PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 2, 1983

MEROMORPHIC FUNCTIONS THAT SHARE TWO FINITE VALUES WITH THEIR DERIVATIVE

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It is shown that if a nonconstant meromorphic function f and its derivative f' share two finite values (counting multiplicities), then $f(z) = ce^z$.

We say two meromorphic functions $f(z)$ and $g(z)$ *share* the finite complex value c if $f(z) - c$ and $g(z) - c$ have the same zeros. We will state whether a value is shared **CM** (counting multiplicities), **IM** (ignoring multiplicities), or by **DM** (by different multiplicities at one point or more). In this paper all functions will be assumed to be meromorphic in the whole complex plane, unless stated otherwise.

R. Nevanlinna [**7**, p. 109] proved that if f and g share five values **IM**, then either $f = g$ or f and g are both constants. He also found [**7**, Chapter V] the particular form of all pairs f, g that share four values **CM** and all pairs f, g that share three values **CM**.

L. A. Rubel and C. C. Yang proved the following result:

THEOREM A. [**8**] *If a nonconstant entire function f and its derivative f' share two finite values* **CM**, *then $f = f'$.*

E. Mues and N. Steinmetz [**6**] have shown that "**CM**" can be replaced by "**IM**" in Theorem A (another proof of this result for nonzero shared values is in [**3**]).

On the other hand, the meromorphic function [**6**]

$$f(z) = \left(\frac{1}{2} - \frac{\sqrt{5}}{2} i \tan\left(\frac{\sqrt{5}}{4} iz\right)\right)^2 \tag{1}$$

shares 0 by **DM** and 1 by **DM** with f'; while the meromorphic function [**3**]

$$f(z) = \frac{2A}{1 - Be^{-2z}}, \qquad A \neq 0, B \neq 0 \tag{2}$$

shares 0 (lacunary) and A by **DM** with f'; and $f \neq f'$ in both (1) and (2).

The purpose of this paper is to prove

THEOREM 1. *If a nonconstant meromorphic function f and its derivative f' share two finite values* **CM**, *then $f = f'$.*

E. Mues has shown the author how he used sums of logarithmic derivatives to prove some results on meromorphic functions that share four values. By choosing the sums in clever ways he was able to effectively combine Nevanlinna theory with the shared value properties. For the proof of Theorem 1 we will use natural extensions of his ideas and combine these extensions with the relationship of f and f'. S. Bank and the author had earlier found a proof of Theorem 1 in the special case when order$(f) < 2$, but this proof cannot be used if order$(f) \geq 2$.

Examples (1) and (2) both have 0 as one of the shared values which is certainly a special case. Hence it is still not known whether there exists a meromorphic function f such that f and f' share two finite nonzero values, by **DM** for at least one value.

It should be mentioned that if a nonconstant meromorphic function f and its derivative f' share three finite values **IM**, then $f = f'$ [**3**, **6**].

We will assume that the reader is familiar with the standard notations and fundamental results of Nevanlinna theory as found in [**5**]. We define $S(r, f)$ to be any quantity that satisfies

$$\frac{S(r, f)}{T(r, f)} \to 0$$

as $r \to \infty$ outside a possible exceptional set of r of finite linear measure.

The next result will follow from Theorem 1.

COROLLARY 1. *If a and b are two distinct complex constants and w is a nonconstant entire function, then the algebraic differential equation*

$$f' = \frac{(a - be^{w})f + ab(e^{w} - 1)}{(1 - e^{w})f + ae^{w} - b} \tag{3}$$

does not possess a nonconstant meromorphic solution f.

We mention that it follows from the general theorems of F. Gackstatter and I. Laine [**2**] and N. Steinmetz [**9**] that there does not exist a nonconstant meromorphic solution f of equation (3) such that $T(r, e^{w}) = S(r, f)$.

The following result is obtained by combining Theorem 1 and [**3**, Theorem 1].

COROLLARY 2. *If h and g are nonconstant entire functions such that* (i) *h', g' share* 0 **CM**, (ii) *h'', g'' share* 0 **CM**, *and* (iii) *$a(1 + h/h') = b(1 + g/g')$ for distinct nonzero numbers a and b, then $h'(z) \equiv Ce^{-z}$ and $g'(z) \equiv Ke^{-z}$ for nonzero constants C, K.*

Entire functions H and G such that (i) H and G share 0 **CM** and (ii) H' and G' share 0 **CM**, are studied in [**4**].

