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ON COMPACT SPACES WITHOUT STRICTLY POSITIVE MEASURE

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In the present paper we construct compact spaces satisfying the countable chain condition and not carrying a strictly positive measure. Our spaces are completely different from already known examples, as it follows from special chain conditions established for them. Spaces satisfying property (*) and not carrying strictly positive family of measures, of given size, are also constructed.

The first example of a compact space X satisfying the countable chain condition and not carrying a strictly positive measure was given by Gaifman [5]. More precisely Gaifman's space satisfies a stronger than c.c.c. property, namely the property $(*)$. Another example, given by Galvin and Hajnal [6], separates property $(*)$ from the existence of caliber for all reasonable cardinals.

The present paper contains examples of compact spaces without strictly positive measure and with or without some special properties.

So, in the first part of the paper we construct a sequence $\{X_n: 2 \leq n < \omega\}$ of compact Hausdorff spaces such that for each $n < \omega$ the space X_n has property $(*)$ and it does not have property $(**)$ (or equivalently it does not have a strictly positive measure).

The space X_n has, also, property $K_n(\alpha)$ for all cardinals α with uncountable cofinality. Finally assuming Martin's axiom we prove that space X_n fails property $K_{n+1}(2^\omega)$.

We notice that both examples of Gaifman and Galvin-Hajnal have property $K_n(\alpha)$ for all cardinals α with uncountable cofinality and for all natural numbers n.

The second part of the paper is devoted to the construction for each cardinal α of a compact Hausdorff space X_α satisfying property $(*)$ but for every family $\{\mu_\xi: \xi < \alpha\}$ of regular positive Borel measures on X_α there exists a non empty open V subset of X_α with $\mu_\xi(V) = 0$ for all $\xi < \alpha$. Also, assuming the generalized continuum hypothesis we show that space X_α fails to have β-measure calibre for certain classes of regular cardinals β.

0. Preliminaries.

0.1. DEFINITION. (a) A topological space X satisfies *countable chain conditions* (c.c.c.) if and only if every family of non empty open, pairwise disjoint subsets of X is countable.

(b) For X topological space, α uncountable cardinal and n natural number, we say that *X has property $K_n(\alpha)$* if and only if every family $\{V_\xi: \xi < a\}$ of non empty open subsets of X contains a subfamily with the same cardinality and with the property: every n elements of it have non empty intersection. If $\alpha = \omega^+$, the first uncountable cardinal, then we denote the $K_n(\omega^+)$ by K_n and $K_2(\omega^+)$ by K.

0.2. DEFINITION. Let X be a topological space and $\mathfrak{T}^*$ the set of all open non empty subsets of X. We say that *X has property* $(*)$ if and only if there exists a decomposition of $\mathfrak{T}^*$ into a sequence $\mathfrak{T}_n$ such that for $n < \omega$ every pairwise disjoint subfamily of $\mathfrak{T}_n$ is finite.

0.3. DEFINITION. Let X be a topological space and $\mathfrak{L}$ be a subfamily of $\mathfrak{T}^*$. The *Kelley number* of the family $\mathfrak{L}$ is denoted by $k(\mathfrak{L})$ and it is defined with the following way: We write $\mathfrak{L}$ as $\{V_i: i \in I\}$ such that for each $i \in I$ there are infinite many $j \in I$ with $V_i = V_j$. Let $J \subset I$ finite the $\mathrm{cal}(J) = \sup\{|J'| : J' \subset I$ and $\bigcap_{i\in J'} V_i \neq 0\}$ then $k(\mathfrak{L}) = \inf\{\mathrm{cal}(J)/|J| : J \subset I$ and $0 < |J| < \omega\}$.

0.4. DEFINITION. A topological space X has property $(**)$ if and only if there exists a decomposition of $\mathfrak{T}^*$ into a sequence $\{\mathfrak{T}_n: n < \omega\}$ such that for each $n < \omega$, $k(\mathfrak{T}_n) > 0$.

Property $(**)$ implies property $(*)$ and the mentioned example of Gaifman shows that the inverse in general fails.

The following result is due to Kelley [7].

0.5. THEOREM. *Let X be a compact Hausdorff space. Then X carries a strictly positive measure (i.e. there exists μ defined on X such that $\mu(V) > 0$ for all non empty open sets (V) if and only if X has property $(**)$).*

0.6. REMARK. It is easy to see that the existence of a strictly positive measure μ on a topological space X implies property $(**)$. The inverse fails for general completely regular spaces. However if Y is a completely regular space with property $(**)$ then the Stone-Čech compactification βY also has property $(**)$ and on βY we have the equivalence. A detailed exposition of the concrete relations between all of them is given in [4] where one can also find the proof of the following result

0.7. PROPOSITION. *If a topological space X has property $(*)$ then it also has property K.*

1. We give in detail the construction of the topological space X_2, that is, the compact space with property $K_2(\alpha)$ for cardinals α with uncountable cofinality and under Martin's axiom fails to have property $K_3(2^\omega)$. The general case follows by analogous arguments and we present an outline of it in Theorem 1.9 at the end of this section.

1.0. *The space* X_2; will be the Stone-Čech compactification of a topological space $Y = (\{0,1\}^\omega, \mathfrak{T})$ where $\mathfrak{T}$ is a subbasis for the topology on Y consisting of all clopen subsets of $\{0,1\}^\omega$, with the usual topology, and of a family $\{V_\Sigma\colon \Sigma \in \boldsymbol{\Sigma}\}$, where $\boldsymbol{\Sigma}$ denotes the set of all branches of an appropriate tree $(T, \prec)$ which we define now.

