PACIFIC JOURNAL OF MATHEMATICS
Vol. 106, No. 1, 1983

THE ROOT SUBGROUPS FOR MAXIMAL TORI IN FINITE GROUPS OF LIE TYPE

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Let G_1 be a finite group of Lie type defined over a field of characteristic p. The results of this paper represent an attempt to achieve a better understanding of the subgroup structure of G_1. It is somewhat surprising how limited our knowledge is, in this regard. For example, while centralizers of semisimple elements (i.e. p'-elements) of G_1 have been studied in detail and are fairly well understood, very little has been written about subgroups of G_1 generated by such centralizers. Even in explicit examples the analysis of such subgroups can be very difficult, the difficulty stemming from an inability to relate the generated group to the Lie structure of G_1. To deal with these situations and others we set up a framework that allows us to effectively study a fairly large class of subgroups of G_1 (those containing a maximal torus), by studying subgroups of the corresponding algebraic group. Essential to the development is a theory of root subgroups for arbitrary maximal tori of G_1.

1. Introduction. The theorems we establish have as their origin Lemma 3 of [**22**], which was later extended in [**7**] to show that if $q \geq 5$ and if H is a Cartan subgoup of G_1 normalizing the p-group V, then V is the product of root subgroups of H. This result is quite useful and provided the starting point for the result in [**21**] which showed that with further field restrictions one could determine all H-invariant subgroups of G_1. For example, if $H \leq L \leq G_1$, then it was shown that L could be generated by $N_L(H)$ together with certain of the root subgroups of H. Hence, L is determined by a subset of the root system of G_1 together with a subgroup of the Weyl group of G_1. One wants to extend these results to cover the case of an arbitrary maximal torus, not just a Cartan subgroup. Therefore, one would like to develop a theory of root subgroups that makes sense for an arbitrary maximal torus and then establish results like those above. The present paper carries out this program.

The group G_1 satisfies $O^{p'}(\bar{G}_\sigma) \leq G_1 \leq \bar{G}_\sigma$, where $\bar{G}$ is a connected simple algebraic group over the closure of $\mathbf{F}_p$, and σ is an endomorphism of $\bar{G}$ whose fixed point set, $\bar{G}_\sigma$, is a finite group. Set $G = \bar{G}_\sigma$ and $G_0 = O^{p'}(\bar{G}_\sigma)$. A *maximal torus* of G_1 is a group of the form $T \cap G_1$, where $T = \bar{T}_\sigma$ and $\bar{T}$ is a σ-invariant maximal torus of $\bar{G}$. The group $\bar{G}$ has a root system, $\bar{\Sigma}$, and for each root $\alpha \in \bar{\Sigma}$, there is a $\bar{T}$-root subgroup $\bar{U}_\alpha$ of $\bar{G}$. These root subgroups are permuted by σ. If Δ is a $\langle \sigma \rangle$-orbit of such root subgroups, let $X = O^{p'}(\langle \Delta \rangle_\sigma)$, a subgroup of G_1. Such a group is called a *T-root subgroup* of $\bar{G}_\sigma$, and these groups are the groups we wish to consider. The T-root subgroups of G are either p-groups or themselves groups of Lie type, and even when they are p-groups their structure can be complicated. Nevertheless, the situation is manageable, as we indicate in the sample results below.

Write $G_1 = G_1(q)$, where q is a power of p and fix a maximal torus T of $G = \bar{G}_\sigma$. Set $T_0 = T \cap G_0$.

THEOREM (6.1). *Suppose $q > 7$, T is a maximal torus of $\bar{G}_\sigma$ and $T_0 \leq Y \leq \bar{G}_\sigma$ with Y solvable. Then $Y = O_p(Y)N_Y(T_0)$ and $O_p(Y)$ is a product of T-root subgroups of $\bar{G}_\sigma$.*

As a consequence of (6.1) we see that for $q > 7$ any T_0-invariant p-subgroup is a product of T-root subgroups; the exact analogue of the result in [7]. When one considers arbitrary subgroups invariant under (or containing) a maximal torus, additional field restrictions must be made. In addition our proofs depend on the classification of finite simple groups.

In each of the following results assume that $p > 3$ and $q > 11$.

THEOREM (12.1). *The map $\bar{X} \to \bar{X}_\sigma$ is a bijection between the collection of all closed, connected, σ-invariant subgroups of $\bar{G}$ containing a maximal torus of $\bar{G}$, and the collection of all subgroups of G generated by maximal tori of G.*

The inverse of the map $\bar{X} \to \bar{X}_\sigma$ is given in §12; it involves the T-root groups described above. We remark that a group of the form $\bar{X}_\sigma$ has known structure (see (2.5)), so by (12.1) we can describe the structure of any subgroup of G generated by maximal tori.

The next result contains parts of (10.1) and (10.2), and concerns those subgroups of G containing a maximal torus of G_0. The result establishes part of the conjecture in [**24**]; the full conjecture follows from (10.1).

THEOREM. *Let $T_0 \leq Y \leq G$. Then*

(i) *The normal closure, $\langle T_0^Y \rangle$, of T in Y_0 is generated by T_0 and those T-root subgroups of G that are contained in Y.*

(ii) *If T_1 is any maximal torus of G_0 with $T_1 \leq Y$, then $\langle T_0^Y \rangle = \langle T_1^Y \rangle$.*

(iii) *T_1 can be chosen so that $Y = \langle T_1^Y \rangle N_Y(T_1)$.*

(iv) *$O_p(Y) \leq \langle T_0^Y \rangle$ and $O^{p'}(\langle T_0^Y \rangle)/O_p(Y) = E(Y/O_p(Y))$ (the product of all quasisimple subnormal subgroups of $Y/O_p(Y)$).*

In view of the above results it is clear that subgroups of G generated by T-root subgroups are of particular importance. The next result indicates such subgroups can be studied by studying $\bar{G}$ and subsets of $\bar{\Sigma}$.

THEOREM (12.9). *Let T be a maximal torus of G and $X_1,\ldots,X_k$ T-root subgroups of G corresponding to $\langle\sigma\rangle$-orbits $\Delta_1,\ldots,\Delta_k$ of $\bar{T}$-root subgroups of $\bar{G}$. Then $\langle X_1,\ldots,X_k\rangle = O^{p'}(\langle\Delta_1,\ldots,\Delta_k\rangle_\sigma)$.*

The following are applications of some of the above results. The second theorem should be compared with the main results in [**23**].

THEOREM (12.10)(ii). *Assume $\bar{G}$ is simply connected and let S be an arbitrary set of p'-elements of G. Then $G_1 = \langle C_{G_1}(s)\colon s \in S\rangle$ if and only if $\bar{G} = \langle C_{\bar{G}}(s)\colon s \in S\rangle$.*

THEOREM (12.12). *Let T_1 be a maximal torus of G_1 and $R \le T_1$. Then $G_1 = \langle E(C_{G_1}(R_1))\colon R_1 \le R$ and R/R_1 cyclic$\rangle$.*

The paper is organized into three chapters, each containing several sections. The first chapter is the basic development of T-root subgroups. In the second chapter we begin the consideration of subgroups of G invariant under a maximal torus, although the classification of finite simple groups does not enter in. The last chapter contains the proofs of several of the main results and it is here where we apply the classification theorem.

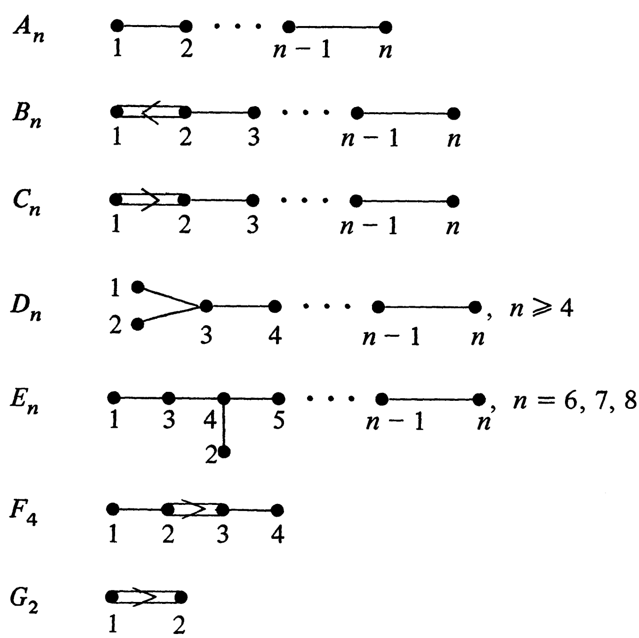
The author would like to thank R. Steinberg for communicating a proof of the main result in §5 much shorter than the original, and M. Kaneda for several helpful comments.

NOTATION. Throughout the paper $\bar{G}$ will denote a connected simple algebraic group over the algebraic closure, K, of the prime field $\mathbf{F}_p$. As before, σ is a surjective endomorphism of $\bar{G}$ with $\bar{G}_\sigma$ finite. Then $G = \bar{G}_\sigma = \bar{G}_\sigma(q)$, where q is a power of p. If $\bar{T}$ is a maximal torus of $\bar{G}$, let $\bar{U}_\alpha$ denote the $\bar{T}$-root subgroup corresponding to the root $\alpha \in \bar{\Sigma}$, the root system of $\bar{G}$. Let $\bar{W} = N_{\bar{G}}(\bar{T})/\bar{T}$, the Weyl group of $\bar{G}$.

If X is a finite group, $\mathrm{Fit}(X)$ denotes the unique largest normal nilpotent subgroup of X and $F^*(X)$ is the product of $\mathrm{Fit}(X)$ and $E(X)$, where $E(X)$ is the (commuting) product of all subnormal quasisimple subgroups of X. $O_p(X)$ denotes the largest normal p-subgroup of X and $O^{p'}(X)$ is the normal subgroup of X generated by all p-elements of X. If d

is a positive integer, then let $\Phi_d(x)$ be the corresponding cyclotomic polynomial of degree $\varphi(d)$. Some additional notation is given at the beginning of §§2, 3, 5, 9, and 10.

We label Dynkin diagrams as follows



I. T-Root subgroups

2. Preliminaries. In this section we establish a number of basic results concerning maximal tori. In addition there are results on subgroups of algebraic groups generated by root subgroups and a somewhat curious number theoretical result.