I would like to thank E. Mues and S. Bank for some helpful discussions.

Proof of Theorem 1. Let a and b be the shared values. If $ab = 0$, then 0 is a Picard value of both f and f' and it follows that $f \equiv f'$ [**3**, Theorem 4(i)].

Now suppose that $a \neq 0$ and $b \neq 0$. Then all a-points and b-points for both f and f' are simple. It is easy to see that f' cannot be a constant. We will first prove

LEMMA 1. *If in addition to the hypothesis of Theorem* 1 *we have* $N(r, f) = S(r, f)$, *then* $f \equiv f'$.

Proof. Set $\phi_1 = f'/(f - b) - f''/(f' - b)$.

Then from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain

$$\begin{aligned} m(r, \phi_1) &\leq m\left(r, \frac{f'}{f - b}\right) + m\left(r, \frac{f''}{f' - b}\right) + O(1) \\ &= S(r, f) + S(r, f'). \end{aligned}$$

Since $T(r, f') \leq 2T(r, f) + S(r, f)$ this means that $m(r, \phi_1) = S(r, f)$. Since ϕ_1 is the logarithmic derivative of $(f - b)(f' - b)^{-1}$, it follows that $N(r, \phi_1) = \bar{N}(r, f) = S(r, f)$. Hence $T(r, \phi_1) = S(r, f)$.

Similarly, if

$$\phi_2 = \frac{f'}{f - a} - \frac{f''}{f' - a},$$

then $T(r, \phi_2) = S(r, f)$.

Suppose first that $\phi_1 \neq 0$ (i.e. $\phi_1(z) \not\equiv 0$) and $\phi_2 \neq 0$. Then from Nevanlinna's fundamental estimate and Jensen's Theorem we obtain

$$\frac{\phi_1}{f-a} = \frac{f'}{(f-a)(f-b)} - \frac{f''}{f'(f'-b)} \cdot \frac{f'}{f-a},$$

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq m\left(r, \frac{1}{\phi_1}\right) + m\left(r, \frac{f'}{(f-a)(f-b)}\right) \\ &\quad + m\left(r, \frac{f''}{f'(f'-b)}\right) + m\left(r, \frac{f'}{f-a}\right) + O(1) \\ &\leq T\left(r, \frac{1}{\phi_1}\right) + S(r,f) + S(r,f') + S(r,f) \\ &\leq T(r, \phi_1) + S(r,f) = S(r,f). \end{aligned}$$

Similarly, by using $\phi_2/(f-b)$ we obtain $m(r, 1/(f-b)) = S(r,f)$.

Now if $f \neq f'$, then from the first fundamental theorem and the fundamental estimate we get that

$$\begin{aligned} 2T(r,f) &= T(r,f,a) + T(r,f,b) + O(1) \\ &= N(r,f,a) + N(r,f,b) + S(r,f) \\ &\leq N\left(r, \frac{f'}{f}, 1\right) + S(r,f) \leq T\left(r, \frac{f'}{f}, 1\right) + S(r,f) \\ &= m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right) + S(r,f) \\ &= S(r,f) + \bar{N}(r,f,0) \leq T(r,f) + S(r,f), \end{aligned}$$

which implies the contradiction $T(r,f) = S(r,f)$. Thus $f = f'$.

On the other hand, if $\phi_1 = 0$ (i.e. $\phi_1(z) \equiv 0$), then from integration of ϕ_1 we get

$$\frac{f-b}{f'-b} = C$$

where C is some nonzero constant. If $C = 1$, then $f = f'$. If $C \neq 1$, then a is a Picard value for both f and f'. This is impossible because $a \neq 0$ [**5**, p. 60]. Similarly, if $\phi_2 = 0$, then $f = f'$. This proves Lemma 1.

Proceeding now with the proof of Theorem 1, we will assume that $f \neq f'$. Consider the following function:

$$\psi = \frac{f'}{f-a} + \frac{f'}{f-b} - \frac{f''-f'}{f'-f}. \tag{4}$$

$m(r, \psi) \leq S(r, f) + S(r, f' - f) \leq S(r, f)$. Since ψ is the logarithmic derivative of

$$F = \frac{(f-a)(f-b)}{f'-f},$$

it follows that $N(r, \psi) = \overline{N}(r, F)$. Hence

$$T(r, \psi) = \overline{N}(r, F) + S(r, f). \tag{5}$$

Suppose that z_0 is a simple pole of f. We will examine the value of $\psi(z_0)$.