1.1. *Construction of the tree* $(T, \prec)$. We define and fix a tree $(T = \bigcup_{n<\omega} T_n, \prec)$ consisting of elements of $[\omega]^2$ in the following way. We choose

$$\{S_{n,j}\colon n<\omega,\ 1 \le j \le 3^n\},$$

such that

$$S_{n,j} \in [\omega]^3$$

and

$$S_{n,j} \cap S_{n',j'} = \varnothing \quad \text{for } (n,j) \neq (n',j').$$

We set

$$T_n = \bigcup_{j=1}^{3^n} [S_{n,j}]^2$$

so that $|T_n| = 3^{n+1}$ and let

$$T_n = \{s_j\colon 1 \le j \le 3^{n+1}\}$$

be an enumeration of T_n for $n<\omega$.

The ordering $\prec$ of the tree $T = \bigcup_{n=1}^{\omega} T_n$ is defined so that the immediate successors of T_n are in T_{n+1} and is completely determined by the rule:

If $s \in T_n$ and $t \in T_{n+1}$ then $s \prec t$ if and only if there is a j $1 \le j \le 3^{n+1}$ such that $s = s_j$ and $t \in [S_{n+1,j}]^2$.

(A *branch* of $(T, \prec)$ is a totally ordered subset Σ of T such that the set $\{t \in T\colon t \prec s\}$ is contained in Σ for all $s \in \Sigma$. A branch may be of finite or infinite length.)

Denote by $\mathbf{\Sigma}$ the family of all branches of T. We set $\Sigma(n) = \Sigma \cap T_n$, and we define the following two sets

$$H_{n,2} = \bigcup_{l \geq n} \bigcup_{s \in T_l} s$$

$$H_{n,1} = \omega \setminus H_{n,2} \quad \text{for } n < \omega.$$

For s an element of T, let K_s be the "anti-diagonal" of $\{0,1\}^s$; i.e. if $s = \{k, l\}$ then $K_s = \{((k,1),(l,0)),((k,0),(l,1))\}$.

We remark, for later use, that if $S \subset \omega$ with $|S| = 3$ and $[S]^2 = \{s_1, s_2, s_3\}$ then the family $\{K_{s_i} \times \{0,1\}^{\omega \setminus s_i}: i = 1,2,3\}$ has empty intersection while any two subsets of the family have non empty intersection.

1.2. *Definition of the space Y.* As set Y is the Cantor set $\{0,1\}^\omega$. The topology on Y is defined by a subbase that contains of

(a) all subsets of $\{0,1\}^\omega$ that are clopen in the usual (product) topology of $\{0,1\}^\omega$; and

(b) the family $\{V_\Sigma: \Sigma \in \mathbf{\Sigma}\}$ where

$$V_\Sigma = \prod_{s \in \Sigma} K_s \times \{0,1\}^{\omega \setminus \cup \Sigma}$$

We note that $(Y, \mathfrak{T})$ has (a subbase and hence) a base consisting of sets that are closed (and hence compact) in the usual topology. Thus, Y is a completely regular Hausdorff space.

Every basic clopen set V of Y has the form

$$V = U \cap \left(\bigcap_{j=1}^{k} V_{\Sigma_j} \right)$$

where U is clopen in the usual topology of $\{0,1\}^\omega$ and $\Sigma_1, \ldots, \Sigma_k$ are branches of the tree $(T, \prec)$; we say, in this case, that V *is determined by* $(U; \Sigma_1, \ldots, \Sigma_k)$ and we say that V *is separated at level* n_0 if

(a) $\Sigma_{j_1}(n_0) \cap \Sigma_{j_2}(n_0) \neq \varnothing$ for $1 \leq j_1 < j_2 \leq k$; and

(b) If F is a finite subset of ω on which U depends, then $F \cap s = \varnothing$ for $s \in T_n$, $n_0 \leq n < \omega$.

If V is separated at level n_0 then there exists a clopen set W (in the usual topology of the Cantor set) with $W \subset \{0,1\}^{H_{1,n_0}}$, such that

$$V = W \times \bigcap_{j=1}^{k} \left(\prod_{l \geq n_0,} K_{\Sigma_j}(l) \times \{0,1\}^{H_{2,n_0} \setminus \bigcup_{l \geq n_0} I_j(l)} \right).$$

1.3. LEMMA. *Let $\{V_i: i \in I\}$ be a family of clopen basic subsets of Y, such that each V_i is determined by $(U_k; \Sigma_1^{(i)}, \ldots, \Sigma_{k(i)}^{(i)})$, V_i is separated at level n_0 and we set $W_i = \pi_{H_1,n_0}(V_i)$.*

Then the following are equivalent:

(a) $\{V_i: i \in I\}$ has the finite intersection property;

(b) 1. $\{W_i: i \in I\}$ has the finite intersection property; and

2. Of any three (distinct) elements of the set

$$A_n = \{\Sigma_j^{(i)}(n): i \in I, 1 \leq j \leq k(i)\}$$

some two of them have empty intersection for all $n \geq n_0$.

Proof. (a) $\Rightarrow$ (b). Condition (b)1 is clearly satisfied. For (b)2 suppose that $n \geq n_0$ and $B = \{s_1, s_2, s_3\} \subset A_n$ with $|B| = 3$ and $s_1 \cap s_2 \neq \varnothing$, $s_1 \cap s_3 \neq \varnothing$, $s_2 \cap s_3 \neq \varnothing$. Then there exists l, $1 \leq l \leq 3^{n+1}$ such that $B = [S_{n,l}]^2$.