The group $\bar{G}$ is as in §1 with root system $\bar{\Sigma}$ and Weyl group $\bar{W}$. We assume that $\bar{\Sigma}$ is indecomposable, so that $\bar{G}$ can be regarded as a Chevalley group over K. σ is a surjective endomorphism of $\bar{G}$ and $G = \bar{G}_\sigma$ is finite. Then G is of Lie type and associated with a field $\mathbf{F}_q$ of characteristic p. The number q will be specified below; in nearly every case it is the order of the center of a root subgroup of G for a long root. Write $G = G(q)$. Usually we will regard σ as an element of the semidirect product $\bar{G}\langle\sigma\rangle$; hence σ acts on $\bar{G}$ by conjugation.

By (10.10) of **[26]** we may choose a σ-stable maximal torus, $\bar{H}$, of $\bar{G}$ contained in a σ-stable Borel subgroup of G. Let $\bar{T}$ be a fixed σ-stable maximal torus. Then $\bar{T} = \bar{H}^g$ for some $g \in \bar{G}$. Therefore, $\bar{H}^{g\sigma} = \bar{H}^g$ so

$\bar{H}^{g\sigma g^{-1}} = \bar{H}$, $g\sigma g^{-1}\sigma^{-1} \in N_{\bar{G}}(\bar{H})$, and we write $g\sigma = n\sigma g$ for $n \in N$. This shows that the diagram

$$\begin{array}{ccc} \bar{H} & \overset{n\sigma}{\rightarrow} & \bar{H} \\ g\downarrow & & \downarrow g \\ \bar{T} & \underset{\sigma}{\rightarrow} & \bar{T} \end{array}$$

commutes. Hence we will identify the action of σ on $\bar{T}$ and on the character group $X = X(\bar{T})$ with the action of $n\sigma$ on $\bar{H}$ and on $X(\bar{H})$, the identification being made via conjugation by g. Now, n induces an element $w \in \bar{W}$ on $X(\bar{H})$ and, except for the Ree and Suzuki groups, σ induces $q\gamma$ on $X(\bar{H})$, where γ is a graph automorphism of $\bar{\Sigma}$. If G is a Ree or Suzuki group, then setting $q_1 = \sqrt{q}$, σ induces $q_1\gamma$ on $\mathbf{R} \otimes X(\bar{H})$ and γ is an isometry (which interchanges long and short roots). So σ induces $q\tau$ or $q_1\tau$ on $\mathbf{R} \otimes X(\bar{H})$, where $\tau = w\gamma$ is an isometry of $\mathbf{R} \otimes X(\bar{H})$ of finite order. We now carry this over to $\bar{T}$ and X, regarding $w \in \bar{W} \cong N_{\bar{G}}(\bar{T})/\bar{T}$ and w, γ acting on X. We then have

(2.1)(i) If G is not a Suzuki or Ree group, then σ induces $q\tau = qw\gamma$ on X.

(ii) If G is a Suzuki or Ree group, then σ acts on X and induces $q_1\tau = q_1w\gamma$ on $\mathbf{R} \otimes X$.

(iii) $\bar{T}_\sigma \cong X/X(\sigma - 1)$.

(iv) $|\bar{T}_\sigma| = |f(q)|$ ($|f(q_1)|$ in the Suzuki or Ree groups), where $f(x)$ is the characteristic polynomial of $w\gamma$ on $\mathbf{R} \otimes X$.

Proof. This follows from the above identification and (1.7) of **[25]**.

The following lemma explains (2.1)(iv) and can be used to obtain the structure of $T = \bar{T}_\sigma$ in certain cases.

(2.2) Let Y be a free Z-module and θ an endomorphism of Y. Suppose that $\mathbf{R} \otimes Y$ is a Euclidean space and θ induces $q_1\varphi$ on $\mathbf{R} \otimes Y$, where φ is an isometry of finite order and $|q_1| \geq 1$. If $q_1 = \pm 1$, assume that $C_Y(\theta) = 0$. Then

(i) $\operatorname{rank}_Z(Y) = \operatorname{rank}_Z(Y_0)$, where $Y_0 = Y(\theta - 1)$.

(ii) $|Y/Y_0| = |f(q_1)|$, where $f(x)$ is the characteristic polynomial of φ on $\mathbf{R} \otimes Y$.

(iii) Suppose that q_1 is an integer, $(Y)\varphi = Y$ and Y has a free basis in which the matrix of φ is in rational form. Then $Y/Y_0 \cong Z_{f_1(q_1)} \times \cdots \times Z_{f_k(q_1)}$, where $f_1(x) | \cdots | f_k(x)$ are the invariant factors of φ on $\mathbf{R} \otimes Y$.

Proof. Let $V = \mathbf{R} \otimes Y$. If $q_1 = \pm 1$, then we are assuming that $C_Y(\theta) = 0$. If $|q_1| > 1$, then use the fact that φ has finite order to conclude $C_V(\theta) = 0$. In either case $C_V(\theta) = 0$ and $\theta - 1$ is injective. This proves (i).

For (ii), choose a basis $\{v_1, \ldots, v_n\}$ of Y and positive integers $z_1, \ldots, z_n$ such that $z_1 | \cdots | z_n$ and $\{z_1 v_1, \ldots, z_n v_n\}$ is a Z-basis of $Y_0 = Y(\theta - 1)$. For $i = 1, \ldots, n$ let $(v_i)(\theta - 1) = \sum a_{ij} z_j v_j$. There is an integral matrix (b_{ij}) such that

$$(b_{ij})(a_{ij})\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}.$$

Then $(b_{ij})(a_{ij}) = 1$, $\det(b_{ij}) = \pm 1$, and $\det(\theta - 1) = \pm z_1 \cdots z_n$. Passing to V, we have $z_1 \cdots z_n = |\det(\theta - 1)| = |\det(q_1\varphi - 1)|$. Since φ is an isometry of V, φ and φ^{-1} have the same eigenvalues with equal multiplicities, hence φ and φ^{-1} have the same characteristic polynomial, say $f(x)$. Since $\det(\varphi) = \det(\varphi^{-1}) = \pm 1$, we have $|\det(q_1\varphi - 1)| = |\det(q_1 - \varphi^{-1})| = |f(q_1)|$, proving (ii).

For (iii), suppose $Y = Y_1 \oplus \cdots \oplus Y_k$, where $Y_i = (Y_i)\varphi = \langle \beta_i \rangle$ and β_i is a Z-basis of Y_i in which the matrix of φ is the companion matrix of $f_i(x)$. Here, $f_1(x) | \cdots | f_k(x)$ are the invariant factors of φ. Fix $1 \leq i \leq k$ and let $\beta_i = \{\beta_1, \ldots, \beta_l\}$. Write $f_i(x) = a_1 + a_2 x + \cdots + a_l x^{l-1} + x^l$. Then $Y_i = \langle \beta_l \rangle \oplus \langle q_1\beta_l - \beta_{l-1} \rangle \oplus \cdots \oplus \langle q_1\beta_2 - \beta_1 \rangle$. Also, $q_1\beta_i - \beta_{i-1} = \beta_{i-1}(q_1\varphi - 1)$ for $i = 1, \ldots, l - 1$. Thus,

$$Y_i(q_1\varphi - 1) = \langle q_1\beta_2 - \beta_1, \cdots, q_1\beta_l - \beta_{l-1} \rangle + \langle \beta_l(q_1\varphi - 1) \rangle.$$

Now,

$$\begin{aligned} \beta_l(q_1\varphi - 1) &= -a_1 q_1 \beta_1 - \cdots - a_{l-1} q_1 \beta_{l-1} - (a_l q_1 + 1)\beta_l \\ &= a_1 q_1 (q_1\beta_2 - \beta_1) + (a_1 q_1^2 + a_2 q_1)(q_1\beta_3 - \beta_2) \\ &\quad + \cdots + (-1)(a_1 q_1^l + \cdots + a_l q_1 + 1)\beta_l. \end{aligned}$$

Therefore, $Y_i / Y_i(q_1\varphi - 1) \cong \langle \beta_l \rangle / \langle g(q_1)\beta_l \rangle$, where $g(x) = a_1 x^l + \cdots + a_l x + 1$. Now $g(x) = x^l f_i(1/x)$. Since $|\varphi| < \infty$, $f_i(x)$ is a product of cyclotomic polynomials, hence the roots of $f_i(x)$ and $g(x)$ are equal and also $a_1 = \pm 1$. Thus, $g(q_1) = \pm f_i(q_1)$ and $Y_i / Y_i(q_1\varphi - 1) \cong Z_{f_i(q_1)}$. From here (iii) is immediate.

(2.3) Let π: $\tilde{G} \to \bar{G}$ be the natural surjection, where $\tilde{G}$ is the universal covering group of $\bar{G}$, and let $\tilde{T}$ be the preimage of $\bar{T}$. Then σ can be viewed as an endomorphism of $\tilde{G}$. Also,

(i) $(\tilde{G}_\sigma)\pi = O^{p'}(\bar{G}_\sigma) = O^{p'}(G)$.

(ii) $\bar{T}_\sigma/(\tilde{T}_\sigma)\pi \cong G/O^{p'}(G)$.

Proof. The first fact is standard (see (12.6) of [**26**]). The second assertion is proved as in (2.12) of [**23**] (or see 5.10.1 in [**10**]).

(2.4) Let $\tilde{G}$ be as in (2.3), $|Z(\tilde{G}_\sigma)| = d$ and $|Z(O^{p'}(\bar{G}_\sigma))| = d_1$. Then

(i) $|\bar{G}_\sigma : O^{p'}(\bar{G}_\sigma)| = d/d_1$.

(ii) If $T_0 = T \cap O^{p'}(\bar{G}_\sigma)$, then $|T_0| = (d_1/d)|f(q)|$ ($f(q_1)$ in the Suzuki and Ree groups), where $f(x)$ is as in (2.1)(iv).

(iii) If $\bar{G}$ has Lie rank r, then

$$(q-1)^r \leq |f(q)| \leq (q+1)^r$$

$$\left((q_1-1)^r \leq |f(q_1)| \leq (q_1+1)^r \text{ for Suzuki and Ree groups}\right).$$

Proof. With notation as in (2.3), $(Z(\tilde{G}_\sigma))\pi = Z(O^{p'}(\bar{G}_\sigma))$. By (2.1)(iv) $|\tilde{T}_\sigma| = |\bar{T}_\sigma|$, so $|f(q)| = |\bar{T}_\sigma| = (d/d_1)|(\tilde{T}_\sigma)\pi| = (d/d_1)|T_0|$ (similar equations in the Suzuki and Ree cases). Then (i) and (ii) follow. For (iii) use the triangle inequality and the fact that the roots of $f(x)$ are roots of unity.

