E_2^{p,q} = H^p(G/N, H^q(N, n))$. The map r is determined by the following: For every element η of $H^k(G, n)$ there is an inhomogeneous continuous normalized cocycle $\tau \in C^k(G, n)$, with class $[\tau] = \eta$, such that for $g_1,\dots,g_k \in G$, the value of $\tau(g_1,\dots,g_k)$ depends only on g_1 and the cosets $g_2N,\dots,g_kN$. For any such τ, $r([\tau]) = [\sigma]$, where the cocycle $\sigma \in C^{k-1}(G/N, H^1(N, n))$ is defined by

$$\sigma(g_2N,\dots,g_kN)(h) = \tau(h, g_2,\dots,g_k), \quad \text{for all } h \in N,\ g_2,\dots,g_k \in G.$$

The map φ^* in (3.2) is the isomorphism induced from φ, using (i) and (ii). ψ was defined above. We compute $\varphi^* \circ r \circ \psi$. Take any $\alpha = [\rho] \in H^{k-1}(G/N, n)$. Then $\psi(\alpha) = [\tau]$, where, for $g_1,\dots,g_k \in G$, $\tau(g_1,\dots,g_k) = \beta(g_1)\cdot\rho(g_2N,\dots,g_kN)$. This cocycle τ has the property given above. Hence, $r(\psi(\alpha)) = [\sigma]$, where

$$\sigma(g_2N,\dots,g_kN)\colon h \mapsto \beta(h)\cdot\rho(g_2N,\dots,g_kN), \quad \text{for } h \in N,\ g_2,\dots,g_k \in G.$$

Thus, $\varphi^*(r(\psi(\alpha))) = [\kappa]$ where

$$\begin{aligned}\kappa(g_2N,\dots,g_kN) &= \varphi([h \to \beta(h)\cdot \rho(g_2N,\dots,g_kN)]) \\ &= \rho(g_2N,\dots,g_kN)\cdot \varphi(\operatorname{res}_{G\to N}(\beta)) \\ &= \rho(g_2N,\dots,g_kN)\cdot 1.\end{aligned}$$

This calculation shows that $\varphi^* \circ r \circ \psi = \mathrm{id}$. Hence, r is surjective, so $d = 0$ in the long exact sequence in (3.2), and this sequence breaks up into short exact sequences. (For $k = 1$, the injectivity of $\inf_{G/N\to G}$ follows from the standard isomorphism $H^1(G, n) \cong \mathrm{Hom}(G, \mathbf{Z}_n)$.) Setting $\rho = \varphi^* \circ r$, we have the split short exact sequences of the theorem. □

REMARKS 3.3. (i) The quoted theorem from [**13**] is proved there for ordinary group cohomology, not continuous cohomology. However, the proof in [**13**] carries over to the profinite setting simply by replacing the cochains and coboundaries by continuous cochains and coboundaries.

(ii) Alternative proofs of [**13**, Th. 3] can be found in [**21**, (11.35), (11.45)] or in [**12**] — Hattori gives a proof without using spectral sequences. However, the specific property of the map r needed to see that ψ splits the short exact sequence in (3.1) (the property given in the sentence earlier beginning "For any such τ...") is not apparent from the arguments in [**12**] or [**21**]. Nor is it very clear in [**13**], either, but it follows from the proof of [**13**, Th. 1, pp. 121–122].

COROLLARY 3.4. *With the hypotheses of* (3.1), *we have, for each* $k \geq 1$,

$$H^k(G, n) = \inf(H^k(G/N, n)) \oplus \left(\beta \cup \inf(H^{k-1}(G/N, n))\right),$$

with $\inf(H^k(G/N, n)) \cong H^k(G/N, n)$ *and* $\beta \cup \inf(H^{k-1}(G/N, n)) \cong H^{k-1}(G/N, n)$.

Proof. This is immediate from (3.1). □

For the rest of this section we adopt the following

Standing Hypotheses 3.5. *Let* (F, v, Γ) *be a field with valuation. We assume* $\mu_n \subseteq F$, *where* $n = p^c$, *p prime, and* $\operatorname{char} \bar{F} \neq p$. *Fix some subset* $\{\pi_j\}_{j\in J} \subseteq \dot{F}$, *such that* $\{\bar{v}(\pi_j)\}$ *is a* $\mathbf{Z}_n$*-base of* $\Gamma/n\Gamma$. *Let* $\mathcal{J}_m$ *denote the collection of subsets of J with m elements* (*as in* (1.9) *above*), $m = 1, 2, \dots$.

Let F_s denote a separable closure of F, and let G_F denote the Galois group $\mathcal{G}(F_s/F)$. Since $\mu_n \subseteq F$, G_F acts trivially on μ_n. Fixing henceforth a generator $\omega \in \mu_n$ to be mapped to 1 in $\mathbf{Z}_n$, we will identify $H^*(G_F, \mu_n)$

with $H^*(G_F, n)$. Let

$$\delta\colon \dot{F} \to H^1(G_F, n)$$

denote the connecting homomorphism arising from the short-exact G_F-module sequence given by the nth power map

$$1 \to \mu_n \to \dot{F}_s \xrightarrow{(\ )^n} \dot{F}_s \to 1.$$

Because $\mu_n \subseteq \bar{F}$ (as char $\bar{F} \neq p$) we have an analogous map $\bar{\delta}\colon \dot{\bar{F}} \to H^1(G_{\bar{F}}, n)$.

Suppose now that (F, v, Γ) is Henselian. Let (F_{nr}, w, Γ) denote the maximal unramified extension of F in F_s. Then $\overline{F_{nr}} \cong (\bar{F})_s$. Since w/v is indecomposed and unramified, $G_{\bar{F}} \cong G_F/\mathcal{G}(F_s/F_{nr})$; after identifying these groups, we have an inflation map inf: $H^*(G_{\bar{F}}, n) \to H^*(G_F, n)$.

THEOREM 3.6. *Suppose, in addition to hypotheses* (3.5), *that* (F, v, Γ) *is Henselian. Let* $\beta_j = \delta(\pi_j)$, *for every* $j \in J$. *Then, for each* $k \geq 0$,

$$H^k(G_F, n) = \inf\big(H^k(G_{\bar{F}}, n)\big)$$
$$\oplus \bigoplus_{m=1}^{k} \bigoplus_{\vec{j} \in \mathcal{J}_m} \big(\inf\big(H^{k-m}(G_{\bar{F}}, n)\big)\big) \cup \beta_{j_1} \cup \beta_{j_2} \cup \cdots \cup \beta_{j_m}.$$

Moreover inf *is injective and for each* $\vec{j} \in \mathcal{J}_m$,

$$\inf\big(H^{k-m}(G_{\bar{F}}, n)\big) \cup \beta_{j_1} \cup \cdots \cup \beta_{j_m} \cong H^{k-m}(G_{\bar{F}}, n).$$

Furthermore, there is a graded ring isomorphism (*in notation* (1.8))

$$\nu\colon H^*(G_F, n) \xrightarrow{\cong} H^*(G_{\bar{F}}, n)[J; \bar{\delta}\,(-\bar{1})], \quad \text{with } \nu(\beta_j) = x_j, j \in J.$$

Proof. To simplify the notation, we write $H^*(G)$ for $H^*(G, n)$ and $H^*(L/K)$ for $H^*(\mathcal{G}(L/K))$. For a field K, $F \subseteq K \subseteq F_s$, let (K, v_K, Γ_K) denote the unique extension of v to K.