Let i_1, i_2, i_3 in I and $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$, $1 \leq j_3 \leq k_{i_3}$ be such that $s_1 = \Sigma_{j_1}^{(i_2)}(n)$, $s_2 = \Sigma_{j_2}^{(i_2)}(n)$, $s_3 = \Sigma_{j_3}^{(i_3)}(n)$. Then

$$V_{i_1} \subset K_{s_1} \times \{0,1\}^{\omega \setminus s_1}$$

$$V_{i_2} \subset K_{s_2} \times \{0,1\}^{\omega \setminus s_2}$$

$$V_{i_3} \subset K_{s_3} \times \{0,1\}^{\omega \setminus s_3}$$

and $K_{s_1} \times \{0,1\}^{\omega \setminus s_1} \cap K_{s_2} \times \{0,1\}^{\omega \setminus s_2} \cap K_{s_3} \times \{0,1\}^{\omega \setminus s_3} \neq \varnothing$ contradicting the finite intersection property.

(b) $\Rightarrow$ (a). Let $i_1, \ldots, i_m$ in I. By (b)1 it follows that

$$\pi_{H_{1,n_0}}(V_{i_1}) \cap \cdots \cap \pi_{H_{1,n_0}}(V_{i_m}) \neq \varnothing.$$

So it is enough to show that

$$(1) \quad \bigcap_{p=1}^{m} \bigcap_{j=1}^{k_{i_p}} \left(\prod_{l=n_0}^{\infty} K_{\Sigma_j^{(i_p)}(l)} \times \{0,1\}^{H_{2,n_0} \setminus \bigcup_{l=n_0}^{\infty} \Sigma_j^{(i_p)}(l)} \right) \neq \varnothing.$$

We note (from definition of the tree T) that if s_1, s_2 are in T and $s_1 \cap s_2 \neq \varnothing$ then there exists $n < \omega$ such that s_1, s_2 in T_n; and if s_1, s_2, s_3 are in T and $s_1 \cap s_2 \neq \varnothing$, $s_1 \cap s_3 \neq \varnothing$, then $s_2 \cap s_3 \neq \varnothing$. Hence, and using (b)2 there is a sequence $\{B_n: n < \omega\}$ such that $\{\Sigma_j^{(i_p)}(l): n_0 \leq l < \omega,$

$1 \leq j \leq kp,\ 1 \leq p \leq m\} = \bigcup_{n<\omega} B_n$, $1 \leq |B_n| \leq 2$ for all $n < \omega$, $B_n \cap B_{n'} = \varnothing$ for $n < n' < \omega$ and if $s \in B_n$, $s' \in B_{n'}$, $n < n' < \omega$ then $s \cap s' = \varnothing$. We choose x_n elements of $\bigcap_{s\in B_n} K_s \times \{0,1\}^{\omega\setminus s}$ for $n < \omega$, and we choose x in $\{0,1\}^{H_{2,n_0}}$ such that

$$\pi_{\bigcup_{s\in B_n} s}(x) = \pi_{\bigcup_{s\in B_n} s}(x_n) \quad \text{for all } n < \omega.$$

Then x is well defined, and is an element of the set given in relation (1).

1.4. *Claim.* Y has property $(*)$ (and hence property (K)).

Proof. We define $\mathfrak{T}_{n,m}$ *a subset of* $\mathfrak{T}^*$ *with the following way.*

$\mathfrak{T}_{n,m} = \{U$: *U is open in Y, and there exists a basic clopen V in Y such that V is contained in U, V is separated at level n, and* $\mu(W) \geq \frac{1}{m}\}$. (Where $W = \pi_{H_{1,n}}(V)$ and μ is the usual Haar measure on $\{0,1\}^{H_{1,n}}$.)

Let $\{V_1, \cdots, V_{m+1}\}$ be a subset of $\mathfrak{T}_{n,m}$ for given n and m. Then there are i_1, i_2 with $1 \leq i_1 < i_2 \leq m+1$ such that

$$\pi_{H_{1,n}}(V_{i_1}) \cap \pi_{H_{1,n}}(V_{i_2}) \neq \varnothing;$$

hence, condition (b)1 of Lemma 1.3 is satisfied. Since V_{i_1}, V_{i_2}, V_{i_2} are separated at level n, it follows that condition (b)2 of Lemma 1.3 is also satisfied (since every branch of T that determines V_{i_1} intersects at most one of the branches of T that determines V_{i_2} at a level greater than or equal to n). Thus

$$V_{i_1} \cap V_{i_2} \neq \varnothing.$$

It clearly follows that Y has property $(*)$.

1.5. *Claim.* Y does not have property $(**)$.

Proof. Suppose that Y has property $(**)$. Then there exists a sequence $\{\mathfrak{T}_n\colon n < \omega\}$ such that $\mathfrak{T}^*(Y) = \bigcup_{n<\omega} \mathfrak{T}_n$ and $k(\mathfrak{T}_n) > 0$ for all $n < \omega$. Without loss of generality we assume also that $\mathfrak{T}_n \subset \mathfrak{T}_{n+1}$ for all $n < \omega$ and if U, V are clopen in Y and U is a subset of V then

$$\min\{n\colon U \in \mathfrak{T}_n\} \geq \min\{n\colon V \in \mathfrak{T}_n\}.$$

We claim that there is $n_0 < \omega$ and a finite branch $\Sigma_0 = \{s_0 < s_1 < \cdots < s_m\}$ such that, if Σ' is a finite branch and $\Sigma_0 \subset \Sigma'$ then $\prod_{s\in\Sigma'} K_s \times \{0,1\}^{\omega\setminus\bigcup_{s\in\Sigma'} s} \in \mathfrak{T}_{n_0}$. (Indeed, otherwise there exists a sequence $k_1 < k_2 < \cdots < k_n < \cdots$ $(n < \omega)$ of natural numbers and a sequence $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n \subset \cdots$ $(n < \omega)$ of finite branches (containing Σ_0), such that $\min\{k\colon V_{\Sigma_n} \in \mathfrak{T}_k\} = k_n$ for all $n < \omega$. We set $\Sigma = \bigcup_{n<\omega} \Sigma_n$ then $V_\Sigma \in \mathfrak{T}_k$ for some k. By our assumptions it follows that $k > k_n$ for all $n < \omega$ and we get a contradiction proving the claim.)