(2.5) Let $\bar{D} = \bar{D}^\sigma$ be a closed connected subgroup of $\bar{G}$ with $\bar{T} \leq \bar{D}$.

(i) $\bar{D} = R_u(\bar{D})\bar{L}$, where $\bar{L} = \bar{L}^\sigma$ is reductive and $\bar{T} \leq \bar{L}$.

(ii) $\bar{L} = \bar{L}'\bar{T}$ and $\bar{L}' = [\bar{L}, \bar{L}]$ is semisimple.

(iii) $R_u(\bar{D})$ is a product of $\bar{T}$-root subgroups of $\bar{G}$ and $\bar{L}'$ is generated by $\bar{T}$-root subgroups, corresponding to a subsystem of $\bar{\Sigma}$.

(iv) $\bar{D}_\sigma = O^{p'}(\bar{D}_\sigma)\bar{T}_\sigma = R_u(\bar{D})_\sigma O^{p'}(\bar{L}'_\sigma)\bar{T}_\sigma$.

(v) $O^{p'}(\bar{L}'_\sigma)$ is a commuting product of groups of Lie type and $\bar{T}$ contains a maximal torus of each factor.

Proof. Set $\bar{Q} = R_u(\bar{D})$ and let $\bar{A}$ be a Borel subgroup of $\bar{D}$ with $\bar{A} \geq \bar{T}$. Embedding $\bar{A}$ in a Borel subgroup of $\bar{G}$ we see that $R_u(\bar{A})$ and $\bar{Q}$ are both products of $\bar{T}$-root subgroups of $\bar{G}$ (one can modify the argument of Lemma 3 of [**22**] to establish this). Let Δ_1 be those roots $\alpha \in \bar{\Sigma}$ with $\bar{U}_\alpha \leq \bar{Q}$ and let Δ be all roots $\alpha \in \bar{\Sigma}$ such that $\bar{U}_\alpha \leq \bar{A}$ and $\bar{U}_{-\alpha} \leq \bar{D}$. We then have $\bar{Q} = \Pi_{\alpha \in \Delta_1}\bar{U}_\alpha$ and $(\bar{D}/\bar{Q})' = \langle \bar{U}_{\pm\alpha} \mid \alpha \in \Delta \rangle \bar{Q}/\bar{Q}$ (from the structure theory of reductive groups).

Let $\bar{E} = \langle \bar{U}_{\pm\alpha} \mid \alpha \in \Delta \rangle$. From the Bruhat decomposition and the fact that Δ is a subsystem of $\bar{\Sigma}$ we conclude that $\bar{L} = \bar{E}\bar{T}$ is reductive, and then $\bar{E}$ is semisimple. Then $\bar{L}' = \bar{E}$, and since $\bar{D}/\bar{Q} = \bar{E}\bar{T}\bar{Q}/\bar{Q}$, we have established (i), (ii), and (iii).

Since $\bar{Q}$ is connected, Lang's theorem implies that $(\bar{D}/\bar{Q})_\sigma = \bar{D}_\sigma\bar{Q}/\bar{Q}$. As $\bar{D}$ is the semidirect product of $\bar{Q}$ and $\bar{L}$, (iv) will be proved once we know that $\bar{L}_\sigma = O^{p'}(\bar{L}'_\sigma)\bar{T}_\sigma$. We first note that $\bar{L}_\sigma = \bar{E}_\sigma\bar{T}_\sigma$. To see this let $\bar{J} = \bar{E} \cap \bar{T}$, a maximal torus of $\bar{E}$, normalized by σ. Suppose $e \in \bar{E}$, $t \in \bar{T}$ and $(et)^\sigma = et$. Since $\bar{E}$ and $\bar{T}$ are both σ-stable, we have $e^\sigma = ej$ and $t^\sigma = tj^{-1}$ for some $j \in \bar{J}$. From Lang's theorem ((10.1) of **[26]**) there is an element $j_1 \in \bar{J}$ with $j_1^\sigma = j_1 j^{-1}$. Then $e = (ej_1)(j_1^{-1}j)$ represents e as an element in $\bar{E}_\sigma\bar{T}_\sigma$. The proof of (iv) has now been reduced to the semisimple group $\bar{E}$, where the result follows from (2.12) of **[23]** (that result concerned a simple adjoint group, but these conditions were never used).

For (v), apply (11.7) of **[26]** to get the structure of $O^{p'}(\bar{L}'_\sigma)$. The remaining part of (v) is obtained by considering orbits of $\langle\sigma\rangle$ on the simple factors of $\bar{L}'$.

(2.6) Let $T_0 = T \cap O^{p'}(G)$ and assume $q \geq 4$. Then $C_{\bar{G}}(T_0)^0 = \bar{T}$.

Proof. From the Bruhat decomposition of $\bar{G}$ (with respect to a Borel subgroup $\bar{B} \geq \bar{T}$) we see that the result holds unless $T_0 \leq C(\bar{U}_\alpha)$ for some root subgroup $\bar{U}_\alpha$ of $\bar{G}$. Let $\Delta = \{\alpha \in \Sigma \mid [T_0, \bar{U}_\alpha] = 1\}$. Then Δ is closed under taking negatives and we set $D = \langle \bar{U}_\alpha \mid \alpha \in \Delta\rangle$.

We have $D^\sigma = D$ and by (2.5) $D = D_1 \cdots D_l$, a commuting product of quasisimple groups D_i, where each D_i is generated by certain of the root subgroups, $\bar{U}_\alpha$, for $\alpha \in \Delta$. Each D_i is a Chevalley group with indecomposable root system. Reorder so that $\{D_1, \ldots, D_k\}$ is a $\langle\sigma\rangle$-orbit. Then σ^k normalizes each D_i, $1 \leq i \leq k$, and $(D_i)_{\sigma^k} = D_i(q^k)$, a group of Lie type associated with $\mathbf{F}_{q^k}$ (see (11.6) of **[26]** and the proof of (2.6) of **[23]**). Also, $(D_1 \cdots D_k)_\sigma$ is obtained as a diagonal copy of $D_1(q^k)$ (except for amalgamation of centers). Let $T_1 = \bar{T} \cap O^{p'}((D_1 \cdots D_k)_\sigma)$. Then $T_1 \leq T_0 \leq C(\bar{U}_\alpha)$ for each $\alpha \in \Delta$, and projecting to D_1, we see that $T_2 = \bar{T} \cap O^{p'}((D_1)_{\sigma^k}) \leq Z(O^{p'}((D_1)_{\sigma^k}))$.

By (2.4)(ii) and (2.4)(iii) $|T_2| = (e_1/e)|f(q^k)| \geq (e_1/e)(q^k - 1)^r$ (replace q by q_1 if $(D_1)_{\sigma^k}$ is a Suzuki or Ree group), where $e_1 = |Z(O^{p'}((D_1)_{\sigma^k}))|$, e is the order of the center of the universal covering group of $(D_1)_{\sigma^k}$, and r is the Lie rank of D_1. But $|T_2| \leq e_1$ (which is 1 in the Suzuki and Ree cases), whereas $q^k - 1 \geq 3$ ($\sqrt{3}$ in the Suzuki and Ree cases). Then $3^r \leq e$ (or $(\sqrt{3})^r \leq 1$), a contradiction.

An immediate consequence of (2.6) is

(2.7) Let $q \geq 4$ and set $T_0 = T \cap O^{p'}(G)$. Then $N_G(T_0) \leq N_{\bar{G}}(\bar{T})$.

For $q > 5$, we also have control over $C_G(T_0)$.

(2.8) Let $q > 5$ and set $G_0 = O^{p'}(G)$, $T_0 = T \cap G_0$. Assume that $\bar{G}$ is adjoint. Then

(i) $C_G(T_0) = T$.
(ii) If $G_0 \neq \mathrm{PSL}(2,9)$, $\mathrm{Sz}(8)$, or ${}^2F_4(8)$, then $C_{\mathrm{Aut}(G_0)}(T_0) = T$.

Proof. $\bar{G}$ is adjoint, so $Z(\bar{G}) = Z(G_0) = 1$. Let $a \in \mathrm{Aut}(G_0)$ and suppose $a \in C(T_0) - T$. Replacing a by a power of a we may assume that $a^s \in T$ for some prime s. Extend a to an endomorphism of $\bar{G}$ commuting with σ. By (2.6) $\bar{T}$ is a-invariant, so a acts on X. $C_X(a)$ is G-invariant and both $C_X(a)$ and $\tilde{X} = X/C_X(a)$ are free Z-modules.

Let X_0 denote the annihilator in X of T_0. Since $T_0 \leq C_{\bar{T}}(\sigma) \cap C_{\bar{T}}(a)$ we have $[X, a] \leq X_0 \geq [X, \sigma]$. Also $|X_0 : [X, \sigma]| = |T : T_0| = d$, where $d = |G : G_0|$ (see (2.3)ii). Both $[X, a]/([X, a] \cap [X, \sigma])$ and $[\tilde{X}, a]/([\tilde{X}, a] \cap [\tilde{X}, \sigma])$ are isomorphic to sections of $X_0/[X, \sigma]$. In particular, each has order a divisor of d and exponent dividing that of $X_0/[X, \sigma]$.

Write $A = \mathrm{Aut}(G)$ and let A_0 be the subgroup of A generated by all inner, diagonal, and graph automorphisms of G_0. The elements of A_0 are precisely the automorphisms of G_0 that can be extended to automorphisms of the algebraic group $\bar{G}$ (elements of $A - A_0$ can be extended to surjective endomorphisms of $\bar{G}$).

Let $\bar{Y}$ be the subgroup of $\mathrm{Aut}(\bar{G})$ normalizing $\bar{T}$ and let F be the Frobenius morphism with respect to $\bar{T}$. Then σ and F commute in their action on X, so σF and $F\sigma$ differ in their action on $\bar{G}$ by an inner automorphism induced from an element of $\bar{T}$. Using Lang's theorem, we modify F so that it commutes with σ. For convenience we postpone discussion of the cases where $\bar{G}$ has type C_2, G_2, F_4 and $p = 2, 3, 2$, respectively. Then there is a power n such that $\sigma \in \bar{Y}F^n$. Consequently, $q = p^n$ and σ induces tp^n on X, where t is an isometry of $\mathbf{R} \otimes X$. Similarly, there is an isometry ε and power p^m, $m \geq 0$, such that a induces εp^m on X. Let $f(x)$, $\tilde{f}(x)$ be the characteristic polynomials of t on $\mathbf{R} \otimes X$, $\mathbf{R} \otimes \tilde{X}$, respectively. Similarly, let $g(x)$, $\tilde{g}(x)$ be the corresponding polynomials of ε.