Note that since f and f' share a and b **CM**, it follows that there is an entire function $w(z)$ such that

$$\frac{(f'-a)(f-b)}{(f-a)(f'-b)} = e^w. \tag{6}$$

(6) can be rewritten as

$$(a-b)(f'-f) = (e^w - 1)(f-a)(f'-b). \tag{7}$$

From (7) we see that $e^w - 1$ has a simple zero at z_0 and the residue of f at z_0 is $(a-b)(w'(z_0))^{-1}$. We emphasize here that the assumption that f has a simple pole implies that $w' \neq 0$. If

$$\frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \sum_{n=0}^{\infty} B_n(z-z_0)^n \tag{8}$$

is the Laurent expansion about z_0, then for any fixed $c \in \mathbf{C}$, we find that $A_0 = B_0 - cw'(z_0)(a-b)^{-1}$ in the Laurent expansion

$$\frac{f'(z)}{f(z)-c} = -\frac{1}{z-z_0} + \sum_{n=0}^{\infty} A_n(z-z_0)^n$$

about z_0. It is easily found that $C_0 = 1$ in the Laurent expansion

$$\frac{f''(z)-f'(z)}{f'(z)-f(z)} = -\frac{2}{z-z_0} + \sum_{n=0}^{\infty} C_n(z-z_0)^n$$

about z_0. Substitution of these calculations into (4) gives

$$\psi(z_0) = 2B_0 - \frac{a+b}{a-b}w'(z_0) - 1. \tag{9}$$

We will further examine the constant B_0. To this end, set

$$f(z) = \frac{(a-b)/w'(z_0)}{z-z_0} + \sum_{n=0}^{\infty} D_n(z-z_0)^n, \tag{10}$$

and substitute the expansion (10) into (7) and equate the coefficient of $(z - z_0)^{-1}$. This yields

$$(11)\qquad -\frac{(a-b)^2}{w'(z_0)} = (D_0 - a)(b - a) - (a-b)^2 \frac{w''(z_0) + (w'(z_0))^2}{2(w'(z_0))^2},$$

$$D_0 = \frac{a-b}{w'(z_0)} + \frac{1}{2}(a+b) + \frac{(b-a)w''(z_0)}{2(w'(z_0))^2}.$$

Now substitute the expansion (10) into (8), multiply by f, and equate the coefficient of $(z - z_0)^{-1}$. Using (11) we obtain

$$(12)\qquad 0 = -D_0 + \frac{(a-b)B_0}{w'(z_0)},$$

$$B_0 = \frac{a+b}{2(a-b)} w'(z_0) + 1 - \frac{w''(z_0)}{2w'(z_0)}.$$

Substitution of (12) into (9) gives

$$(13)\qquad \psi(z_0) = 1 - \frac{w''(z_0)}{w'(z_0)}.$$

We have two cases.

Case 1. $\psi(z) \not\equiv 1 - w''(z)/w'(z)$.

Note that if f has a pole of order k at z_1 then from (7), $e^w - 1$ has a zero of order k at z_1; if $k \geq 2$, then w' has a zero of order $k - 1$ at z_1. Combining this observation with (13) gives

$$(14)\qquad N(r, f) \leq \overline{N}\left(r, \psi - 1 + \frac{w''}{w'}, 0\right) + N(r, w', 0).$$

From (6) we obtain

$$m(r, e^w) \leq 2T(r, f') + 2T(r, f) + O(1)$$
$$\leq 6T(r, f) + S(r, f).$$

Since w' is the logarithmic derivative of e^w, this means that

$$(15)\qquad m(r, w') = S(r, f).$$

Hence from (14), (5), and (15) we obtain

$$(16)\quad \begin{aligned} N(r,f) &\le T\left(r, \psi - 1 + \frac{w''}{w'}, 0\right) + T(r, w', 0) \\ &= T\left(r, \psi - 1 + \frac{w''}{w'}\right) + T(r, w') + O(1) \\ &\le T(r,\psi) + T\left(r, \frac{w''}{w'}\right) + S(r,f) \\ &\le \bar{N}(r,F) + S(r,w') + S(r,f) \le \bar{N}(r,F) + S(r,f). \end{aligned}$$