We choose $k > 0$ such that $(2/3)^k < k(\mathfrak{T}_{n_0})$, and we set $\zeta = \{\Sigma\colon \Sigma$ is a finite branch of T, $\Sigma_0 \subset \Sigma$ and the length of Σ is $\eta_0 + k\}$. It is easy to see that

$$\begin{aligned}&\zeta \subset \mathfrak{T}_{n_0} \quad \text{(from the above claim)}\\ &|\zeta| = 3^k \quad \text{and}\\ &\operatorname{cal}\{\Sigma\colon \Sigma \in \zeta\} = 2^k\end{aligned}$$

a contradiction, proving that Y does not have $(**)$.

1.6. Proposition. *For every cardinal* α *with* $\operatorname{cf}(\alpha) > \omega$ *the space* Y *has property* $K_2(\alpha)$.

Proof. Let $\{V_\xi\colon \xi < \alpha\}$ be a family of basic clopen subsets of Y. Since cardinal α has uncountable cofinality there exists set I subset of α with $|I| = \alpha$, a $n_0 < \omega$ and $W \subset \{0,1\}^\omega$ clopen in the usual topology such that for all ξ in I

(i) V_ξ is separated at level n_0

(ii)

$$V_\xi = W \times \bigcap_{j=1}^{k} \left(\prod_{l \geq n_0} K_{\Sigma_j^\xi(l)} \times \{0,1\}^{H_{2,n_0} \setminus \bigcup_{l \geq n_0} \Sigma_j^\xi(l)} \right).$$

(We use the terminology of Definition 1.2.) Now we easily verify that the family $\{V_\xi\colon \xi \in I\}$ has the two-intersection property and the proof is complete.

1.7. *Claim*. Assume Martin's axiom. Then the density character of Y is equal to the cardinal 2^ω.

Proof. Assume the contrary. Then there exists a cardinal α less than 2^ω and a family $\mathfrak{D} = \{x_\xi\colon \xi < \alpha\}$ of elements of $\{0,1\}^\omega$ dense in the space Y. We construct a partial ordered set $(\mathfrak{P}, \prec)$ with the following way. $\mathfrak{P}$, a set, contains elements of the form (F, Σ) where F is a finite subset of $\mathfrak{D}$ and Σ is a finite branch of the tree T, such that for each x_ξ in F, x_ξ does not belong to the set V_Σ. The order is defined on $\mathfrak{P}$ with the natural way namely $(F_1, \Sigma_1) < (F_2, \Sigma_2)$ if and only if $F_1 \subset F_2$, $\Sigma_1 \subset \Sigma_2$.

Fact 1. $(\mathcal{P}, \prec)$ satisfies c.c.c. Indeed, for $\{(F_\xi, \Sigma_\xi): \xi < \omega^+\}$ there are $\Sigma_{\xi_1} = \Sigma_{\xi_2} = \Sigma$. Then the pair $(F_{\xi_1} \cup F_{\xi_2}, \Sigma)$ is an element of $(\mathcal{P}, \prec)$ extending $(F_{\xi_i}, \Sigma_{\xi_i})$ for $i = 1, 2$.

Fact 2. Let x_ξ in $\mathcal{D}$ and (F, Σ) in $\mathcal{P}$ be given. Then there is Σ_1 finite branch with $I \subset \Sigma_1$ and $(F \cup \{x_\xi\}, \Sigma_1)$ be in $(\mathcal{P}, \prec)$. Indeed, let $\{s_1, s_2, s_3\}$ be the set of the immediate successors of I; then there exists i element of $\{1, 2, 3\}$ such that $(F \cup \{x_\xi\}, I \cup \{s_i\})$ is an element of $(\mathcal{P}, \prec)$.

We set D_ξ be the set of all (F, Σ) such that $x_\xi \in F$. The previous argument shows that D_ξ is a dense subset of $(\mathcal{P}, \prec)$ for all $\xi < \alpha$.

Martin's axiom implies the existence of a filter $\mathcal{F}$ of elements of $(\mathcal{P}, \prec)$ with $\mathcal{F} \cap D_\xi \neq \varnothing$ for all $\xi < \alpha$.

Now setting $\tilde{\Sigma} = \bigcup\{\Sigma$: there is (F, Σ) in $\mathcal{F}\}$ we remark that $\tilde{\Sigma}$ is a branch (maybe infinite) of the tree T and for all x_ξ in $\mathcal{D}$, x_ξ does not belong to the open non empty set $V_{\tilde{\Sigma}}$ a contradiction, and the proof is complete.

1.8. *Claim*. Assume Martin's axiom. The space Y does not have property $K_3(2^\omega)$.