Assume $a \notin A_0$. The group A/A_0 is cyclic of order n and generated by γA_0, where $\gamma = F|_{G_0}$. Replacing a by a suitable power, we may assume that $a \in A_0 \cdot \gamma^{n/s}$. That is, $a \in \bar{Y} \cdot F^{n/s}$ and so a induces $\varepsilon p^{n/s}$ on X. By (2.2)(ii) $|f(p^n)| = |X : [X, \sigma]|$ which divides $|X : [X, \sigma] \cap [X, a]|$. Another application of (2.2)(ii) and previous remarks show that the latter number divides $d|g(p^{n/s})|$. So (2.4)(iii) yields the inequality $(p^n - 1)^r \leq d(p^{n/s} + 1)^r$. Using this together with the inequality $d \leq r + 1$ we calculate and obtain $r = 1$, $p^n = 9$, $s = 2$. But then $G_0 \cong \mathrm{PSL}(2,9)$, which is excluded in (ii). So $a \in A_0$ and a induces ε on X.

One checks that $C_{\tilde{X}}(a) = 0$, so by (2.2)(ii) we argue as above that $|\tilde{f}(q)| \leq d_0 |\tilde{g}(1)| \leq d |\tilde{g}(1)|$, where d_0 is the order of

$$[\tilde{X}, a] / ([\tilde{X}, a] \cap [\tilde{X}, \sigma]) = D.$$

Then (2.4)(iii) yields $(q-1)^{\tilde{r}} \leq d_0 2^{\tilde{r}} \leq d 2^{\tilde{r}}$. Since $d \leq q+1$, we are led to $\tilde{r} = 1, \tilde{f}(x) = x \pm 1, \tilde{g}(x) = x + 1$, and $6 \leq q - 1 \leq 2d_0 \leq 2d$.

If $\bar{G} = D_r$, then T/T_0 is a subgroup of $Z_2 \times Z_2$,* so D is isomorphic to a cyclic (as $\tilde{r} = 1$) subgroup of $X_0/[X, \sigma] \cong T/T_0$. Hence $d_0 \leq 2$, against the above inequality. Suppose a is an inner automorphism. Then a centralizes T/T_0 and hence a centralizes D. However, a inverts $\tilde{X}$ (since $\tilde{g}(x) = x + 1$) and $\tilde{X}$ is cyclic. We conclude $|D| \leq 2$, so again $d_0 \leq 2$, giving a contradiction. At this point we have a in the coset of an involutory graph automorphism of $\bar{G}$, and the inequality of the previous paragraph shows that $\bar{G}$ is either of type E_6 (and $q = 7$) or of type A_r. Also, (i) has been proved (except for the excluded possibilities of $\bar{G}$).

Write $\bar{W} = \bar{N}/\bar{T}$ and consider the action of a on $\bar{W}$. Then $\bar{N}\langle a \rangle = \bar{N}\langle b \rangle$, where $\bar{N}\langle b \rangle / \bar{T} = \bar{W} \times \langle b\bar{T} \rangle$, b sends each root to its negative and inverts X. a acts as $w_1 b$ for $w_1 \in \bar{W}$, and the eigenspace of w_1 for eigenvalue -1 has codimension 1 in $\mathbf{R} \otimes X$. If $\bar{W} \cong S_{r+1}$, use the cycle decomposition of w_1 to see that no such involution exists for $r > 3$. If $r = 2$, then $G_0 \cong \mathrm{PSL}(3,7)$, while if $r = 3$, then $G_0 \cong \mathrm{PSL}(4,9)$ or $\mathrm{PSU}(4,7)$. If $G_0 = \mathrm{PSL}(3,7)$ or $\mathrm{PSL}(4,9)$ we can take w_1 to be s_{α_1}, $s_{\alpha_1}s_{\alpha_3}$, respectively. If $G_0 = \mathrm{PSU}(4,7)$, take w_1 to be $s_1 s_3$, $(s_1 s_3 s_2)^2$, or $s_2 s_1 s_3 s_2$. In all cases we can explicitly compute $[X, a]$, $[X, \sigma]$, and contradict the earlier observation that $[X, a]/[X, a] \cap [X, \sigma]$ has order dividing d.

Suppose $\bar{W} \cong E_6$. Then w_1 has determinant -1. Now $\bar{W} \cong \mathrm{Aut}(\mathrm{PSU}(4,2))$ and by (19.5) of [1] $\bar{W} - \bar{W}'$ has two classes of involutions, represented in $\bar{W}$ by reflections and the product of 3 commuting reflections. This contradicts the condition on eigenvalues.

At this point we have proved the lemma for all cases except $\bar{G}$ of type C_2, G_2, F_4 and $p = 2, 3, 2$, respectively. We indicate the necessary adjustments in the previous arguments. First note that $d = 1$ in all cases. As in (2.1) σ acts on $\mathbf{R} \otimes X$ as $t\hat{q}$, with t an isometry and $\hat{q} = q$ or $\sqrt{q}$. If $a \in A_0$, proceed as before to get the inequality $|\tilde{f}(\hat{q})| \leq d |\tilde{g}(1)| = |\tilde{g}(1)|$. By (2.4) we then obtain $(\hat{q} - 1)^r \leq 2^r$, a contradiction. So $a \notin A_0$.

Let $\hat{F}$ be the endomorphism of $\bar{G}$ such that $\hat{F}^2 = F$ is the Frobenius map (with respect to $\bar{T}$). If $\hat{q} = \sqrt{q} = p^{l+1/2}$, then A/A_0 is cyclic of order $2l + 1$, with quotient generated by a field automorphism. Since $F|_{G_0}$ induces a field automorphism of order $2l + 1$, we argue as before that a

Added in proof. If r is odd, T/T_0 could be Z_4 with $q = 7, 9$. In this case multiplying a by an element of T one can assume $a \in \mathrm{PO}^{\pm}(2r, q)$ and argue within the linear group.

can be taken in $\bar{Y}F^{n/s}$. Now argue as before to get a contradiction (recall that $G_0 \neq \mathrm{Sz}(8)$, ${}^2F_4(8)$). Finally, assume $\hat{q} = q = p^n$. Here $\gamma = \hat{F}|_{G_0}$ has order $2n$ and $A/A_0 = \langle A_0\gamma \rangle$. We may then assume that a induces $\varepsilon p^{n/2s}$ on X, for ε an isometry ($n/2s$ need not be an integer). The usual inequalities give a contradiction and complete the proof of (2.8).

(2.9) Let $x \in T$. Then $C_G(x)$ contains normal subgroups Y_0 and Y such that

(i) $Y = Y_0T$.

(ii) $Y_0 = D_1 \cdots D_kX$, a commuting product, where for each $i = 1, \ldots, k$ there exists a power, q^{l_i}, of q such that $D_i = D_i(q^{l_i})$ is a quasisimple group of Lie type defined over $\mathbf{F}_{q^{l_i}}$. Also $Z(Y_0) \geq X \leq T$.

(iii) If each $q^{l_i} \geq 4$, then $D_1, \ldots, D_k$ are the components of $E(C_G(x))$ and $X = C_Y(E(C_G(x))) \leq T$.

(iv) $C_G(x)/Y$ is isomorphic to a subgroup of the center of the universal covering group of G_0.

Proof. Since $x \in T \leq \bar{T}$, $C_{\bar{G}}(x)$ can be computed from the Bruhat decomposition of $\bar{G}$ (with respect to the root subgroups of $\bar{T}$). We have $\bar{T} \leq C_{\bar{G}}(x)^0$ and $C_{\bar{G}}(x)^0 = Y_1 \cdots Y_lZ$, where the product is a commuting product, each Y_i is a Chevalley group defined with respect to an indecomposable subsystem of $\bar{\Sigma}$, and $Z \leq \bar{T}$. Let $Y = (C_{\bar{G}}(x)^0)_\sigma$. Since $Y_1 \cdots Y_l$ is connected, Lang's theorem (see (10.1) of [**26**]) implies that $Y = (Y_1 \cdots Y_l)_\sigma \bar{T}_\sigma$.

Let $R = Y_1 \cdots Y_l$. The argument in (2.13) of [**23**] shows that $R_\sigma = O^{p'}(R_\sigma)(\bar{T} \cap R)_\sigma$. Moreover, the proof of (2.6)(ii) of [**23**] shows that $O^{p'}(R_\sigma) = D_1 \cdots D_k$ has the required structure. So setting $Y_0 = O^{p'}(R_\sigma)Z_\sigma$, we have (i), (ii), and (iii) holding. For (iv) see (4.4) of Springer-Steinberg [**25**].

The following number theoretical result will be needed in §7.

(2.10) Let $p > 3$ be prime, $x = \prod_{i=1}^m \Phi_{d_i}(p)$, and $y = \prod_{i=1}^n \Phi_{f_j}(p)$. Suppose that

(a) $x \mid y$;

(b) $d_1 < \cdots < d_m$; and

(c) $\Sigma\, \varphi(d_i) \geq \Sigma\, \varphi(f_j)$.

Then $m = n$ and $\{d_i \mid 1 \leq i \leq m\} = \{f_j \mid 1 \leq j \leq n\}$. In particular, $x = y$.

Proof. Suppose false. Factoring out common factors we may assume $d_i \neq f_j$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Suppose it is not the case that $d_t = 2$ and p a Mersenne prime. Then by Zsigmondy [**28**], for each $1 \leq i \leq m$ there is a prime r_i such that $r_i \mid p^{d_i} - 1$, but $r_i \nmid p^d - 1$ for $d < d_i$. We call these primes primitive divisors. In the exceptional case,

$\Phi_{d_i}(p) = p + 1$ is a power of 2 and we set $r_i = 2$. If this case occurs and $d_{i'} = 1$ for some i', then $d_{i'} = d_1$ and $\Phi_{d_{i'}}(p)$ is divisible by some odd prime, which we may take to be r_1. Therefore, $r_i \neq r_{i'}$ for $i \neq i'$. Choose $r_1 \neq 2$, if possible.