Let $q = \operatorname{char} \bar{F}$ (so $q \neq p$), and let F_{tr} denote the maximal tamely ramified extension of F in F_s. Note that for the valuation extension v_{F_s}/v, F_{tr} is the ramification field, F_{nr} is the inertia field, and F itself is the decomposition field (cf. [**11**]). If $q = 0$, of course $F_{tr} = F_s$. If $q \neq 0$, the Galois group $\mathcal{G}(F_s/F_{tr})$ is a pro-q-group ([**11**, (20.11)]); hence, as q is prime to $n = p^c$, $H^k(F_s/F_{tr}) = 0$ for all $k \geq 1$. It follows by the inflation-restriction sequence [**7**, p. 101] that inf: $H^*(F_{tr}/F) \to H^*(G_F)$ is an isomorphism. Thus, it suffices to prove the theorem with $\mathcal{G}(F_{tr}/F)$ replacing G_F (and the β_j modified correspondingly).

By valuation theory F_{tr}/F_{nr} is an abelian Galois extension [**11**, (20.14)] (see also [**22**, (1.1)]) and every intermediate field K is completely determined by the image of Γ_K in $\Gamma_{F_{tr}}$ [**11**, (20.19)]. Furthermore, by [**11**, (20.20)], for all fields K and M with $F_{nr} \subseteq K \subseteq M \subseteq F_{tr}$ and $[M:K] < \infty$, we have $[M:K] = |\Gamma_M : \Gamma_K|$.

For every $\vec{j} = \{j_1, \ldots, j_m\} \in \mathcal{J}_m$, let $K_{\vec{j}} = F_{nr}(\{\pi_{j_i}^{1/p^a} \mid i = 1, 2, \ldots, m, a = 1, 2, \ldots\})$. Since $\mu_{p^a} \subseteq F_{nr}$ for each a, $K_{\vec{j}}$ is a Galois extension of F. Let $L = \cup_{\vec{j} \in \mathcal{J}} K_{\vec{j}}$. Then Γ_L is p-divisible, as $\{\bar{v}(\pi_j)\}$ spans $\Gamma/n\Gamma$. So $\Gamma_{F_{tr}}/\Gamma_L$ has no p-torsion, and the previous paragraph shows that $\mathcal{G}(F_{tr}/L)$ has order prime to p (as a supernatural number — cf. [**23**, p. I-4]). Therefore, arguing as before, $H^k(F_{tr}/L) = 0$ for all $k \geq 1$, so it suffices to prove the theorem with $\mathcal{G}(L/F)$ replacing G_F. Since $L = \varinjlim K_{\vec{j}}$, it suffices to analyze $H^*(G_{\vec{j}})$, where $G_{\vec{j}} = \mathcal{G}(H_{\vec{j}}/F)$.

For each $c \in \dot{F}$, the character $\delta(c) \in H^1(G_F)$ is given by $\delta(c)$: $\sigma \mapsto \sigma(c^{1/n})/c^{1/n} \in \mu_n$ $(\sigma \in G_F)$; so, clearly $\delta(c)$ lies in the image of $\inf_{\mathcal{G}(F(c^{1/n})/F) \to G_F}$. For each $j \in J$, let β_j' be the unique element of $H^1(F(\pi_j^{1/n})/F)$ such that $\inf_{\mathcal{G}(F(\pi_j^{1/n})/F) \to G_F}(\beta_j') = \beta_j$.

We claim that for every $\vec{j} = \{j_1, \ldots, j_m\} \in \mathcal{J}$, and every $k \geq 0$,

(3.7)

$$H^k(G_{\vec{j}}) = \inf H^k(G_{\bar{F}}) \oplus \bigoplus_{l=1}^{\min(k,m)} \bigoplus_{\vec{i} \in \mathcal{J}_l, \vec{i} \subseteq \vec{j}} \left(\inf H^{k-l}(G_{\bar{F}})\right) \cup \gamma_{i_1} \cup \cdots \cup \gamma_{i_l},$$

where $\gamma_{i_s} = \inf \beta_{i_s}' \in H^1(G_{\vec{j}})$. Further, inf: $H^k(G_{\bar{F}}) \to H^k(G_{\vec{j}})$ *is injective and each term* $(\inf H^{k-l}(G_{\bar{F}})) \cup \gamma_{i_1} \cup \cdots \cup \gamma_{i_l} \cong H^{k-l}(G_{\bar{F}})$. (There are various inflation maps here — each is the only one that makes sense in its context.)

We prove the claim by induction on m. To simplify notation write $\vec{j} = \{1, 2, \ldots, m\}$ and $\vec{h} = \{1, 2, \ldots, m-1\}$, and set $\pi = \pi_m$. (If $m = 1$, set $K_{\vec{h}} = F_{nr}$.) Then $K_{\vec{j}} = K_{\vec{h}}(\{\pi^{1/p^a}\}_{a=1}^{\infty})$, and $\pi^{1/p} \notin K_{\vec{h}}$. (Otherwise, by Kummer theory, $\pi \in$ span of $\{\pi_1, \ldots, \pi_{m-1}\}$ in $\dot{F}/\dot{F}^{p^a}$, for some a, contradicting the choice of the π_j's.) Let $G = G_{\vec{j}}$ and $N = \mathcal{G}(K_{\vec{j}}/K_{\vec{h}})$, a closed normal subgroup of G. Note that for each a, $K_{\vec{h}}(\pi^{1/p^a})$ is a cyclic Galois extension of $K_{\vec{h}}$ of degree p^a. Since $K_{\vec{j}} = \cup_{a=1}^{\infty} K_{\vec{h}}(\pi^{1/p^a})$, $N \cong \varprojlim \mathbf{Z}/p^a\mathbf{Z} = \hat{\mathbf{Z}}_p$, which is a free pro-$p$-group of rank 1. Hence, there is an isomorphism φ: $H^1(N) \cong \mathbf{Z}_n$, and $H^i(N) = 0$ for $i > 1$ (see [**23**, pp. I-6, I-37] or [**17**, pp. 43, 45]). Furthermore, as $\mu_n \subseteq F$, G acts trivially on $H^1(N)$. (For, recall that the action of G on $H^1(N) = \operatorname{Hom}(N, n)$ is given by

$$(\tau \cdot \chi)(\sigma) = \tau[\chi(\tau^{-1}\sigma\tau)] = \chi(\tau^{-1}\sigma\tau),$$

for $\tau \in G$, $\chi \in H^1(N)$, and $\sigma \in N$. But $\chi(\tau^{-1}\sigma\tau) = \chi(\sigma)$, since $\tau^{-1}\sigma\tau \equiv \sigma$ (mod $\mathcal{G}(K_{\vec{j}}/K_{\vec{h}}(\pi^{1/n}))$), and this group lies in ker χ.) Also, if we choose $\sigma \in N$ mapping to a generator of $\mathcal{G}(K_{\vec{h}}(\pi^{1/n})/K_{\vec{h}})$, then $\text{res}_{G\to N}(\gamma_m(\sigma)) = \sigma(\pi^{1/n})/\pi^{1/n}$, which is a generator of μ_n. Thus, $\text{res}_{G\to N}(\gamma_m)$ is a generator of $H^1(N)$; for some $b \in \mathbf{Z}$ prime to n, $\varphi(\text{res}_{G\to N}(b\gamma_m)) = 1$ in $\mathbf{Z}_n$. Since $G/N \cong G_{\vec{h}}$ and $(-1)^{k-1}\gamma_m$ generates the same group as $b\gamma_m$ it follows by (3.1) that, for every $k \geq 1$,