Proof. Martin's axiom implies that cardinal 2^ω is regular and from the Proposition 1.7 we have that $|Y| = d(Y) = 2^\omega$. Let $\{x_\xi: \xi < 2^\omega\}$ be a well-ordering of X. For every $\xi < 2^\omega$ we choose a basic clopen set V_ξ in Y such that

$$V_\xi \cap \{x_\zeta: \zeta \leq \xi\} = \varnothing.$$

We claim that for each I subset of 2^ω with $|I| = 2^\omega$ the family $\{V_\xi: \xi < 2^\omega\}$ does not have the finite intersection property. (Indeed, if it does, for some such I, then since V_ξ is a closed (and hence compact) subset of $\{0, 1\}^\omega$ with the usual topology we would have

$$x \in \bigcap_{\xi \in I} V_\xi \neq \varnothing.$$

But $x = x_{\xi_0}$ for some $\xi_0 < 2^\omega$ and if $\xi \in I$, $\xi > \xi_0$ then $x \notin V_\xi$.)

Since 2^ω is regular cardinal there exists $I \subset 2^\omega$, $n_0 < \omega$ and W such that

$$|I| = 2^\omega,$$

V_ξ is separated at the level n_0 for $\xi \in I$, W depends on a finite subset of $H_{n_0,1}$ and $\pi_{H_{n_0,1}}(V_\xi) = W$ for all ξ in I.

Since $\{V_\xi: \xi \in I\}$ does not have the finite intersection property, and condition (b)1 of Lemma 1.3 is satisfied, it follows that there is $n_1 > n_0$ and B subset of A_{n_1} (in the notation of Lemma 1.3) such that

$$B = \{s_1, s_2, s_3\} \quad |B| = 3 \quad \text{and}$$
$$B \text{ has the two-intersection property.}$$

Choose ξ_1, ξ_2, ξ_3 in I such that V_{ξ_i} is determined by $(U_{\xi_i}, \Sigma_j^{(\xi_i)}, 1 \le j \le K_{\xi_i}$ for $i = 1, 2, 3)$ and

$$s_1 = \Sigma_{j_1}^{(\xi_1)}(n_1)$$
$$s_2 = \Sigma_{j_2}^{(\xi_2)}(n_1)$$
$$s_3 = \Sigma_{j_3}^{(\xi_3)}(n_1)$$

for some $1 \le j_i \le k_{\xi_i}, i = 1, 2, 3$.

It is clear that $V_{\xi_1} \cap V_{\xi_2} \cap V_{\xi_3} = \varnothing$.

1.9. THEOREM. *For each $n \ge 2$ there exists a compact Hausdorff space X_n satisfying the following properties*

(a) *X_n does not carry a strictly positive measure*

(b) *X_n satisfies property* $(*)$

(c) *X_n satisfies property $K_n(\alpha)$ for all cardinals α with* $\mathrm{cf}(\alpha) > \omega$

(d) *Assuming Martin's axiom X_n does not have property $K_{n+1}(2^\omega)$.*

Proof. We construct the space X_n following similar steps as in the construction of the space X_2. For the sake of completeness we give here the main steps for the general case.

We choose a positive number λ in such a way that there exists a family

$$\Lambda = \{\Lambda_1, \ldots, \Lambda_{n+1}\}$$

of subsets of $\{0, 1\}^\lambda$ satisfying the property: $\bigcap_{i=1}^{n+1} \Lambda_i = \varnothing$ while for each $1 \le j \le n + 1$, $\bigcap_{i \ne j} \Lambda_i \ne \varnothing$.

Now for $k = 0, 1, 2, \ldots$ we define a family

$$\left\{S_j^k : j = 1, \ldots, (n+1)^k\right\}$$

where $S_j^k \subset \omega$, $|S_j^k| = \lambda$, $S_{j_1}^{k_1} \cap S_{j_2}^{k_2} = \varnothing$ for $(k_1, j_1) \ne (k_2, j_2)$.

Let (k, j) be given with $j \le (n+1)^k$. We choose $\Lambda_j^k = \{(\Lambda_i)_j^k : i = 1, \ldots, n + 1\}$ family of subsets of $\{0, 1\}^{S_j^k}$ satisfying the properties of the family Λ.

We produce a tree $(T, \prec)$ such that for each $k < \omega$

$$T_k = \bigcup_{j=1}^{(n+1)^k} \Lambda_j^k.$$

We enumerate T_k as $\{\Lambda_1^k, \ldots, \Lambda_{(n+1)^k}^k\}$ and we consider this enumeration fixed for the rest. Now the immediate successors of Λ in T_k belong to T_{n+1} and they are defined by the following rule, if $\Lambda \equiv \Lambda_j^k$ then $\Lambda \prec \Lambda_\sigma$ for all Λ_σ elements of Λ_j^{k+1}. Using the tree $(T, \prec)$ we define a family $\{V_\Sigma\colon \Sigma \in \mathbf{\Sigma}\}$ of subsets of $\{0,1\}^\omega$ such that $\mathbf{\Sigma}$ is the set of all branches of the tree T and

$$V_\Sigma = \prod_{\Lambda \in \Sigma} \Lambda \times \{0,1\}^{\omega \setminus W_\Sigma}$$

where W_Σ is the subset of ω on which the family $\{V_\Sigma\colon \Sigma \in \mathbf{\Sigma}\}$ depends. Now we can prove that Y_n has property $(*)$, and property $K_n(\alpha)$ for all cardinals α with $\mathrm{cf}(\alpha) > \omega$. Also Y_n fails property $(**)$ and if in addition we assume Martin's axiom it fails property $K_{n+1}(2^\omega)$. To get space X_n we simply take the Stone-Čech compactification of the space Y_n.

1.10. REMARK. It is proved in [3] that every compact space X carrying a strictly positive measure has property $K_n(\alpha)$ for all $n < \omega$ and cardinals α with $\mathrm{cf}(\alpha) > \omega$. Our example (Theorem 1.9) shows that the above result fails if we consider instead of strictly positive measure the weaker assumption namely X has property $(*)$.