In the primitive divisor situation, d_i is the order of p modulo r_i. So $d_i \mid r_i - 1$ and $\varphi(d_i) \leq d_i \leq r_i - 1$. Fix $1 \leq i \leq m$. There exists $j \in \{1,\ldots,n\}$ such that $r_i \mid \Phi_{f_j}(p)$. Then $d_i \mid f_j$. Set $g(t) = (t^{f_j} - 1)/(t^{d_i} - 1)$ and expand $g(t)$ in powers of t^{d_i}. Letting $t = p$ and using the congruence $p^{d_i} \equiv 1 \pmod{r_i}$ we have $r_i \mid (f_j/d_i) \mid f_j$.

We estimate $\Sigma\, \varphi(f_j)$ as follows. If $1 \leq j \leq m$ is fixed and $r_{i_1},\ldots,r_{i_k}$ are the primes satisfying $r_{i_l} \mid \Phi_{f_j}(p)$, then, by the above, $\varphi(f_j)$ is divisible by $(r_{i_1} - 1) \cdots (r_{i_k} - 1)$. If $c > 1$ and $d > 1$ are integers, then $cd \geq c + d$. Suppose that no $r_i = 2$. Using these facts we have $\Sigma(r_i - 1) \leq \Sigma\, \varphi(f_j)$. This inequality combined with our hypothesis and the remarks of the previous paragraph yield

$$(*) \qquad \sum \varphi(f_j) \leq \sum \varphi(d_i) \leq \sum d_i \leq \sum (r_i - 1) \leq \sum \varphi(f_j).$$

Therefore all inequalities are equalities. In particular, $\varphi(d_i) = d_i$ for $i = 1,\ldots,m$. But this forces $d_i = 1$, and so $1 = d_i = r_i - 1$, against our supposition. Therefore, either $d_1 = 1$ and $r_1 = 2$ (p a Fermat prime) or $d_i = 2 = r_i$, for $i = 1$ or 2 (p a Mersenne prime).

To deal with these cases we slightly modify the above argument. Say $d_i = 2 = r_i$. Choose j with $r_i \mid \Phi_{f_j}(p)$ and $f_j \neq 1$. Then $r_i \mid f_j/d_i$, so $4 \mid f_j$. If $r_{i_1},\ldots,r_{i_k}$ are the other primes among $r_1,\ldots,r_m$ that are factors of f_j, then $\varphi(f_j) \geq 2(r_{i_1} - 1) \cdots (r_{i_k} - 1)$. So we again have the inequality $\Sigma(r_i - 1) \leq \Sigma\, \varphi(f_j)$. So we will again obtain $(*)$ provided $\Sigma\, d_i \leq \Sigma(r_i - 1)$. Suppose this fails and let $1 \leq k \leq m$, $k \neq i$. Since $d_i = (r_i - 1) + 1$ and $d_k \mid r_k - 1$ we necessarily have $d_k = r_k - 1$. So d_k is even and $\varphi(d_k) \leq \frac{1}{2}d_k$. On the other hand, if we add 1 to each of the last two terms in $(*)$, the resulting inequalities hold. Thus $\varphi(d_k) \geq d_k - 1$, which is impossible. We conclude that no such k exists, $m = 1$, $d_i = d_1$, and $\Sigma\, \varphi(f_j) \leq \Sigma\, \varphi(d_i) = 1$. So $n = 1$, $f_j = 1$, and $p + 1 \mid p - 1$ (as x divides y). This is absurd. Therefore $(*)$ holds and $\Sigma\, \varphi(d_i) = \Sigma\, d_i$. This is a contradiction.

Finally, suppose $d_1 = 1$ and $r_1 = 2$. Then $d_i \mid r_i - 1$ for each $i = 1,\ldots,m$. If $(*)$ holds, then $m = 1$ and $\Sigma\, \varphi(f_j) \leq 1$. Therefore, $n = 1$, $f_1 = 2$, and $p - 1 \mid p + 1$, a contradiction. So we suppose $(*)$ to be false. Then the inequality $\Sigma(r_i - 1) \leq \Sigma\, \varphi(f_j)$ must fail to hold. Let notation be as in the previous paragraph. Then $2r_{i_1} \cdots r_{i_k}$ is a factor of f_j and $\varphi(f_j)$ is divisible by $(r_{i_1} - 1) \cdots (r_{i_k} - 1)$. So $\varphi(f_j) \geq (r_{i_1} - 1) + \cdots + (r_{i_k} - 1)$. Combining this with the other values, $\varphi(f_1),\ldots,\varphi(f_n)$, we do have

$\Sigma(r_i - 1) \leq \Sigma\, \varphi(f_j) + 1$, where the 1 corresponds to $r_1 - 1$. In view of our assumption, equality must hold. The previously used inequality, $cd \geq c + d$, is strict unlesss $c = d = 2$, and this forces $f_j = 2r_i$ for some r_i, while for $k \neq j$, $f_k = r_{i(k)}$ or $2r_{i(k)}$ with $r_{i(k)} \in \{r_2, \ldots, r_m\}$. However, we have seen that $r_{i(k)} \mid (f_k/d_{i(k)})$. So the only possibilities are $m = 1$ or $m = 2 = d_2$. That is $x = p - 1$ or $x = (p-1)(p+1)$. Considering the possibilities for y, we have a contradiction.

The next several lemmas concern subgroups of $\overline{G}$ generated by $\overline{T}$-root subgroups.

(2.11) Let $S \subset \overline{\Sigma}$, $X = \langle \overline{U}_\alpha \mid \alpha \in S \rangle$, and $\Delta = \{\delta \in \overline{\Sigma} \mid \overline{U}_\delta \leq X\}$. Suppose $\Delta \cap -\Delta = \varnothing$. Then $X = \Pi_{\delta \in \Delta} \overline{U}_\delta$ and X is unipotent.

Proof. Let S, X, Δ be as in the statement. It will be convenient to exclude the case $\overline{\Sigma}$ of type G_2. This case can be handled by a direct check. For a fixed ordering on $\overline{\Sigma}$, let $\Delta^+ = \Delta \cap \overline{\Sigma}^+$, $\Delta^- = \Delta \cap \overline{\Sigma}^-$, $X^+ = \langle \overline{U}_\alpha \mid \alpha \in \Delta^+ \rangle$, and $X^- = \langle \overline{U}_\alpha \mid \alpha \in \Delta^- \rangle$. Then $X^+ \leq \overline{U}$ and $X^- \leq \overline{U}^-$, where $\overline{U}^-$ is opposite to $\overline{U}$. We then have $X^+ = \Pi_{\delta \in \Delta^+} \overline{U}_\delta$ and $X^- = \Pi_{\delta \in \Delta^-} \overline{U}_\delta$.

If $\overline{\Sigma}$ has two root lengths, let $\overline{\Sigma}_0$ be the subsystem of long roots. Then $\overline{G}_0 = \langle \overline{U}_\alpha \mid \alpha \in \overline{\Sigma}_0 \rangle$ is proper in $\overline{G}$, so by induction (on $|\overline{\Sigma}|$) we have $\overline{X}_0 = \langle \overline{U}_\alpha \mid \alpha \in \Delta \cap \overline{\Sigma}_0 \rangle$ unipotent. So, in this case, we may use a different ordering, if necessary, so that $\Delta \cap \overline{\Sigma}_0 \subset \overline{\Sigma}^-$. That is, $X_0 \leq X^-$.

We claim that $X = X^+ X^- = X^- X^+$. It will suffice to show $X^- X^+ \subseteq X^+ X^-$. Let $\alpha \in \Delta^+$ and $\beta \in \Delta^-$. The Chevalley commutator relations give $[\overline{U}_\beta, \overline{U}_\alpha] \leq \Pi_{i,j>0} \overline{U}_{i\alpha + j\beta}$. If $\overline{U}_{i\alpha+j\beta}$ is contained in the commutator, then $i\alpha + j\beta \in \Delta$. Suppose this occurs. If $i > 1$, then since $\overline{\Sigma}$ is not of type G_2, α is short and $i\alpha + j\beta$ long. By our convention this gives $i\alpha + j\beta \in \Delta^-$ and $\overline{U}_{i\alpha+j\beta} \leq X^-$. If $i = 1$, either $U_{i\alpha+j\beta} \leq X^-$ or $\alpha + j\beta = i\alpha + j\beta \in \Delta^+$, but $\mathrm{ht}(i\alpha + j\beta) < \mathrm{ht}(\alpha)$. From these remarks we conclude that for $u_\alpha \in \overline{U}_\alpha$ and $u_\beta \in \overline{U}_\beta$, $u_\beta u_\alpha \in u_\alpha u_\beta X^-$ or $u_\alpha u_\beta \overline{U}_\gamma X^-$, with $\gamma \in \Delta^+$ and $\mathrm{ht}(\alpha) > \mathrm{ht}(\gamma)$. To prove the claim, let $\alpha \in \Delta^+$ and show $X^- \overline{U}_\alpha \subseteq X^+ X^-$ by induction on $\mathrm{ht}(\alpha)$. Therefore, $X = X^+ X^-$.

Let τ be a field automorphism of $\overline{G}$ with respect to $\overline{\Sigma}$, $\overline{T}$, $\overline{U}$, and such that $\overline{G}_\tau = G(q_0)$, where $q_0 > 4$. Repeat the above to show that $Y = \langle (\overline{U}_\alpha)_\tau \mid \alpha \in \Delta \rangle = \Pi_{\delta \in \Delta} (\overline{U}_\delta)_\tau$. Therefore, Y is a p-subgroup of $\overline{G}_\tau$ normalized by the split torus $H = \overline{T}_\tau$. By (3.12) of **[4]** we embed YH in a proper parabolic subgroup P of $\overline{G}_\tau$ such that $Y \leq O_p(P)$. Embedding YH in a Borel subgroup of P, we see that in some new ordering of $\overline{\Sigma}$, each root $\delta \in \Delta$ is positive. In particular, X is unipotent, proving the result.

(2.12) Let $S \subseteq \bar{\Sigma}$ and $X = \langle \bar{U}_\alpha \mid \alpha \in S \rangle$. Let $\Delta = \{\alpha \in \bar{\Sigma} \mid \bar{U}_\alpha \leq X\}$. Set $\Delta_1 = \{\alpha \in \Delta \mid -\alpha \notin \Delta\}$ and $\Delta_2 = \{\alpha \in \Delta \mid -\alpha \in \Delta\}$. Then

(i) $X_1 = \langle \bar{U}_\alpha \mid \alpha \in \Delta_1 \rangle = \prod_{\alpha \in \Delta_1} \bar{U}_\alpha$ is unipotent.