$$H^k(G_{\vec{j}}) = \inf H^k(G_{\vec{h}}) \oplus \left(\inf H^{k-1}(G_{\vec{h}})\right) \cup \gamma_m, \tag{3.8}$$

with $H^k(G_{\vec{h}})$ and $H^{k-1}(G_{\vec{h}})$ mapping injectively into the respective summands. If $m = 1$, $G_{\vec{h}} \cong G_{\bar{F}}$ and this establishes the claim. If $m > 1$, we may assume by induction that the claim holds for $H^*(G_{\vec{h}})$; substituting (3.7) for $H^*(G_{\vec{h}})$ into (3.8) yields (3.7) for $H^k(G_{\vec{j}})$. Thus, the claim is proved for all $\vec{j} \in \mathcal{J}$.

Since all the inflation maps inf: $H^*(G_{\vec{i}}) \to H^*(G_{\vec{j}})$ are injective for $\vec{i}, \vec{j} \in \mathcal{J}$, $\vec{i} \subseteq \vec{j}$, and $H^*(L/F) = \varinjlim H^*(G_{\vec{j}})$, it follows that inf: $H^*(G_{\vec{j}}) \to H^*(L/F)$ is injective for every $\vec{j}$. Hence, we have for $H^*(L/F)$ a direct sum formula corresponding to (3.7); that formula is just the one in (3.6) with $\mathcal{G}(L/F)$ replacing G_F (and $\inf(\beta'_{j_i})$ replacing β_{j_i}). As shown above, this proves the direct sum formula in (3.6), and the injectiivity assertions for the maps. Hence, $H^*(G_F)$ is a free left $H^*(G_{\bar{F}})$-module. It follows that there is a well-defined degree-preserving $H^*(G_{\bar{F}})$-module isomorphism ν: $H^*(G_F) \to H^*(G_{\bar{F}})[J; \bar{\delta}(-\bar{1})]$ given by

$$\nu\left((\inf \alpha) \cup \beta_{j_1} \cup \cdots \cup \beta_{j_m}\right) = \alpha x_{j_1} \cdots x_{j_m}, \quad \text{and} \quad \nu(\inf \alpha) = \alpha,$$

for any $\alpha \in H^i(G_{\bar{F}})$. Since distinct β_j's anticommute, $\beta_j \cup \beta_j = \delta(-1) \cup \beta_j$, and $\delta(-1) = \inf \bar{\delta}(-\bar{1})$, we see that ν is actually a ring isomorphism. . □

REMARK 3.9 (cf. (2.5)). If n is composite, say $n = p_1^{c_1} \cdots p_k^{c_k}$, with the p_i distinct primes, then $H^*(G_F, n) \cong \oplus_{i=1}^k H^*(G_F, p_i^{c_i})$. Hence, there was no loss in restricting to a prime power in (3.6).

For any field F with $\mu_n \subseteq F$ it is known (cf. [**20**, §15] and [**19**, §6]) that there is a well-defined graded ring homomorphism φ_F: $(K_*F)_n \to H^*(G_F, n)$ such that,

$$\varphi_F(l(a_1) \cdots l(a_k)) = \delta(a_1) \cup \cdots \cup \delta(a_k), \quad \text{for any } a_i \in \dot{F}.$$

PROPOSITION 3.10. *Let (F, v, Γ) be a field with valuation for which hypotheses* (3.5) *hold. Then there is a graded ring homomorphism*

$$\nu\colon H^*(G_F, n) \to H^*(G_{\bar{F}}, n)\left[J; \bar{\delta}\,(-\bar{1})\right] \quad \textit{with } \delta(\pi_j) \mapsto x_j.$$

If ν is injective, then F is p-Henselian. Furthermore, the following diagram commutes:

$$\text{(3.11)}\qquad \begin{array}{ccc} K_*(F)_n & \xrightarrow{\ \theta\ } & (K_*\bar{F})_n[j;\, l(-\bar{1})] \\ {\scriptstyle \varphi_F}\downarrow & & \downarrow{\scriptstyle \varphi'_{\bar{F}}} \\ H^*(G_F, n) & \xrightarrow{\ \nu\ } & H^*(G_{\bar{F}}, n)[J;\, \bar{\delta}\,(-\bar{1})]. \end{array}$$

Proof. Let (E, w, Δ) be any Henselization of (F, v, Γ). Then, we may identify $\bar{E}$ with $\bar{F}$ and Δ with Γ. Let ν_E denote the map of (3.6) for E (using the same $\{\pi_j\}$ for E as for F). Define $\nu := \nu_E \circ (\mathrm{res}_{G_F \to G_E})$. For any $a \in U_v$, $\nu(\delta(a)) = \bar{\delta}(\bar{a})$. Hence, if ν is injective, then $1 + \mathfrak{m}_v \subseteq \ker \delta = \dot{F}^n$; so v is p-Henselian, by (1.2). The map $\varphi'_{\bar{F}}$ is the canonical extension of $\varphi_{\bar{F}}$: $(K_*\bar{F})_n \to H^*(G_{\bar{F}}, n)$. It suffices to check the commutativity of the diagram on generators $l(a)$, $a \in \dot{F}$, and this is easily verified. □

REMARK 3.12. As in (2.2), if $|J| = r < \infty$, the map ν of (3.10) induces a "nearly canonical" residue map ∂': $H^*(G_F, n) \to H^*(G_{\bar{F}}, n)$ of degree $-r$. Furthermore, if ∂: $(K^*F)_n \to (K^*\bar{F})_n$ is the corresponding residue map defined in (2.2), we have $\varphi_{\bar{F}} \circ \partial = \partial' \circ \varphi_F$.

COROLLARY 3.13. *Suppose the field with valuation (F, v, Γ) is Henselian and satisfies hypotheses* (3.5). *Then φ_F is an isomorphism* (resp. *injective, surjective*) *in degree $\leq k$* iff *$\varphi_{\bar{F}}$ is an isomorphism* (resp. *injective, surjective*) *in degree $\leq k$.*

Proof. This is immediate from the commutative diagram (3.11), since by (2.4) and (3.6) the rows of the diagram are isomorphisms. □

REMARK 3.14. There is a p-Henselian version of (3.6) and (3.10)–(3.13). Let $\tilde{G}_F$ denote $\mathcal{G}(\tilde{F}(p)/F)$, where $\tilde{F}(p)$ is the p-closure of F, and let $\tilde{\delta}$: $\dot{F} \to H^1(\tilde{G}_F, n)$ be the analogue of the earlier δ (when $\mu_n \subseteq F$). Then the analogue to (3.6) holds for (F, v, Γ) p-Henselian (for $n = p^c$), with $\tilde{G}_F$, $\tilde{G}_{\bar{F}}$, and $\tilde{\delta}(\pi_j)$ replacing G_F, $G_{\bar{F}}$, and $\delta(\pi_j)$. The proof can be carried out in the same way as for (3.6), and is a little easier, because the reduction from F_s to L is not needed. Likewise, (3.10)–(3.13) hold with the corresponding changes.