1.11. REMARK. In [8] it is given another example, under C.H. of a space X_n with $K_n(\omega^+)$ and without $K_{n+1}(\omega^+)$.

2.

2.1. DEFINITION. A topological space X carries a *strictly positive family of measures of cardinality* α if and only if there exists a family $\{\mu_\xi\colon \xi < \alpha\}$ of positive measures on X such that for each non empty open V subset of X there is $\xi_0 < \alpha$ with $\mu_{\xi_0}(V) > 0$.

2.2. DEFINITION. (i) A topological space X has property $(*(\alpha))$ if and only if there exists a decomposition of the family $\mathfrak{T}^*$ into a family $\{\mathfrak{T}_\xi\colon \xi < \alpha\}$ such that for each $\xi < \alpha$ every pairwise disjoint subfamily of $\mathfrak{T}_\xi$ is finite.

(ii) The space X has property $(**(\alpha))$ if and only if there exists a decomposition of $\mathfrak{T}^*$ into a family $\{\mathfrak{T}_\xi\colon \xi < \alpha\}$ such that for each $\xi < \alpha$ the Kelley number $k(\mathfrak{T}_\xi)$ be strictly positive.

2.3. REMARK. Kelley's characterization of the existence of strictly positive measures implies actually that, a compact space X carries a strictly positive family of measures of cardinality α if and only if it has property $(**(\alpha))$.

2.4. THEOREM. *Let β be a strong limit cardinal with* $\mathrm{cf}(\beta) = \omega$. *Then there exists a compact space X_β with property* $(*)$ *and without property* $(**(\beta))$.

Proof. The construction of the space X_β is, essentially, given in [**1**]. For the sake of completeness we give here the main steps of this construction.

We start by choosing a sequence $\{\beta_n: n < \omega\}$ of cardinals such that

(i) $\Sigma_{n<\omega}\beta_n = \beta$,

(ii) β_n is infinite regular cardinal for all $n < \omega$,

(iii) $2^{\beta_n} < \beta_{n+1}$.

We define a tree $T = \bigcup_{n<\omega} T_n$ consisting of elements of $[\beta]^2$ in the following way: for every $n < \omega$ we let $\{B_{n+1,i}: i < \beta_n\}$ be such that

(a) $B_{n+1,i} \subset \beta_{n+1}\setminus\beta_n$ and $|B_{n+1,i}| = \beta_{n+1}$ for all $i < \beta_n$.

(b) $B_{n+1,i} \cap B_{n+1,j} = \varnothing$ for $i < j < \beta_n$. We set

$$T_{n+1} = \bigcup_{i<\beta_n} \left([B_{n+1,i}]^2\right) \quad \text{and} \quad T_0 = [\beta_0]^2.$$

For every $n < \omega$ let $T_n = \{S_i^{(n)}: i < \beta_n\}$ be a one-to-one well-ordering of T_n fixed for the rest. Furthermore, if $s \in T_n$ and $t \in T_{n+1}$ then $s \prec t$ if and only if there exists $i < \beta_n$ such that $s = s_i^{(n)}$ and $t \in [B_{n+1,i}]^2$. The definition of T is, now, complete. Let $\mathbf{\Sigma}$ be the set of all branches of T. For Σ in $\mathbf{\Sigma}$, we set $\Sigma(n) = \Sigma \cap T_n$. For s in T let K_s be the "anti-diagonal" of $\{0,1\}^s$ i.e. if $s = \{k, l\}$

$$K_s = \{((k,1),(l,0)),((k,0),(l,1))\}.$$

We next define the topological space Y_β. As a set Y_β is the set $\{0,1\}^\beta$. The topology $\mathfrak{T}$ on Y_β is defined by a subbasis that contains of

(a) all subsets of $\{0,1\}^\beta$ that are clopen in the usual topology of $\{0,1\}^\beta$.

(b) the family $\{V_\Sigma: \Sigma \in \mathbf{\Sigma}\}$ where

$$V_\Sigma = \prod_{s\in\Sigma} K_s \times \{0,1\}^{\beta\setminus\bigcup_{s\in\Sigma}s}.$$

Since, each V_Σ is closed subset of $\{0,1\}^\beta$ with the usual topology and the topology $\mathfrak{T}$ is finer than the usual, it follows that V_Σ is clopen subset of Y_β. So Y_β is completely regular and we set X_β be the Stone-Čech compatification of Y_β.

2.4. LEMMA. *The space Y_β (and hence space X_β) has property* $(*)$.

The proof of the lemma goes along the same steps as the proof of Claim 1.4.

2.6. LEMMA. *Space Y_β (and hence space X_β) does not have property* $(**(\beta))$.

Proof. We prove it by contradiction. So we assume that the conclusion fails for some cardinal β. Let $\{\mathcal{T}_\xi : \xi < \beta\}$ be a $(**(\beta))$ decomposition of open non empty subsets of Y_β. Then we assert the following.

Claim 1. There is a finite branch Σ_0 and ξ_0 ordinal such that for every finite branch Σ that contains Σ_0 ($\Sigma_0 \subset \Sigma$) there exists $\xi < \xi_0$ such that V_Σ belongs to the family $\mathcal{T}_\xi$.

The proof of this claim as well as the proof of the next one is analogous to the proof of Claim 1.5.

Claim 2. For every $0 < n < \omega$ we set

$$I_n = \left\{\xi < \xi_0 : k\left(\mathcal{T}_\xi\right) > \tfrac{1}{n}\right\}$$

and we assert that there exists an Σ_1 finite branch and n_0 natural number such that $\Sigma_0 \subset \Sigma_1$ and for every Σ finite branch with $\Sigma_1 \subset \Sigma$ there is $\xi \in I_{n_0}$ with $V_\Sigma \in \mathcal{T}_\xi$.