We will now prove an inequality in the opposite direction. First differentiate (6) logarithmically to get

$$w' = \frac{f''}{f'-a} + \frac{f'}{f-b} - \frac{f'}{f-a} - \frac{f''}{f'-b}.$$

Multiplying by f' and using (15) gives

$$f' = \frac{b-a}{w'} \cdot \left[\frac{f'}{f-a} \cdot \frac{f'}{f-b} - \frac{f'f''}{(f'-a)(f'-b)}\right],$$

$$\begin{aligned} m(r,f') &\le m\left(r, \frac{1}{w'}\right) + m\left(r, \frac{f'}{f-a}\right) + m\left(r, \frac{f'}{f-b}\right) \\ &\quad + m\left(r, \frac{f'f''}{(f'-a)(f'-b)}\right) + O(1) \\ &\le T(r,w') + S(r,f) + S(r,f') \le S(r,f). \end{aligned}$$

Hence

$$(17)\qquad T(r,f') = N(r,f') + S(r,f).$$

We note also that [**5**, p. 33]

$$(18)\quad \begin{aligned} & m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) \\ &\le m\left(r, \frac{1}{f-a} + \frac{1}{f-b}\right) + O(1) \\ &\le m\left(r, \frac{f'}{f-a} + \frac{f'}{f-b}\right) + m\left(r, \frac{1}{f'}\right) + O(1) \\ &\le m\left(r, \frac{f'}{f-a}\right) + m\left(r, \frac{f'}{f-b}\right) + m\left(r, \frac{1}{f'}\right) + O(1) \\ &= m\left(r, \frac{1}{f'}\right) + S(r,f). \end{aligned}$$

Now by the second fundamental theorem we have

$$2T(r,f') \le \overline{N}(r,f') + N(r,f',0) + N(r,f',a) + N(r,f',b) + S(r,f').$$

Therefore, by using $N(r,f,a) = N(r,f',a)$, $N(r,f,b) = N(r,f',b)$, (17), and (18), we obtain

$$\begin{aligned} 2N(r,f') &\le \tfrac{1}{2}N(r,f') + T(r,f',0) - m(r,f',0) \\ &\quad + N(r,f,a) + N(r,f,b) + S(r,f) \\ &\le \tfrac{1}{2}N(r,f') + T(r,f') - m(r,f,a) - m(r,f,b) \\ &\quad + N(r,f,a) + N(r,f,b) + S(r,f) \\ &= \tfrac{3}{2}N(r,f') - T(r,f,a) - T(r,f,b) \\ &\quad + 2N(r,f,a) + 2N(r,f,b) + S(r,f); \end{aligned}$$

hence

$$\tfrac{1}{2}N(r,f') + 2T(r,f) \le 2N(r,f,a) + 2N(r,f,b) + S(r,f). \tag{19}$$

Since a pole of f of order $k \ge 2$ is a zero of w' of order $k-1$, then by using (15) we get

$$\begin{aligned} N(r,f) &\le \overline{N}(r,f) + N(r,w',0) \\ &\le \tfrac{1}{2}N(r,f') + T(r,w') + O(1) = \tfrac{1}{2}N(r,f') + S(r,f). \end{aligned}$$

Hence from (19) we obtain

$$\tfrac{1}{2}N(r,f) + T(r,f) \le N(r,f,a) + N(r,f,b) + S(r,f). \tag{20}$$

Then from (20) we can deduce that

$$\begin{aligned} &\overline{N}(r,F) + \tfrac{1}{2}N(r,f) + T(r,f) \\ &\quad \le \overline{N}(r,F) + N(r,f,a) + N(r,f,b) + S(r,f) \\ &\quad \le N\left(r,\frac{f'}{f},1\right) + \overline{N}(r,w',0) + N(r,f,0) - \overline{N}(r,f,0) + S(r,f) \\ &\quad \le T\left(r,\frac{f'}{f}\right) + T(r,w') + N(r,f,0) - \overline{N}(r,f,0) + S(r,f) \\ &\quad \le \overline{N}(r,f) + N(r,f,0) + S(r,f) \le N(r,f) + T(r,f) + S(r,f), \end{aligned}$$

which gives

$$\overline{N}(r,F) \le \tfrac{1}{2}N(r,f) + S(r,f). \tag{21}$$

But (16) and (21) together imply that

$$N(r, f) = S(r, f).$$

By Lemma 1, $f = f'$, which we have been assuming is not true.