(ii) $X_2 = \langle \bar{U}_\alpha \mid \alpha \in \Delta_2 \rangle$ is semisimple.

(iii) $X = X_1 X_2$ with $X_1 \trianglelefteq X$.

Proof. (i) follows immediately from (2.11). Consider the set Δ_2. Let $\Delta_2^+ = \Delta_2 \cap \bar{\Sigma}^+$ and $X_2^+ = \langle \bar{U}_\alpha \mid \alpha \in \Delta_2^+ \rangle$. Then $X_2^+ \leq \bar{U}$, so X_2^+ is unipotent and $X_2^+ = \prod_{\alpha \in \Delta_2^+} \bar{U}_\alpha$. It is easy to verify that Δ_2 is a root system. It follows from the Bruhat decomposition that $X_2 \bar{T}$ is a group with a (B, N)-pair and $X_2^+ \bar{T}$ is a Borel subgroup of $X_2 \bar{T}$. This implies (ii).

For (iii) it will suffice to show $[\bar{U}_\alpha, \bar{U}_\beta] \leq X_1$, whenever $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Suppose $Y = [\bar{U}_\alpha, \bar{U}_\beta]$ and $Y \neq 1$. If α, β are long roots, then $Y = \bar{U}_{\alpha+\beta}$ and $\bar{U}_{-\alpha} = [\bar{U}_{-(\alpha+\beta)}, \bar{U}_\beta]$. So if $\alpha + \beta \in \Delta_2$ we have $\alpha \in \Delta_2$, which is not the case. Now consider the general case, but exclude $\bar{\Sigma}$ of type G_2. Then $\bar{U}_{\alpha+\beta} \leq Y$ and the only possible difficulty is when $[\bar{U}_{-(\alpha+\beta)}, \bar{U}_\beta] = 1$. This forces $\alpha + \beta$ and β to be short, K of characteristic 2 and α, β fundamental roots in a system of type B_2. But here $X \geq \langle \bar{U}_{\pm\beta}, \bar{U}_{\pm(\alpha+\beta)}, \bar{U}_\alpha \rangle$ and *a* direct shows the latter group contains $\bar{U}_{-\alpha}$. Again we have a contradiction. Similar arguments work if $\bar{\Sigma}$ has type G_2, and we omit the details.

(2.13) Let $\delta_1, \ldots, \delta_k \in \bar{\Sigma}^+$ and assume that for each $i \neq j$, $(Z\delta_i + Z\delta_j) \cap \bar{\Sigma}$ is a root system with $\{\delta_i, \delta_j\}$ as a fundamental set of roots. Assume that the corresponding graph (with vertices $\delta_1, \ldots, \delta_k$) is a Dynkin diagram. Then $\bar{X} = \langle \bar{U}_{\pm\delta_1}, \ldots, \bar{U}_{\pm\delta_k} \rangle$ is a Chevalley group associated with the same Dynkin diagram. Moreover, $\{\delta_1, \ldots, \delta_k\}$ is a fundamental system for the root system of $\bar{X}$.

Proof. For each $1 \leq i \leq k$ let t_i denote the reflection associated with δ_i, and let $W_0 = \langle t_1, \ldots, t_k \rangle$. The roots $\delta_1, \ldots, \delta_k$ are pairwise obtuse, so a standard argument implies that they are linearly independent. Comparing the action of W_0 on the Z-span of $\{\delta_1, \ldots, \delta_k\}$ with the action of the appropriate Weyl group on the underlying lattice we see that $\Delta = \{\delta_i^{W_0} : i = 1, \ldots, k\}$ is a root system with $\delta_1, \ldots, \delta_k$ as fundamental system.

Since each t_i can be realized by conjugation by an element of $\langle U_{\pm\alpha_i} \rangle$, we have $\bar{X} = \langle \bar{U}_{\pm\delta} : \delta \in \Delta \rangle$. We claim that Δ is a closed system of roots. We have Δ locally closed in the sense that the root subsystem of Δ spanned by $\pm\delta_i$, $\pm\delta_j$ is closed in $\bar{\Sigma}$ for all i, j. On the other hand, any pair of roots in Δ can be conjugated by an element of W_0 into such a local system. This proves the claim. At this stage the result follows from the Bruhat decomposition and the classification of reductive groups.

In view of the claim we see that Γ is a root system and $\{\delta_1, \ldots, \delta_k\}$ a fundamental set for Γ. This completes the proof.

The following lemma shows that the maximal tori of G are defined unambiguously and will be used implicitly throughout the paper.

(2.14) Let L be of Lie type over $\mathbf{F}_q$. Suppose $\bar{L}_1$, $\bar{L}_2$ are semisimple algebraic groups over $\bar{\mathbf{F}}_q$ and τ_1, τ_2 are surjective endomorphisms of $\bar{L}_1$, $\bar{L}_2$, respectively, such that $L = O^{p'}((\bar{L}_i)_{\tau_i})$ for $i = 1, 2$. For $i = 1, 2$ let J_i denote the set of maximal tori of L defined with respect to τ_i-invariant maximal tori of $\bar{L}_i$. Then $J_1 = J_2$.

Proof. Write $\bar{L}_1 = \bar{L}_{i1} \cdots \bar{L}_{in_i}$, where the product is a commuting product of simple algebraic groups. Then $\langle \tau_i \rangle$ acts transitively on $\{\bar{L}_{i1}, \ldots, \bar{L}_{in_i}\}$ for $i = 1, 2$. Let $\bar{X}_i = \langle ll^{\tau_i} \cdots l^{\tau_i^{n_i - 1}} : l \in \bar{L}_{i1} \rangle$. Then $\bar{X}_1$, $\bar{X}_2$ are $\tau_i^{n_i}$-invariant images of $\bar{L}_{11}$, $\bar{L}_{21}$, respectively, so are connected simple algebraic groups. Moreover, $L \leq \bar{X}_i Z(\bar{L}_i)$ for $i = 1, 2$. On the other hand $L = O^{p'}(L)$, so $L \leq \bar{X}_i$ for $i = 1, 2$ and we may now replace $\bar{L}_1$, $\bar{L}_2$ by $\bar{X}_1$, $\bar{X}_2$.

Next argue that we may assume $\bar{X}_1$, $\bar{X}_2$ are simply connected. It then follows that there is a surjective endomorphism γ from $\bar{X}_1$ to $\bar{X}_2$ satisfying $\tau_1 \gamma = \gamma \tau_2$. Then γ induces a bijection between the set of τ_1-invariant maximal tori of $\bar{X}_1$ and the set of τ_2-invariant maximal tori of $\bar{X}_2$. It follows that $\gamma|_L$ is an isomorphism with $J_1^\gamma = J_2$. On the other hand, any isomorphism of L can be lifted to an endomorphism of $\bar{L}_1$ commuting with τ_1. It follows that $J_1^\gamma = J_1$, proving the result.

(2.15) Let A be an abelian p'-group acting on a commuting product $Y_1 \cdots Y_k$, where for $1 \leq i \leq k$, Y_i is a group of Lie type over a field of order p^{e_i}. Assume that $p \geq 5$. Then A normalizes a maximal torus of $Y_1 \cdots Y_k$ (the product of maximal tori, one from each Y_i).

Proof. Argue by induction on $|Y_1 \cdots Y_k||A|$. Clearly we may assume that A is transitive on $\{Y_1, \ldots, Y_k\}$ and that $Z(Y_1 \cdots Y_k) = 1$. If $A_1 = N_A(Y_1)$ and $A_1 < A$, then $A_1 = N_A(Y_i)$ for $1 \leq i \leq k$. Inductively, there is a maximal torus T_1 of Y_1 with $T_1^{A_1} = T_1$. Then A normalizes $T = \langle T_1^a : a \in A \rangle$ and T is a maximal torus of $Y_1 \cdots Y_k$. So we now assume $k = 1$.

Regard $A/C_A(Y_1)$ as a subgroup of $\mathrm{Aut}(Y_1)$ and let $A_0/C_A(Y_1) = A/C_A(Y_1) \cap \tilde{Y}_1$, where $\tilde{Y}_1$ denotes the subgroup of $\mathrm{Aut}(Y_1)$ generated by inner and diagonal automorphisms. Assume $A_0 > C_A(Y_1)$ and choose a subgroup B of A_0 with $B \trianglelefteq A$ and $|B/C_A(Y_1)|$ a prime. Then B centralizes maximal tori of Y_1 and we let D be the subgroup of Y_1 generated by all such maximal tori. By (2.9) we have $D = D_1 \cdots D_l I$, where the D_i are commuting groups of Lie type over extension fields of $\mathbf{F}_{p^{e_i}}$ and I can be taken as any maximal torus centralizing B.

By induction we may assume $A \leq N(J_1 \cdots J_l)$, where for $1 \leq i \leq l$, J_i is a maximal torus of D_i. Write $Y_1 = O^{p'}(\bar{Y}_\tau)$, where $\bar{Y}$ is the corresponding adjoint algebraic group and τ an endomorphism of $\bar{Y}$. Then A can be extended to a group of endomorphisms of $\bar{Y}$ (automorphisms of the abstract group $\bar{Y}$). Hence A normalizes the group $\bar{C} = C_{\bar{Y}}(B)^0$. Write $\bar{C} = \bar{L}_1 \cdots \bar{L}_s \bar{I}$, where the $\bar{L}_i$ are commuting quasisimple algebraic groups and $\bar{I}$ is the τ-invariant maximal torus of $\bar{Y}$ containing I. We may assume $J_i \leq \bar{I}$ for $i = 1, \ldots, l$. Each of the groups $D_1, \ldots, D_l$ is the group generated by all p-elements fixed by τ in a particular orbit product of $\langle \tau \rangle$ on $\{\bar{L}_1, \ldots, \bar{L}_s\}$. Using (2.6) we see that $\bar{I} = C_{\bar{C}}(J_1 \cdots J_l)^0$. Hence, $A \leq N(\bar{I})$ and so $A \leq N(\bar{I} \cap Y_1) = N(I)$. Consequently, we may now assume $A_0 = C_A(Y_1)$.

Now $A/C_A(Y_1) = \langle a \rangle \times \langle b \rangle$, where no element of $\langle a \rangle$ is in the coset of a nontrivial graph automorphism of Y_1 and $|b| = s^k$ for $s = 2$ or 3. If $k > 0$ then $\langle b \rangle$ contains the coset of a graph automorphism (either a or b could be trivial). By Lang's theorem ((10.1) of [**26**]) a induces a field or graph-field automorphism of Y_1. It follows from the fact $p \geq 5$ that A centralizes an element c of Y_1 with $|c| = 2$ or 3. Consequently, A normalizes $C_{Y_1}(c)$ and we can argue as in the preceding paragraph, replacing B by $\langle c \rangle$. This completes the proof of (2.14).