We choose Σ_2 finite branch with $\Sigma_1 \subset \Sigma_2$ such that if λ is the cardinality of the immediate successors of Σ_2 then

$$\lambda \to (3)^2_{|\xi_0|}.$$

(I.e. if we define a decomposition $\{A_\sigma : \sigma < |\xi_0|\}$ of the set $[\lambda]^2$ then there is an S subset of λ with $|S| \geq 3$ such that $[S]^2 \subset A_{\sigma_0}$ for some $\sigma_0 < |\xi_0|$.) For a finite branch Σ we set

$$A_\Sigma = \left\{\tilde{\Sigma} : \tilde{\Sigma} = \Sigma \cup \{s\} \text{ where } s \in \left[B_\Sigma\right]^2\right\}$$

(That is, the set of all immediate successors of Σ.)

So the set A_Σ coincides with the set $[B]^2$ and we will use the set $[B_\Sigma]^2$ instead of A_Σ.

Every finite branch Σ with $\Sigma_2 \subset \Sigma$ has the form $\Sigma = \Sigma_2 \cup \{s_1, \ldots, s_r\}$ and in the following we denote by $|\Sigma|$ the number r. With this notation we have that $|\Sigma_2| = 0$.

Let $k < \omega$ be such that $(2/3)^k < 1/n_0$. For every Σ finite branch extending the branch Σ_2 ($\Sigma_2 \subset \Sigma$) and $|\Sigma| = k - 1$ we define a decomposition of the set $[B_\Sigma]^2$ into a family $\{(T_k^\Sigma)_\zeta : \zeta < \xi_0\}$ by the rule $s \in (T_k^\Sigma)_\zeta$ iff $V_{\Sigma \cup \{s\}} \in \mathfrak{T}_\zeta$.

We note that from our assumption on the cardinality λ of $[B_{\Sigma_2}]^2$ it follows that there is a three-point set D_k^Σ and a $\zeta_0 < \xi_0$ such that $[D_k^\Sigma]^2 \subset (T_k^\Sigma)_{\zeta_0}$.

Let $1 < \mu \leq k$ and we assume that for every finite branch Σ with $\Sigma_2 \subset \Sigma$ and $k - \mu < |\Sigma| \leq k - 1$ we have already defined a decomposition of $B_{|\Sigma|}^\Sigma$ into a family $\{(T_{|\Sigma|}^\Sigma)_\zeta : \zeta < \xi_0\}$ and let Σ be a finite branch with $\Sigma_2 \subset \Sigma$ and $|\Sigma| = k - \mu$. Then we define a decomposition of $[B_{k-\mu+1}^\Sigma]^2$ into a family $\{(T_{k-\mu+1}^\Sigma)_\zeta : \zeta < \xi_0\}$ by the rule $s \in (T_{k-\mu+1}^\Sigma)_\zeta$ if and only if there exists $D_{k-\mu+2}^{\tilde{\Sigma}}$ with $|D_{k-\mu+2}^{\tilde{\Sigma}}| = 3$ and

$$\left[D_{k-\mu+2}^{\tilde{\Sigma}}\right]^2 \subset \left(T_{k-\mu+2}^{\tilde{\Sigma}}\right)_{\zeta_0}$$

where $\tilde{\Sigma} = \Sigma \cup \{s\}$.

The inductive definition of the families $\{(T_{|\Sigma|+1}^\Sigma)_\zeta : \zeta < \xi_0\}$ for Σ a finite branch with $\Sigma_2 \subset \Sigma$ and $|\Sigma_2| \leq |\Sigma| \leq k - 1$ is now complete.

Since the cardinal $\lambda = |B_{\Sigma_2}|$ satisfies the relation

$$\lambda \to (3)_{|\xi_0|}^2$$

it follows that there exists a $D_1^{\Sigma_2}$ subset of B_{Σ_2} and a $\zeta_0 < \xi_0$ such that

$$\left[D_1^{\Sigma_2}\right]^2 \subset \left(T_1^{\Sigma_2}\right)_{\zeta_0}.$$

Now using the definition of the families $\{(T_{|\Sigma|+1}^\Sigma)_\zeta : \zeta < \xi_0\}$ we choose a finite tree $T_k = \bigcup_{\mu=1}^k T_\mu$ where $T_1 = [D_1^{\Sigma_2}]^2$.

The level T_2 is defined in conjunction with the elements of the level T_1 by the next rule: for every s in T_1, $s \in (T_1^{\Sigma_2})_{\zeta_0}$ means that there exists a $D_2^s \subset B_{\Sigma_2} \cup \{s\}$ such that $[D_2^s]^2 \subset (T_2^\Sigma)_{\zeta_0}$ and $|D_2^s| = 3$.

We set $T_2 = \bigcup_{s \in T_1} [D_2^s]^2$. In the same way we produce level $T_\mu + 1$ from level T_μ for $1 \leq \mu \leq k - 1$, and so the inductive definition of the finite tree T_k is now complete.

Now it is easily verified that if we consider the family

$$\mathfrak{L} = \left\{ V_{\Sigma_2 \cup \Sigma} \text{ where } \Sigma \text{ is a branch of } T_k \text{ with } |\Sigma| = k \right\}$$

then

(i) $|\mathfrak{L}| = 3^k$
(ii) $\mathrm{cal}(\mathfrak{L}) = 2^k$

(iii) each element of $\mathfrak{L}$ belongs to $\mathfrak{T}_{\xi_0}$. Comparing the above three conclusions we get a contradiction and this finishes the proof of the lemmas.