Case 2. $\psi(z) \equiv 1 - w''(z)/w'(z)$.

Then integration of (4) gives

$$F = K\frac{e^z}{w'} \tag{22}$$

where K is a nonzero constant. If z_0 is a simple pole of f, then from (7), the residue of f at z_0 is $(a - b)(w'(z_0))^{-1}$. Thus from (22) we obtain $e^{z_0} = (b - a)K^{-1}$. On the other hand, a pole of order $n \geq 2$ of f is a pole of order $n - 1$ of F. Since 0 and ∞ are Picard values for e^w, we have that $T(r, e^w) = N(r, e^w, 1) + S(r, e^w)$ from the second fundamental theorem. Combining these observations with (7) and (22) we see that

$$\begin{aligned} T(r, e^w) &= N(r, e^w, 1) + S(r, e^w) \\ &= N\left(r, \frac{(f-a)(f'-b)}{f'-f}\right) + S(r, e^w) \\ &\leq \overline{N}\left(r, e^z, \frac{b-a}{K}\right) + N(r, w', 0) + N(r, F) + S(r, e^w) \\ &= \frac{r}{\pi} + 2N(r, w', 0) + S(r, e^w) \\ &\leq \frac{r}{\pi} + 2T(r, w') + S(r, e^w) \leq \frac{r}{\pi} + S(r, e^w). \end{aligned}$$

Hence outside a set of finite linear measure we have

$$T(r, e^w) \leq \frac{2r}{\pi}. \tag{23}$$

Now we will invoke the following lemma due to S. Bank.

LEMMA 2. [**1**, p. 68] *If $g(r)$ and $h(r)$ are monotone nondecreasing functions on $(0, \infty)$ such that $g(r) \leq h(r)$ for all r outside a set of finite linear measure, then for any given real number $\lambda > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\lambda r)$ for all $r \geq r_0$.*

Lemma 2 (with $\lambda = 2$) applied to the inequality (23) gives

$$T(r, e^w) \leq \frac{4r}{\pi} \quad \text{for } r \geq r_0.$$

Hence the order of e^w is at most one. It follows that $e^w = Ae^{Bz}$ for some nonzero constants A and B. Since $w' = B$, (22) reduces to

$$F = Ce^z \tag{24}$$

where C is a nonzero constant. Eliminating f' between (24) and (6) yields

$$f = a - Ce^z + \frac{a-b}{Ae^{Bz}-1}. \tag{25}$$

Then

$$f' = -Ce^z + \frac{(b-a)ABe^{Bz}}{(Ae^{Bz}-1)^2}. \tag{26}$$

Substitution of (26) and (25) into (24) gives

$$\begin{aligned} bCe^z(Ae^{Bz}-1)^2 + C^2e^{2z}(Ae^{Bz}-1)^2 + (a-b)^2Ae^{Bz} \\ + (b-a)Ce^z(Ae^{Bz}-1) + C(a-b)ABe^{(B+1)z} = 0. \end{aligned} \tag{27}$$

If the constant B is not real, then by equating the coefficient of e^z on the left side of (27) to zero, we get $aC = 0$, which is a contradiction. If $B > 0$, then the coefficient of $e^{(2B+2)z}$ in (27) gives $A^2C^2 = 0$, which is a contradiction. If $B < 0$, then the coefficient of e^{2z} gives the contradiction $C^2 = 0$. Thus $B = 0$ which is a contradiction.

Cases 1 and 2 have both led to a contradiction. Therefore f cannot have any simple poles.

If $w' \neq 0$, then from (15),

$$N(r, f) \leq N(r, w', 0) \leq T(r, w') + O(1) = S(r, f).$$

Then $f \equiv f'$ by Lemma 1, which contradicts the original assumption. If $w' \equiv 0$, then $e^w \equiv C$ where C is a nonzero constant. If $C = 1$ then $f \equiv f'$ from (7), a contradiction. If $C \neq 1$, then f has no poles and $f \equiv f'$ from Lemma 1, a contradiction.

Therefore, the original assumption that $f \not\equiv f'$ has yielded a contradiction, and the proof of Theorem 1 is now complete.

Proof of Corollary 1. Since equation (3) is merely a rewriting of equation (6), the result easily follows.

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Received July 30, 1981.

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