REMARK 3.15. Jacob points out an alternative approach to (3.6) and its p-Henselian analogue (cf. [**14**, pp. 266–267]). With the hypotheses of (3.6), and the notation as in its proof, let $G = \mathcal{G}(L/F)$ and $K = \mathcal{G}(L/F_{nr})$. So $H^*(G_F) \cong H^*(G)$, as noted above, and K is an inverse limit of free abelian pro-p-groups. The short exact sequence

$$1 \to K \to G \to G_{\bar{F}} \to 1$$

is split exact, since $G_{\bar{F}} \cong \mathcal{G}(L/L_r)$, where L_r is a maximal totally ramified extension of F in L. Hence, in the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_{\bar{F}}, H^q(K)) \Rightarrow H^{p+q}(G),$$

the d_2 maps are all 0 [**13**, Th. 4], so the d_i maps are all 0 for $i \geq 2$, yielding $E_\infty^{p,q} = E_2^{p,q}$. Therefore, $H^k(G) \cong \oplus_{p+q=k} E_2^{p,q}$. From the description of K, $H^q(K)$ is a direct sum of copies of $\mathbf{Z}_n$; using that $H^q(K)$ is generated by cup products of terms from $H^1(K)$, one can check that $G_{\bar{F}}$ acts trivially on $H^q(K)$. This determines $E_2^{p,q}$, and shows that $H^k(G)$ decomposes into a direct sum of copies of $H^m(G_{\bar{F}})$, $0 \leq m \leq k$, as in (3.6). One is left with the task of identifying the summands within $H^k(G)$; this further information is needed for proving (3.10) and (3.13).

(This argument, or (3.6) and (3.10) can be used to prove the group extension case of [**14**, Th. 6], replacing the argument on the bottom of p.266 of [**14**]; the reduction given there using composite valuations is invalid.)

REMARK 3.16. Whereas $H^2(G_F, n)$ is the n-torsion in the Brauer group of F (when $\mu_n \subseteq F$), the full Brauer group is $H^2(G_F, \dot{F}_s)$. It is natural to ask for a description like (3.6) of $H^*(G_F, \dot{F}_s)$ in terms of $H^*(G_{\bar{F}}, (\dot{\bar{F}})_s)$ when (F, v, Γ) is Henselian. Let Ω denote the set of all roots of unity in F_s. Then, in the short exact sequence

$$1 \to \Omega \to \dot{F}_s \to \dot{F}_s/\Omega \to 1,$$

the right-hand term is uniquely p-divisible for each prime $p \neq \operatorname{char} F$. Hence, (with $A(p)$ denoting the p-primary component of the abelian group A),

$$H^*(G_F, \dot{F}_s)(p) \cong H^*(G_F, \Omega)(p) \cong \varinjlim_a H^*(G_F, \mu_{p^a}).$$

For F as in (3.6) with $\mu_{p^a} \subseteq F$ for each a, this direct limit is calculable from (3.6). The question then arises whether there is an analogue to (3.6) for $H^*(G_F, \mu_n)$, $n = p^a$, when $\mu_n \not\subseteq \dot{F}$. For this, the argument using (3.1) does not seem to work, since for the G and N in the proof of (3.6), one has $H^1(N, \mu_n) \cong \mathbf{Z}_n$ (with trivial G action), and this is not G-isomorphic to μ_n. Nonetheless, the spectral sequence argument of (3.15) shows that some sort of direct decomposition of $H^*(G_F, \mu_n)$ is still possible.

For an extensive discussion of $H^2(G_F, \dot{F}_s)$ for F Henselian, see Scharlau's paper [**22**]. But note that, as Becker observes in [**4**, p. 130] Satz 4.1 in [**22**] is incorrect. (The error arises in ll.-8--6 of [**22**, p. 247]. In fact, in Scharlau's terminology, the image of $Br(K_{tr}/K)$ in $Br(K_{tr}/K_{nr})$ is the union of the n-torsion parts of $Br(K_{tr}/K_{nr})$, for those n with $\mu_n \subseteq K$.)

4. The graded Witt ring. In this section we will obtain results like those of the preceding sections (with $n = p = 2$) for the graded Witt ring GWF of quadratic forms of a field with valuation (F, v, Γ). This is a matter of translating to associated graded rings known theorems about the Witt ring of F, which we will first review briefly.

For a field F, with $\operatorname{char} F \neq 2$, let WF denote the Witt ring of anisotropic quadratic forms over F. Let $I^kF := (IF)^k$, for $k \geq 0$, where IF is the fundamental ideal in WF of even-dimensional forms. For $a_1, \ldots, a_m \in \dot{F}$, $\langle a_1, \ldots, a_m \rangle$ denotes the diagonal quadratic form $a_1X_1^2 + \cdots + a_mX_m^2$ (or its image in WF). We write $\langle\langle a_1, \ldots, a_m \rangle\rangle$ for the m-fold Pfister form $\otimes_{i=1}^m \langle 1, a_i \rangle$. For background and undefined terminology, see [**18**]. At times we will abuse notation by not distinguishing between an element in $\dot{F}/\dot{F}^2$ and an inverse image in $\dot{F}$.

Let (F, v, Γ) be a fixed field with valuation. *We will assume throughout this section that* $\operatorname{char} \bar{F} \neq 2$. Let T be a subgroup of $\dot{F}/\dot{F}^2$ mapped bijectively to $\Gamma/2\Gamma$ by $\bar{v}$; let $W\bar{F}[T]$ denote the group algebra of T over the Witt ring $W\bar{F}$. Knebusch has shown [**16**, Th. 3.1] that there is a well-defined additive group homomorphism $\rho: WF \to W\bar{F}$ determined by: $\rho(\langle a \rangle) = \langle \bar{a} \rangle$ if $a \in U_v$ and $\rho(\langle a \rangle) = 0$ if $v(a) \notin 2\Gamma$. From ρ one obtains (cf. [**2**, p. 174]) a surjective map

$$(4.1) \qquad \lambda: WF \to W\bar{F}[T], \quad \text{given by } \lambda(q) = \sum_{t \in T} \rho(q\langle t \rangle)t.$$

λ is well-defined, since $\rho(q\langle t \rangle) = 0$ for almost all $t \in T$; one easily verifies that λ is a ring homomorphism. Knebusch has shown [**15**, §12.2] that λ is an isomorphism if and only if $1 + \mathfrak{m}_v \subseteq F^2$ (iff F is 2-Henselian, by (1.2)). (The proof is like that for (2.3) above: if F is 2-Henselian, the map $\dot{\bar{F}}/\dot{\bar{F}}^2 \to \dot{F}/\dot{F}^2$ of (1.5) induces a canonical map $W\bar{F} \to WF$, from which one can build up an inverse of λ.) Of course, if v is a complete discrete valuation the isomorphism of (4.1) is a restatement of Springer's Theorem [**26**].