2.7. Definition. If $\alpha \geq \omega$ is a cardinal and X is a compact Hausdorff space, then X has *α-measure calibre* if for every family $\{V_\xi: \xi < \alpha\}$ of open non empty subsets of X there is a measure μ in $\mu^+(X)$ and I subset of α with $|I| = \alpha$ such that $\mu(V_\xi) > 0$ for all ξ in I.

2.8. Remark. The above definition, that extends the definition of α-calibre is due to N. Kalamidas.

2.9. Remark. If a compact space X carries a strictly positive measure then it has α-measure calibre for all infinite cardinals α. Tsaralias and the author in [2] have proved that, under G.C.H., every cardinal α with α does not have the form β^+ for some β with $\mathrm{cf}(\beta) = \omega$ is a calibre for every compact space satisfying the c.c.c.

In the same paper for each cardinal of the form β^+ with $\mathrm{cf}(\beta) = \omega$ we construct a compact space Z_β carrying a strictly positive measure and not having β^+ calibre.

In the sequel we will prove under G.C.H. that for every such cardinal α (i.e. $\alpha = \beta^+$ and $\mathrm{cf}(\beta) = \omega$) space X_β, defined before, does not have α-measure calibre.

2.10. Proposition. *Assume the* G.C.H. *and let β be a cardinal with* $\mathrm{cf}(\beta) = \omega$. *Then the space X_β does not have β^+-measure calibre.*

Proof. We first note the following two facts:

(a) Since every subbasic open set in $Y_\beta = (\{0,1\}^\beta, \mathfrak{T})$ is either clopen in the usual topology of $\{0,1\}^\beta$ or it has the form

$$V_\Sigma = \bigcap_{s \in \Sigma} K_s \times \{0,1\}^{\beta \setminus s}$$

it follows that there exists a basis for the topology $\mathfrak{T}$ consisting of elements V satisfying the property: each V is the countable intersection of clopen subsets of $\{0,1\}^\beta$ in the usual product topology.

(b) From the proof of Kelley's theorem [7] follows the proof of the next statement. Let $\Lambda = \{\Lambda_i: i \in I\}$ be a family of clopen subsets of a compact space X and $\partial > 0$ be a real number. Then the following are equivalent.

(i) The Kelley number $k(\Lambda)$ is greater or equal to ∂.

(ii) There is a probability measure μ on X with $\mu(\Lambda_i) \geq \partial$ for all $i \in I$.

We, now, pass in the proof of the proposition.

The Banach space $M(\{0,1\}^\beta)$ of all regular Borel measures on $\{0,1\}^\beta$ is the conjugate of the Banach space $C(\{0,1\}^\beta)$ of the continuous functions on $\{0,1\}^\beta$ and density character of $C(\{0,1\}^\beta)$ is equal to β.

Therefore density character of $M\{0,1\}^\beta$ is $2^\beta = \beta^+$ and $|M(\{0,1\}^\beta)| = \beta^+$.

Let $\{\mu_\xi: \xi < \beta^+\}$ be a well-ordering of $M(\{0,1\}^\beta)$. Since Y_β fails to carry a strictly positive family of measures of cardinality β it follows that for each $\xi < \beta^+$ there is a basic clopen subset of Y_β, say V_ξ such that for every $\zeta \leq \xi$ $\mu_\zeta(V_\xi) = 0$.

For $\xi < \beta^+$ we denote by $\overline{V_\xi}$ the closure of V_ξ into the space X_β, the Stone-Čech compatification of Y_β and we notice that $\overline{V_\xi}$ is a clopen subset of X_β.

Claim. For every μ in $M^+(X_\beta)$

$$\left|\left\{\xi < \beta^+ : \mu\left(\overline{V_\xi}\right) > 0\right\}\right| < \beta^+ .$$

Assume the contrary. Then there exist μ in $M^+(X_\beta)$, an I subset of β^+ and $\partial > 0$ such that $\mu(\overline{V_\xi}) > \partial$ for all ξ in I and $|I| = \beta^+$.

For each ξ in I let $\{U_\xi^n: n < \omega\}$ be a decreasing sequence of clopen subsets of $\{0,1\}^\beta$ with $\bigcap_{n<\omega} U_\xi^n = V_\xi$.

Our assumption implies that the Kelley number of the family $\{V_\xi: \xi \in I\}$ is at least ∂ and therefore the same is true for the Kelley number of the family $\{U_\xi^n: \xi \in I, n < \omega\}$, which means that there exists a measure μ in $M(\{0,1\}^\beta)$ such that $\mu(U_\xi^n) \geq \partial$ for all ξ in I and $n < \omega$. This is a contradiction since $\mu = \mu_{\xi_0}$ and if ξ is in I with $\xi > \xi_0$ then $\mu_{\xi_0}(V_\xi) = 0$, and so $\lim_{n\to\infty} \mu_{\xi_0}(U_\xi^n) = 0$. The proof is now complete.

2.11. Remark. As Professor Negrepontis established extending the methods of the first paragraph in higher cardinal, one can prove the following result that extends Theorem 1.9.

2.12. Theorem. *For every infinite cardinal α and every $n > 1$ there exists a compact space $X_n(\alpha)$ with property $K_n(\alpha^+)$ but, under* G.C.H., *$X_n(\alpha)$ fails property $K_{n+1}(\alpha^+)$.*

2.13. Remark. Professor Galvin informed me that from a variation of the Galvin-Hajnal example follows the existence of a space X_α, for each cardinal α with β-calibre for all cardinals β with $\text{cf}(\beta) > \omega$ and without property $(*(\alpha))$.

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