3. Basic properties. In this section we begin the discussion of T-root groups. We maintain the notation of §2 and introduce additional notation and terminology as follows. Set $G_0 = O^{p'}(G)$ and $T_0 = T \cap G_0$. If $G_0 \leq G_1 \leq G$, then a group of the form $T \cap G_1$ is called a maximal torus of G_1. Let $\Delta = \{\bar{U}_\alpha \mid \alpha \in \bar{\Sigma}\}$, the root subgroups of $\bar{G}$ with respect to $\bar{T}$. Set $\bar{U} = \langle \bar{U}_\alpha \mid \alpha \in \bar{\Sigma}^+ \rangle$ and $\bar{B} = \bar{U} \cdot \bar{T}$, a Borel subgroup of $\bar{G}$ (not necessarily σ-invariant).

For $\alpha \in \bar{\Sigma}$, regard $\bar{U}_\alpha$ as a 1-dimensional K representation of $\bar{T}$. Let φ_α denote this representation (φ_α equals α if we regard α as a character of $\bar{T}$).

Since Δ is the set of minimal $\bar{T}$-invariant unipotent subgroups of $\bar{G}$, $\Delta = \Delta^\sigma$, so $\Delta = \Delta_1 \cup \cdots \cup \Delta_v$, a union of $\langle \sigma \rangle$-orbits. Correspondingly, we set $\bar{\Sigma}_i = \{\alpha \mid \bar{U}_\alpha \in \Delta_i\}$. For $i = 1,\ldots,v$ we set $\bar{X}_i = \langle \Delta_i \rangle$ and $X_i = O^{p'}((\bar{X}_i)_\sigma)$. The groups $X_1,\ldots,X_v$ are the *T-root subgroups of G*. We note that for $i = 1,\ldots,v$, $X_i \leq O^{p'}(G) = G_0$. For $1 \leq i \leq v$, there is a unique $j \in \{1,\ldots,v\}$ such that $\bar{\Sigma}_j = -\bar{\Sigma}_i$. We set $\bar{X}_j = \bar{X}_i^*$, $X_j = X_i^*$, and $\Delta_j = \Delta_i^*$.

The first result is that a T-root subgroup is either a p-group or a group of Lie type defined over an extension field of $\mathbf{F}_q$.

(3.1) Fix $i = 1,\ldots,v$. The group $\bar{X}_i$ is either unipotent or semisimple. Correspondingly, X_i is either a p-group or X_i is a group of Lie type defined over an extension field of $\mathbf{F}_q$.

Proof. Consider the group $\bar{X}_i$. We may assume that $\bar{X}_i$ is not unipotent. But $\bar{X}_i = \bar{X}_i^0$ and $\bar{X}_i$ is $\bar{T}$-invariant. Let $\bar{X}_i\bar{T} = \bar{Q} \cdot \bar{L}$, where $\bar{Q} = R_u(\bar{X}_i)$ and $\bar{L}$ is the product of $\bar{T}$ with those root subgroups $\bar{U}_\alpha$ such that $\bar{U}_\alpha$ and $\bar{U}_{-\alpha}$ are both contained in $\bar{X}_i$ (see (2.12)). We have $\bar{L} \cap \bar{Q} = 1$. Also, each root subgroup of $\bar{X}_i$ is contained in either $\bar{L}$ or $\bar{Q}$. Since σ normalizes each of $\bar{L}$ and $\bar{Q}$, we conclude that $\langle \Delta_i \rangle \leq \bar{L}$, hence $\bar{X}_i = \bar{L}$. Write $\bar{L} = \bar{L}_1 \cdots \bar{L}_k$, a central product of the components of $\bar{L}$. For each $U_\alpha \in \Delta_i$, $U_\alpha \leq \bar{L}_j$ for some j. Therefore $\langle \sigma \rangle$ is transitive on $\{\bar{L}_1,\ldots,\bar{L}_k\}$ and $X_i/Z(X_i) \cong O^{p'}((\bar{L}_1)_{\sigma^k})/Z(O^{p'}((L_1)_{\sigma^k}))$. The argument in (2.6) of [**23**] shows that $X_i/Z(X_i)$ is associated with $\mathbf{F}_{q^k}$, completing the proof of (3.1).

Order the T-root groups so that $X_1,\ldots,X_t$ are p-groups and $X_{t+1},\ldots,X_v$ are groups of Lie type. We note that if $\bar{T}$ is contained in a σ-stable Borel subgroup of $\bar{G}$, then $t = v$ and $\{X_1,\ldots,X_v\}$ are the usual root subgroups of G. We also point out that there may be containments among the X_i's. This even occurs when T is a Cartan subgroup of G. For example, in the case of $\mathrm{PSU}(n, q)$ with n odd, there is a non-abelian root subgroup E of order q^3 and $Z(E)$ is also a root group.

The next result gives bounds on the nilpotence class of the groups $X_1,\ldots,X_t$. First, we require the following (temporary) notation. Let $\bar{H} = \bar{H}^\sigma$ be a maximal torus of $\bar{G}$ with $\bar{H} \leq \bar{B}_1 = \bar{B}_1^\sigma$, where $\bar{B}_1$ is a Borel subgroup of $\bar{G}$. Let $\hat{\Sigma}$ be the root system of $\bar{G}$, with respect to $\bar{H}$, and $\{\hat{\alpha}_1,\ldots,\hat{\alpha}_n\}$ a fundamental set of roots. For $\alpha \in \hat{\Sigma}$, write $\hat{\alpha} = \sum n_i \hat{\alpha}_i$ and let $c(i, \hat{\alpha}) = \sum n_j$, the sum ranging over those j with $\alpha_j \in \alpha_i^{\langle \sigma \rangle}$. Let $c = \max\{c(i, \hat{\alpha})\colon \hat{\alpha} \in \hat{\Sigma},\ 1 \leq i \leq n\}$.

(3.2) If $1 \leq i \leq t$, then X_i has nilpotence class at most c.

Proof. By either (3.9) of [**3**] or by [**4**] there exist a parabolic subgroup $\bar{P}_1 \leq \bar{G}$ such that $\bar{X}_i \leq R_u(\bar{P}_1)$ and $N_{\bar{G}}(\bar{X}_i) \leq \bar{P}_1$. Moreover, $\bar{P}_1$ is obtained canonically from $\bar{X}_i = \bar{X}_i^\sigma$, so $\bar{P}_1 = \bar{P}_1^\sigma$. Therefore, we may assume that $\bar{B}_1 \leq \bar{P}_1$.

Let $\bar{P}_1 \leq \bar{P}_0$, where $\bar{P}_0$ is a maximal parabolic subgroup of $\bar{G}$. Let $\bar{P} = \bigcap_i P_0^{\sigma^i}$. Notice that $\bar{B}_1 \leq \bar{P}$ and $P_0^{\langle\sigma\rangle}$ has size 1, 2, or 3, with the latter case possible only for $\bar{G} = D_4(K)$. Now $R_u(\bar{P}) \leq R_u(P_1)$. Indeed, $R_u(\bar{P}) = \prod R_u(P_0^{\sigma^i})$. $R_u(\bar{P}_1)$ is a product of root subgroups for $\bar{T}$. Choose $\bar{P}_0$ such that some element $\bar{U}_\alpha$ of Δ_i is such that α has non-zero coefficient of the fundamental root defining $\bar{P}_0$. It follows that $\bar{U}_\alpha \leq R_u(\bar{P})$, and hence $\bar{X}_i \leq R_u(\bar{P})$.

We now know that X_i has nilpotence class bounded by that of $R_u(\bar{P})$. $R_u(\bar{P})$ is also the product of root groups of $\bar{G}$ with respect to the maximal torus $\bar{H}$. Viewing $R_u(\bar{P})$ in this way and using the Chevalley commutator relations, the result follows.

The table below gives bounds on the nilpotence class of the groups X_i, $1 \leq i \leq t$, for the various groups G. This bound is c except for the cases of $G = \mathrm{Sz}(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$. In the latter cases the bounds are less than c. This is because the characteristic restrictions needed to define G force the appropriate parabolic subgroups of the above proof to have unipotent radicals of smaller nilpotence class. In these cases direct computations give the bounds. Otherwise, the number c is computed easily once the root system is given; for σ induces a (possibly trivial) graph automorphism on the root system for $\bar{H}$.

TABLE 1

$G_0/Z(G_0)$	bound on class X_i
$\mathrm{PSL}(n,q)$	1
$\mathrm{Psp}(n, q)$	2
$\mathrm{PSU}(n, q)$	2
$PO^{\pm}(n, q)'$	2
$E_6(q)$	3
$E_7(q)$	4
$E_8(q)$	6
$G_2(q)$	3
$F_4(q)$	4
${}^3D_4(q)$	3
${}^2E_6(q)$	4
$\mathrm{Sz}(q)$	2
${}^2G_2(q)$	3
${}^2F_4(q)$	5

It will be a consequence of later work that the above bounds are best possible. Also, we will discuss the embedding of T-root groups in G, and for the classical groups we describe the action of T-root groups on the natural module.

Our next two results concern the embedding of X_i in G and the embedding of $\bar{X}_i$ in $\bar{G}$, for $i = 1,\ldots,t$.

(3.3) Let $1 \neq \bar{C}$ be a unipotent group generated by a subset of $\{\bar{X}_1,\ldots,\bar{X}_t\}$, and let C be the subgroup of G generated by the corresponding subset of $\{X_1,\ldots,X_t\}$. Then C is a p-group. There is a parabolic subgroup $\bar{P} = \bar{P}^\sigma \geq \bar{T}$ of $\bar{G}$, such that $\bar{C} \leq R_u(\bar{P})$. The group $\bar{P}_\sigma = P$ is a parabolic subgroup of G satisfying $C \leq O_p(P)$, and T is contained in a conjugate of a Levi factor of P.