For a ring R and an ideal $\mathfrak{A}$ of R, the associated graded ring of R with respect to $\mathfrak{A}$ is denoted $G_{\mathfrak{A}}R$ (or just GR): $G_{\mathfrak{A}}R = \bigoplus_{k=0}^{\infty} \mathfrak{A}^k/\mathfrak{A}^{k+1}$. Taking $R = WF$, F a field, and $\mathfrak{A} = IF$, the associated graded ring GWF is the *graded Witt ring* of F. The summands of GWF are written, for short $\bar{I}^kF := I^kF/I^{k+1}F$.

The map λ of (4.1) induces a surjective graded ring homomorphism

$$\eta\colon GWF \to G_{\mathfrak{A}}(W\bar{F}[T]), \quad \text{where } \mathfrak{A} = \lambda(IF). \tag{4.2}$$

Let $\{\pi_j\}_{j\in J} \subseteq \dot{F}$ map to a $\mathbf{Z}_2$-base of T in $\dot{F}/\dot{F}^2$. Then, in terms of the ring construction of (1.8), we have

LEMMA 4.3. *$G_{\mathfrak{A}}(W\bar{F}[T]) \cong GW\bar{F}[J;\ \overline{\langle\langle\bar{1}\rangle\rangle}]$, a graded ring isomorphism.*

Proof. (Here, $\overline{\langle\langle\bar{1}\rangle\rangle}$ is the image of $\langle\langle\bar{1}\rangle\rangle$ in $\bar{I}^1\bar{F}$.) Viewing $\pi_j \in T$, let $\tau_j = \langle\bar{1}\rangle - \pi_j \in W\bar{F}[T]$. Then, in notation (1.9), we have

$$W\bar{F}[T] = W\bar{F} \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\vec{j}\in\mathfrak{J}_m} W\bar{F}\tau_{j_1}\cdots\tau_{j_m}, \tag{4.4}$$

with each summand isomorphic to $W\bar{F}$. (To see that $\{1\} \cup \{\tau_{j_1}\cdots\tau_{j_m} \mid \vec{j} \in \mathfrak{J}\}$ is a $W\bar{F}$-module basis of $W\bar{F}[T]$, note that for any finite subset J_0 of J (and corresponding $\mathfrak{J}_0 \subseteq \mathfrak{J}$), the coefficient matrix for expressing 1 and $\{\tau_{j_1}\cdots\tau_{j_m} \mid \vec{j} \in \mathfrak{J}_0\}$ in terms of 1 and $\{\pi_{j_1}\cdots\pi_{j_m} \mid \vec{j} \in \mathfrak{J}_0\}$ is a triangular matrix with diagonal entries ± 1; so the matrix is invertible.) The description of $\mathfrak{A}^k$, $k \geq 1$ relative to (4.4) is:

$$\mathfrak{A}^k = I^k\bar{F} \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\vec{j}\in\mathfrak{J}_m} I^{k-m}\bar{F}\tau_{j_1}\cdots\tau_{j_m}. \tag{4.5}$$

Here, set $I^{k-m}\bar{F} := W\bar{F}$ when $m \geq k$. Formula (4.5) is easy to see for $k = 1$, since $|W\bar{F}[T]/\mathfrak{A}| = 2$; it follows for $k > 1$ by induction, noting that $\tau_j^2 = \langle\langle\bar{1}\rangle\rangle\tau_j$. Since the decompositions of $\mathfrak{A}^k$ and $\mathfrak{A}^{k+1}$ are compatible, we have

$$\mathfrak{A}^k/\mathfrak{A}^{k+1} = \bar{I}^k\bar{F} \oplus \bigoplus_{m=1}^{k} \bigoplus_{\vec{j}\in\mathfrak{J}_m} (\bar{I}^{k-m}\bar{F})\bar{\tau}_{j_1}\cdots\bar{\tau}_{j_m}, \tag{4.6}$$

where $\bar{\tau}_j$ is the image of τ_j in $\mathfrak{A}/\mathfrak{A}^2$, and each term $(\bar{I}^{k-m}\bar{F})\bar{\tau}_{j_1}\cdots\bar{\tau}_{j_m} \cong \bar{I}^{k-m}\bar{F}$.

Equation (4.6) shows that $G_{\mathfrak{A}}(W\bar{F}[T])$ is a free $GW\bar{F}$-module with base $\{1\} \cup \{\bar{\tau}_{j_1}\cdots\bar{\tau}_{j_m} \mid \vec{j} \in \mathfrak{J}\}$. Hence, there is a $GW\bar{F}$-module isomorphism

$$G_{\mathfrak{A}}(W\bar{F}[T]) \to GW\bar{F}\big[J;\ \overline{\langle\langle\bar{1}\rangle\rangle}\big]$$

given by $1 \mapsto 1$ and $\bar{\tau}_{j_1}\cdots\bar{\tau}_{j_m} \mapsto x_{j_1}\cdots x_{j_m}$. Since each of these rings is commutative and $\bar{\tau}_j^2 = \overline{\langle\langle\bar{1}\rangle\rangle}\bar{\tau}_j$, we see that this map is also a ring isomorphism. □

PROPOSITION 4.7. *There is a surjective graded ring homomorphism*

$$\gamma\colon GWF \to GW\bar{F}\big[J;\ \overline{\langle\langle \bar{1}\rangle\rangle}\big],$$

such that $\gamma(\overline{\langle\langle a\rangle\rangle}) = \overline{\langle\langle \bar{a}\rangle\rangle}$ *if* $a \in U_v$ *and* $\gamma(\overline{\langle\langle -\pi_j\rangle\rangle}) = x_j$. *The map* γ *is an isomorphism if and only if* v *is* 2-*Henselian.*

Proof. γ is the composition of η of (4.2) with the isomorphism of (4.3). If v is 2-Henselian, λ is an isomorphism, hence also are η and γ. Conversely, if γ is an isomorphism, then for any $a \in 1 + \mathfrak{m}_v$, $\overline{\langle\langle -a\rangle\rangle} = 0$ in $\bar{I}^1F \cong \dot{F}/\dot{F}^2$. Hence, $1 + \mathfrak{m}_v \subseteq F^2$; so F is 2-Henselian, by (1.2). □

REMARKS 4.8. (i) Milnor has defined **[19**, §4**]** a graded ring homomorphism $s_F\colon (K_*F)_2 \to GWF$. It is easy to see that for any field with valuation (F, v, Γ) (with char $\bar{F} \neq 2$), there is a commutative diagram like (3.11) above for s_F, i.e., $\gamma \circ s_F = s'_{\bar{F}} \circ \theta$ (where the same $\{\pi_j\}_{j\in J}$ is used for both γ and θ).

(ii) Just as in (2.2) above, if $|\Gamma/2\Gamma| = 2^r < \infty$, we obtain from γ a canonical residue map $\partial\colon GWF \to GW\bar{F}$ of degree $-r$. The formula for ∂ is like the one in (2.2), with $\overline{\langle\langle -a\rangle\rangle}$ replacing $l(a)$.

(iii) Let R be the group algebra $\mathbf{Z}[T]$, for T a 2-torsion abelian group. Then R is the Witt ring of a field with 2-Henselian valuation (E, w, Δ), such that $\Delta/2\Delta \cong T$ and $\bar{E}$ is Euclidean (so $W\bar{E} \cong \mathbf{Z}$). The fundamental ideal $\mathfrak{A}$ of R is $2R + \Sigma_{t\in T}(1 - t)R$. As (4.3) shows, for the associated graded ring with respect to $\mathfrak{A}$, $GR \cong G\mathbf{Z}[J;\ \bar{2}]$, where J is an index set for a $\mathbf{Z}_2$-base of T and $\bar{2} = 2 + 4\mathbf{Z} \in (G\mathbf{Z})_1$. Here, $G\mathbf{Z} = G_{2\mathbf{Z}}\mathbf{Z}$, so $G\mathbf{Z} \cong \mathbf{Z}_2[X]$ as graded rings (with $\bar{2} \mapsto X$). See (1.12)(ii) for another description of GR.

Now assume (F, v, Γ) is a field with 2-Henselian valuation with $\Gamma/2\Gamma \cong T$. The isomorphism λ of (4.1) can be repeated: $WF \cong W\bar{F} \otimes_{\mathbf{Z}} R$. For graded Witt rings we have analogously from (4.7) and (1.12)(ii),

$$GWF \cong GW\bar{F} \otimes_{G\mathbf{Z}} GR.$$

(Here $GW\bar{F}$ is made into a $G\mathbf{Z}$-algebra by mapping $\bar{2} \mapsto \overline{\langle\langle \bar{1}\rangle\rangle}$.)

(iv) Suppose (F, v, Γ) is 2-Henselian, with $\{\pi_j\}_{j\in J} \subseteq \dot{F}$ mapping bijectively to a $\mathbf{Z}_2$-base of $\Gamma/2\Gamma$. For $j \in J$, let $\varepsilon_j = \langle\langle -\pi_j\rangle\rangle \in IF$. From the isomorphisms λ (4.1) and γ (4.7) above, the direct decompositions in (4.4)–(4.6) above correspond to direct decompositions of WF, I^kF, and

$\bar{I}^k F$. After identifying $W\bar{F}$ with its canonical image in WF induced from i: $\dot{\bar{F}}/\dot{\bar{F}}^2 \to \dot{F}/\dot{F}^2$ of (1.5), formula (4.5) translates to

$$(4.9) \qquad I^k F = I^k \bar{F} \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\vec{j} \in \mathfrak{J}_m} (I^{k-m}\bar{F})\varepsilon_{j_1} \cdots \varepsilon_{j_m}, \quad \text{for } k \geq 0.$$

There are similar decompositions for other ideals of WF. For example, consider qWF, with q a Pfister form. We can write $q = \langle\langle -a_1, \ldots, -a_m, -\pi_1, \ldots, -\pi_l \rangle\rangle$, where $a_1, \ldots, a_m \in U_v$ and $\bar{v}(\pi_1), \ldots, \bar{v}(\pi_l)$ are $\mathbf{Z}_2$-independent in $\Gamma/2\Gamma$. Set $q' = \langle\langle -\bar{a}_1, \ldots, -\bar{a}_m \rangle\rangle \in W\bar{F}$. If $l = 0$, we have

$$(4.10) \qquad qWF = q'W\bar{F} \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\vec{j} \in \mathfrak{J}_m} q'W\bar{F}\varepsilon_{j_1} \cdots \varepsilon_{j_m}.$$

If $l > 0$ we can assume our basic set $\{\pi_j\}_{j \in J}$ contains $\pi_1, \ldots, \pi_l$. Then, with $\vec{i} = \{1, 2, \ldots, l\}$, we have

$$(4.11) \qquad qWF = \bigoplus_{m=1}^{\infty} \bigoplus_{\vec{j} \in \mathfrak{J}_m, \vec{i} \subseteq \vec{j}} q'W\bar{F}\varepsilon_{j_1} \cdots \varepsilon_{j_m}.$$

5. Multiquadratic extensions of a 2-Henselian field. Throughout this section, let (F, v, Γ) be a field with 2-Henselian valuation for which $\operatorname{char} \bar{F} \neq 2$. Let B be a finite subgroup of $\dot{F}/\dot{F}^2$ and let $M = F(\sqrt{B})$, a multiquadratic (i.e., 2-Kummer) extension of F. (That is, $M = F(\{\sqrt{b} \mid b\dot{F}^2 \in B\}$.) We continue to blur the distinction between elements of $\dot{F}$ and their images in $\dot{F}/\dot{F}^2$. Let (M, w, Γ_M) denote the unique extension of v to M; w is also a 2-Henselian valuation. We will show how the image and kernel of the canonical map of Witt rings

$$r\colon WF \to WM$$

can be described in terms of the corresponding map $W\bar{F} \to W\bar{M}$. This will illustrate how one can work with the decompositions of WF and WM given by (4.9), with $k = 0$. Then we will indicate the proofs of a few results mentioned without proof in **[10]** and **[25]**. A key part of the argument involves lining up $\mathbf{Z}_2$-bases of $\Gamma/2\Gamma$ and $\Gamma_M/2\Gamma_M$; since this clearly generalizes to p-Kummer extensions of a p-Henselian field, analogous results hold for K-theory and Galois cohomology, for any prime p.

Let $Y = \Gamma/2\Gamma$ and $Y_M = \Gamma_M/2\Gamma_M$, and let $y\colon Y \to Y_M$ be the map induced by the injection (which we treat as an inclusion) $\Gamma \to \Gamma_M$. Let $Q(F) = \dot{F}/\dot{F}^2$. Identify $Q(\bar{F})$ with its canonical image in $Q(F)$ (see

(1.4)). Let $\bar{B} = B \cap Q(\bar{F})$. Then we have a commutative diagram with exact rows by (1.4):

$$
(5.1)\qquad
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \bar{B} & \to & B & \to & \bar{v}(B) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Q(\bar{F}) & \to & Q(F) & \xrightarrow{\bar{v}} & Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow y & & \\
1 & \to & Q(\bar{M}) & \to & Q(M) & \xrightarrow{\bar{w}} & Y_M & \to & 0
\end{array}
$$

The middle column is exact by Kummer theory, and the left and right columns are clearly zero-sequences. So, $\bar{F}(\sqrt{\bar{B}}) \subseteq \bar{M}$. Note that $\ker y = (\Gamma \cap 2\Gamma_M)/2\Gamma$.

Let $f = [\bar{M} : \bar{F}]$ and $e = |\Gamma_M : \Gamma| = |2\Gamma_M : 2\Gamma|$ (as Γ_M is torsion-free). Then,

$$
(5.2)\qquad
\begin{aligned}
|B| = |\bar{B}| \cdot |\bar{v}(B)| &\leq f \cdot |\ker y| = f \cdot |\Gamma \cap 2\Gamma_M : 2\Gamma| \\
&\leq f \cdot |2\Gamma_M : 2\Gamma| = f \cdot e \leq [M : F].
\end{aligned}
$$

But, by Kummer theory $[M : F] = |B|$, so equality holds throughout (5.2). Hence, $\bar{M} = \bar{F}(\sqrt{\bar{B}})$, $2\Gamma_M \subseteq \Gamma$, $\bar{v}(B) = \ker y = 2\Gamma_M/2\Gamma$ (with order e), and the outer columns of (5.1) are exact.

Choose $b_1, \ldots, b_s \in B$ which map to a $\mathbf{Z}_2$-base of $\bar{v}(B)$, and choose $\{a_j\}_{j \in J} \subseteq Q(F)$ which map to a $\mathbf{Z}_2$-base of $Y/\bar{v}(B)$. Further, for each b_i, choose a fixed square root $\sqrt{b_i} \in Q(M)$. (That is, let $\sqrt{b_i}$ be $\sqrt{c}\dot{M}^2 \in Q(M)$, where $\sqrt{c}$ is a square root of some $c \in \dot{F}$ with $c\dot{F}^2 = b_i \in Q(F)$. Since c is determined only mod $\dot{F}^2$, $\sqrt{b_i}$ can be any element of $\sqrt{c}\,\dot{F}\dot{M}^2/\dot{M}^2$.)

PROPOSITION 5.3. *$\{\bar{v}(a_j)\}_{j \in J} \cup \{\bar{v}(b_1), \ldots, \bar{v}(b_s)\}$ is a $\mathbf{Z}_2$-base of Y. Further, $\{\bar{w}(a_j)\}_{j \in J} \cup \{\bar{w}(\sqrt{b_1}), \ldots, \bar{w}(\sqrt{b_s})\}$ is a $\mathbf{Z}_2$-base of Y_M.*

Proof. Clearly the first set is a base for Y. Since $\bar{v}(B) = \ker y$, $\{\bar{w}(a_j)\}_{j \in J}$ is a base of im y. Let B_0 be the subgroup of B generated by $\{b_1, \ldots, b_s\}$, and let C be the subgroup of $Q(M)$ generated by $\{\sqrt{b_1}, \ldots, \sqrt{b_s}\}$. Then,

$$|\bar{w}(C)| \leq |C| \leq 2^s = |\bar{v}(B)| = e = |Y_M/\text{im } y|$$

(the last equality because $2\Gamma_M \subseteq \Gamma$). But since $\bar{v}(B_0) = \bar{v}(B) = 2\Gamma_M/2\Gamma$, dividing by 2 (in Γ_M) shows that $\overline{w}(C)$ maps onto $Y_M/\mathrm{im}\ y$. A comparison of orders shows that this map must be bijective and $\{\overline{w}(\sqrt{b_1}),\dots,\overline{w}(\sqrt{b_s})\}$ maps to a base of $Y_M/\mathrm{im}\ y$. So, combining this set with our base of im y yields a base of Y_M. □

For $j \in J$, let $\alpha_j = \langle\langle -a_j \rangle\rangle \in WF$ and $\alpha'_j = \langle\langle -a_j \rangle\rangle_M \in WM$; let $\beta_i = \langle\langle -b_i \rangle\rangle \in WF$ and $\gamma_i = \langle\langle -\sqrt{b_i} \rangle\rangle \in WM$, $1 \le i \le s$. Let $\mathfrak{J}$ (resp. $\mathfrak{L}$) denote the set of nonempty subsets of J (resp. of $\{1,2,\dots,s\}$). For $\vec{j} = \{j_1,\dots,j_m\} \in \mathfrak{J}$ (as in notation (1.9)), let $\alpha_{\vec{j}} = \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_m}$; likewise for $\beta_{\vec{l}}$ if $\vec{l} \in \mathfrak{L}$. Formula (4.9) for $k = 0$ becomes a little more complicated because we have split up the base for Y. It now reads:

$$(5.4) \qquad WF = W\overline{F} \oplus \bigoplus_{\vec{j}\in\mathfrak{J}} W\overline{F}\alpha_{\vec{j}} \oplus \bigoplus_{\vec{l}\in\mathfrak{L}} W\overline{F}\beta_{\vec{l}} \oplus \bigoplus_{\vec{j}\in\mathfrak{J},\vec{l}\in\mathfrak{L}} W\overline{F}\alpha_{\vec{j}}\beta_{\vec{l}}.$$

Let us adopt the following notation as an abbreviation for (5.4):

$$(5.5) \qquad WF = \left(W\overline{F}, W\overline{F}\alpha_{\vec{j}}, W\overline{F}\beta_{\vec{l}}, W\overline{F}\alpha_{\vec{j}}\beta_{\vec{l}}\right).$$

Correspondingly,

$$(5.6) \qquad WM = \left(W\overline{M}, W\overline{M}\alpha'_{\vec{j}}, W\overline{M}\gamma_{\vec{l}}, W\overline{M}\alpha'_{\vec{j}}\gamma_{\vec{l}}\right).$$

The map r: $WF \to WM$ sends α_j to α'_j and β_l to 0, so we can read off the image and the kernel (denoted $W(M/F)$) of r:

$$(5.7) \qquad r(WF) = \left(r(W\overline{F}), r(W\overline{F})\alpha'_{\vec{j}}, 0\gamma_{\vec{l}}, 0\alpha'_{\vec{j}}\gamma_{\vec{l}}\right),$$

$$(5.8) \qquad W(M/F) = \left(W(\overline{M}/\overline{F}), W(\overline{M}/\overline{F})\alpha_{\vec{j}}, W\overline{F}\beta_{\vec{l}}, W\overline{F}\alpha_{\vec{j}}\beta_{\vec{l}}\right).$$

In the notation of [**10**, §2], $WD(M/F)$ is the obvious part of $W(M/F)$, i.e., $WD(M/F) = \Sigma_{b\in B}\langle\langle -b \rangle\rangle WF$. Since $\bar{B} \cup \{b_1,\dots,b_s\}$ generates B, we have, using (4.10) and (4.11),

$$\begin{aligned}(5.9)\ WD(M/F) &= \sum_{\bar{b}\in\bar{B}} \langle\langle -\bar{b} \rangle\rangle WF + \sum_{i=1}^{s} \beta_i WF \\ &= \left(WD(\overline{M}/\overline{F}), WD(\overline{M}/\overline{F})\alpha_{\vec{j}}, WD(\overline{M}/\overline{F})\beta_{\vec{j}}, WD(\overline{M}/\overline{F})\alpha_{\vec{j}}\beta_{\vec{l}}\right) \\ &\quad + \left(0, 0\alpha_{\vec{j}}, W\overline{F}\beta_{\vec{l}}, W\overline{F}\alpha_{\vec{j}}\beta_{\vec{l}}\right) \\ &= \left(WD(\overline{M}/\overline{F}), WD(\overline{M}/\overline{F})\alpha_{\vec{j}}, W\overline{F}\beta_{\vec{l}}, W\overline{F}\alpha_{\vec{j}}\beta_{\vec{l}}\right).\end{aligned}$$

Setting $h_2(M/F) := W(M/F)/WD(M/F)$ as in [**10**, (2.2)], (5.8) and (5.9) yield

$$(5.10) \qquad h_2(M/F) \cong h_2(\overline{M}/\overline{F}) \oplus \bigoplus_{\vec{j} \in \mathcal{J}} h_2(\overline{M}/\overline{F}).$$

Likewise, let $r'(WF)$ be the obvious part of $r(WF)$, i.e., $r'(WF) = \bigcap_L \operatorname{im}(WL \to WM)$, as L ranges over fields $F \subseteq L \subsetneq M$ with $[M : L] = 2$. Using compatible direct sum decompositions for WL and WM, one can compute $\operatorname{im}(WL \to WM)$; this image has two forms, depending on whether M/L is ramified or unramified. One obtains from this,

$$(5.11) \qquad r'(WF) = \left(r'(W\overline{F}),\, r'(W\overline{F})\alpha_{\vec{j}},\, 0\gamma_{\vec{l}},\, 0\alpha_{\vec{j}}\gamma_{\vec{l}}\right).$$

Hence, from (5.7) and (5.11),

$$(5.12) \quad h_3(M/F) := r(WF)/r'(WF) \cong h_3(\overline{M}/\overline{F}) \oplus \bigoplus_{\vec{j} \in \mathcal{J}} h_3(\overline{M}/\overline{F}).$$

Note that the index set for the direct sums in (5.10) and (5.12) is in one-to-one correspondence with the elements of $Y/\ker y$; so when $|Y| < \infty$ the number of summands is $|Y|/e$. Formulas (5.10) and (5.12) yield [**10**, (2.15)] and show that F is 1-amenable (resp. strongly 1-amenable) iff $W\overline{F}$ is 1-amenable (resp. strongly 1-amenable).

The formulas for the direct decompositions of $\bar{I}^kF$ (and $(K_kF)_n$ and $H^k(G_F, n)$) are slightly different from (5.4), and this is illustrated by the analogue to (5.4) for I^kF. For $\vec{j} = \{j_1, \ldots, j_m\} \in \mathcal{J}$, set $|\vec{j}| = m$. Then (4.9) becomes, in the shortened notation of (5.5), for any $k \geq 0$,

$$(5.13) \quad I^kF = \left(I^k\overline{F},\, (I^{k-|\vec{j}|}\overline{F})\alpha_{\vec{j}},\, (I^{k-|\vec{l}|}\overline{F})\beta_{\vec{l}},\, (I^{k-|\vec{j}|-|\vec{l}|}\overline{F})\alpha_{\vec{j}}\beta_{\vec{l}}\right).$$

Again, set $I^m\overline{F} = WF$ when $m < 0$. For the corresponding formula $(\overline{5.13})$ for $\bar{I}^kF = I^kF/I^{k+1}F$, simply place bars over the I's, α's and β's in (5.13). Note, though, that some of the summands in the formula for $\bar{I}^kF$ will drop out because $\bar{I}^m\overline{F} = 0$ when $m < 0$. From (5.13) and $(\overline{5.13})$ one easily obtains the analogues to (5.7) and (5.8) for the image and kernel of $I^kF \to I^kM$ and $\bar{I}^kF \to \bar{I}^kM$. By using the corresponding image and kernel formulas for the case of $H^2(-, 2)$, one can show that, in Tignol's notation [**27**, p. 6],

$$(5.14) \qquad N_i(M/F) \cong N_i(\overline{M}/\overline{F}), \quad \text{for } i = 1, 2, 3.$$

(Formula (5.14) corresponds to (5.10) and (5.12) above; but the summands indexed by $\mathcal{J}$ in (5.14) are all 0's because they come from terms involving $H^0(-, 2)$ and $H^1(-, 2)$, where there is complete cancellation.) From (5.14) it follows that F has Tignol's property $P_i(n)$ [**27**, p. 6] iff $\overline{F}$

has property $P_i(n)$, for $i = 1, 2, 3$, $n = 1, 2, \ldots$ (cf. [**25**, (3.10)]. This was proved by Tignol [**27**, §6] for v a complete discrete valuation.

References

1. J. Kr. Arason, *Cohomologische Invarianten quadratischer Formen*, J. Algebra, **36** (1975), 448–491.
2. J. Kr. Arason and M. Knebusch, *Über die Grade quadratischer Formen*, Math. Ann., **234** (1978), 167–192.
3. H. Bass and J. Tate, *The Milnor ring of a global field*, pp. 349–446, in *Algebraic K-Theory II*, (H. Bass, ed.), Lecture Notes in Math. **342**, Springer-Verlag, Berlin, 1973.
4. E. Becker, *Hereditarily-Pythagorean Fields and Orderings of Higher Level*, Monografias de Matemática, No. 29, IMPA, Rio de Janeiro, 1978.
5. L. Bröcker, *Characterization of fans and hereditarily Pythagorean fields*, Math. Z., **151** (1976), 149–163.
6. R. Brown, *Superpythagorean fields*, J. Algebra, **42** (1976), 483–494.
7. J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, London, 1967.
8. A. Dress, *Metrische Ebenen über quadratisch perfekten Körpern*, Math. Z., **92** (1966), 19–29.
9. R. Elman, *On Arason's theory of Galois cohomology*, Comm. Algebra, **10** (1982), 1449–1474.
10. R. Elman, T. Y. Lam, J.-P. Tignol, and A. R. Wadsworth, *Witt rings and Brauer groups under multiquadratic extensions, I*, to appear in Amer. J. Math.
11. O. Endler, *Valuation Theory*, Springer-Verlag, Berlin, 1972.
12. A. Hattori, *On exact sequences of Hochschild and Serre*, J. Math. Soc. Japan, **7** (1955), pp. 312–321.
13. G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc., **74** (1953), 110–134.
14. B. Jacob, *On the structure of Pythagorean fields*, J. Algebra, **68** (1981), 247–267.
15. M. Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, Sitz.-ber. Heidelberg. Akad. Wiss., 1969/70, 3. Abh., pp. 93–157.
16. ______, *Specialization of quadratic and symmetric bilinear forms, and a norm theorem*, Acta Arithmetica, **24** (1973), pp. 279–299.
17. H. Koch, *Galoissche Theorie der p-Erweiterungen*, Math. Mon., No. 10, V.E.B. Deutscher Verlag der Wiss., Berlin, 1970.
18. T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin, Reading, Mass., 1973.
19. J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math., **9** (1970), 318–344.
20. ______, *Introduction to Algebraic K-Theory*, Annals of Math. Studies, No. 72, Princeton University Press, 1971.
21. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
22. W. Scharlau, *Über die Brauer-Gruppe eines Hensel-Körpers*, Abh. Math. Sem. Univ. Hamburg, **33** (1969), 243–249.
23. J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math., **5**, Springer-Verlag, Berlin, 1965.
24. J.-P. Serre, *Local Fields*, Springer-Verlag, Berlin, 1979.
25. D. B. Shapiro, J.-P. Tignol, and A. R. Wadsworth, *Witt rings and Brauer groups under multiquadratic extensions, II*, to appear in J. Algebra.
26. T. A. Springer, *Quadratic forms over a field with a discrete valuation*, Indag. Math., **17** (1955), 352–362.

27. J.-P. Tignol, *Corps à involution neutralisés par une extension abélienne élémentaire*, pp. 1–34, in *Groupe de Brauer* (M. Kervaire and M. Ojanguren, eds.), Lecture Notes in Math., **844**, Springer-Verlag, Berlin, 1981.
28. R. Ware, *Valuation rings and rigid elements in fields*, Canad. J. Math., **33** (1981), 1338–1355.
29. E. Witt, *Schiefkörper über diskret bewerteten Körpern*, J. Reine Angew. Math., **176** (1936), 153–156.

Received September 8, 1981. Supported in part by the National Science Foundation.

UNIVERSITY OF CALIFORNIA AT SAN DIEGO
LA JOLLA, CA 92093