Proof. Since $\bar{C}$ is unipotent, there is a canonical parabolic subgroup $\bar{P} \leq \bar{G}$ with $\bar{C} \leq R_u(\bar{P})$ (Borel-Tits, (3.9) of [3]). Then $\bar{P} = \bar{P}^\sigma$ and $\bar{T} \leq N_{\bar{G}}(\bar{P}) = \bar{P}$. To see that $P = \bar{P}_\sigma$ is a parabolic subgroup of G first use Lang's theorem to get a Borel subgroup $\bar{J}$ of $\bar{P}$ (hence of $\bar{G}$) stabilized by σ and then use (2.12) of [25] to conclude $\bar{J}_\sigma$ is a Borel subgroup of G. This forces P to be a parabolic subgroup of G. Clearly $C \leq R_u(\bar{P})_\sigma \leq O_p(P)$. Choose $x \in \bar{P}$ such that $\bar{T} \leq \bar{J}^x$. Then $R_u(\bar{J}^x)$ is a product of some of the root subgroups $\bar{U}_\alpha$ for $\alpha \in \Sigma$ and the Levi factor of $\bar{P}$ is generated by $\bar{T}$ together with those $\bar{U}_\alpha \leq \bar{P}$ such that $\bar{U}_{-\alpha} \leq \bar{P}$. So σ stabilizes the Levi factor and the result follows.

(3.4) Let $i \in \{1,\ldots,t\}$, let $\bar{X} = \bar{X}_i$, and $X = \bar{X}_\sigma$. Choose $j \in \{1,\ldots,t\}$ so that $\bar{\Sigma}_j = -\bar{\Sigma}_i$, and set $X^* = (\bar{X}_j)_\sigma$. Then

(i) $\bar{Y} = \langle \bar{X}_i, \bar{X}_j \rangle = \bar{D}_1 \cdots \bar{D}_k$, a commuting product of $\langle\sigma\rangle$-conjugate, semisimple groups, each generated by certain root subgroups of $\bar{T}$.

(ii) $Y = \langle \bar{X}_i, \bar{X}_j \rangle_\sigma = Y(q^k)$, a group of Lie type defined over $\mathbf{F}_{q^k}$.

(iii) There exists a unique $\bar{T}\langle\sigma\rangle$-stable parabolic subgroup $\bar{P}_0$ of $\bar{Y}$ such that $\bar{X} \leq R_u(\bar{P}_0)$. Also, $\bar{T} \leq N_{\bar{G}}(\bar{P}_0)$.

(iv) $\bar{P}_0$ is the intersection of a $\langle\sigma\rangle$-orbit of maximal parabolic subgroups of $\bar{Y}$. If σ induces a field automorphism of $\bar{G}$, then $\bar{P}_0$ is a maximal parabolic subgroup of $\bar{Y}$.

(v) Suppose $q \geq 4$. Then $P_0 = (\bar{P}_0)_\sigma$ is the unique parabolic subgroup of Y normalized by T and satisfying $X \leq O_p(P_0)$.

(vi) Suppose $q \geq 4$. Then $T \cap Y$ is a maximal torus in Y, $T \cap Y/Z(Y)$ is cyclic and there is a Levi factor of P_0 in which $T \cap Y$ is a minisotropic torus.

Proof. Let $\bar{Y} = \langle \bar{X}_i, \bar{X}_j \rangle$. Then $\bar{Y}$ is normalized by $\bar{T}$. Let Γ be the collection of all roots $\alpha \in \bar{\Sigma}$ such that $\bar{U}_{\pm\alpha} \leq \bar{Y}$. Then $\Gamma^\sigma = \Gamma$ and by

(2.12) $\langle \bar{U}_\alpha \mid \alpha \in \Gamma \rangle$ is a semisimple subgroup of $\bar{Y}$. Since $\bar{\Sigma}_i, \bar{\Sigma}_j \leq \Gamma$ we have $\bar{Y} = \langle \bar{U}_\alpha \mid \alpha \in \Gamma \rangle$ and $\bar{Y}$ can be expressed $\bar{Y} = \bar{D}_1 \cdots \bar{D}_k$, with each $\bar{D}_i$ quasisimple and generated by certain of the root subgroups in Γ. As $\bar{T}$ is connected, $\bar{T} \leq N_{\bar{G}}(\bar{D}_i)$ for $i = 1, \ldots, k$. Also, each $\gamma \in \bar{\Sigma}_i$ satisfies $\bar{U}_\gamma \leq \bar{D}_l$ for some l. As $\bar{U}_{-\gamma} \leq \bar{D}_l$ as well, we conclude that $\langle \sigma \rangle$ is transitive on $\{\bar{D}_1, \ldots, \bar{D}_k\}$. This gives (i). The argument in (2.6) of **[23]** shows that (ii) holds.

We may write $\bar{Y} \cdot \bar{T} = \bar{D}_1 \cdots \bar{D}_k \bar{Z}$, where $\bar{Z}$ is a subtorus of $\bar{T}$ and $\bar{Z} = Z(\bar{Y} \cdot \bar{T})^0$. Now σ^k stabilizes each of $\bar{D}_1, \ldots, \bar{D}_k$ and we observe that for the purpose of proving the remaining parts of (3.4) we may replace (G, T, σ) by $(\bar{D}_1/Z(\bar{D}_1), \bar{T} \cap \bar{D}_1/Z(\bar{D}_1), \sigma^k)$. Therefore, we now assume that $\bar{Y} = \bar{G}$. In particular, $C_{\bar{T}}(\bar{X}) = 1$.

Let $\bar{P}$ be a parabolic subgroup of $\bar{G}$ such that $\bar{X} \leq R_u(\bar{P}) \leq \bar{P} = \bar{P}^\sigma \geq \bar{T}$ (see (3.3)). Conjugating, if necessary, we may assume that $\bar{X} \leq \bar{B} \leq \bar{P}$, where $\bar{B}$ is the Borel subgroup $\bar{T}\langle \bar{U}_\alpha \mid \alpha \in \bar{\Sigma}^+ \rangle$. So $\bar{P} = \langle \bar{B}, \bar{U}_{\pm\alpha_j} \mid \alpha_j \notin S \rangle$ and $S = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ is a subset of $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Now we may write $\sigma = w\gamma$, where $w \in \bar{W} = \bar{N}/\bar{T}$ and γ is a field or graph-field automorphism of $\bar{G}$ defined with respect to $\bar{\Sigma}$, $\bar{B}$, $\bar{T}$. Then $\bar{P}^\sigma = \bar{P}^{w\gamma} = \bar{P}$ and so $\bar{P}^w = \bar{P}^{\gamma^{-1}}$. Since $\bar{B}^{\gamma^{-1}} = \bar{B}$, $\bar{P}^{\gamma^{-1}}$ is also a standard parabolic subgroup for $\bar{B}$. Thus $\bar{P}$, $\bar{P}^w$ are both parabolics containing $\bar{B}$ and this forces $\bar{P} = \bar{P}^w$, whence $w \in \bar{P}$. Also, $\gamma \in N(\bar{P})$. We claim that the permutation of $\bar{\Sigma}$ associated with γ is transitive on S.

If G is a Suzuki or Ree group, then this is clear from $\bar{P} = \bar{P}^\gamma$, unless $\bar{G}$ is of type $F_4(K)$ and $\bar{P}$ a Borel subgroup. But in this case X is an ordinary root group of G, contradicting $\langle \bar{X}_i, \bar{X}_j \rangle = G$. We now exclude the Ree and Suzuki groups for purposes of establishing the claim. For $\alpha \in \bar{\Sigma}_i$, $\bar{U}_\alpha \leq R_u(\bar{P})$ and we let $(\alpha)_{i_j}$ be the coefficient of α_{i_j} in α. Since $w \in \bar{P}$, $(\alpha)_{i_j} = (\alpha^w)_{i_j}$ for each $\alpha \in \bar{\Sigma}_i$.

Write $S = S_1 \cup \cdots \cup S_m$, where the union is disjoint, each S_i is a $\langle \gamma \rangle$-orbit, and suppose $m > 1$. For $\alpha \in \bar{\Sigma}_i$, let $\alpha(k) = \Sigma_{\alpha_{i_j} \in S_k} (\alpha)_{i_j}$. Then $\alpha(k) = \beta(k)$ for each $\alpha, \beta \in \bar{\Sigma}_i$. Choose $\chi(\alpha_j) = 1$ for each $\alpha_j \in S_1 \cup S_2$, $\chi(\alpha_j) = \varphi$ for each $\alpha_j \in S_1$, and $\chi(\alpha_j) = \eta$ for each $\alpha_j \in S_2$. Consideration of the numbers $\alpha(1)$ and $\alpha(2)$, for $\alpha \in \Delta_1$, shows that it is possible to choose $1 \neq \chi$ such that $\chi(\alpha) = 1$ for each $\alpha \in \bar{\Sigma}_i$. But then $h(\chi) \in C_{\bar{T}}(\bar{X}) = 1$, a contradiction. This proves the claim.

The claim gives (iv), once (iii) is checked. So suppose $\bar{P}_1$ is another $\bar{T}\langle \sigma \rangle$-stable parabolic subgroup of $\bar{G}$ with $\bar{X} \leq R_u(\bar{P}_1)$. Then $\bar{P}_1 = \bar{P}_2^g$ where $\bar{B} \leq \bar{P}_2$. So $\bar{T}, \bar{T}^g \leq \bar{P}_2^g$ and, conjugating, we may assume that $g \in N_{\bar{G}}(\bar{T}) = \bar{N}$. So $\bar{P}_1 = \bar{P}_2^w$ for some $w \in W$. But now consider $\bar{P} \cap \bar{P}_2^w$. This group is $\bar{T}\langle \sigma \rangle$-invariant and we apply the results in §2 of **[8]**. Write

$\bar{P} = \bar{P}_{J_1}$ and $\bar{P}_2 = \bar{P}_{J_2}$ where $J_1, J_2 \subseteq \Pi$, and $w = w_2 w' w_1$, where $w_2 \in W_{J_2}$, $w_1 \in W_{J_1}$, and $w' \in W_{J_2,J_1}$, the distinguished set of double coset representatives. Then $\bar{P} \cap \bar{P}_2^w = (\bar{P}_{J_1} \cap \bar{P}_{J_2}^{w'})^{w_1}$, and we consider the group $P_{J_1} \cap P_{J_2}^{w'}$. Let $K = J_1 \cap J_2^{w'}$. By (2.4) of [8], $\bar{P}_K = (\bar{P}_{J_1} \cap \bar{P}_{J_2}^{w'}) R_u(\bar{P}_{J_1})$, so $\bar{X} \leq \bar{P} \cap \bar{P}_2^w = (\bar{P}_{J_1} \cap \bar{P}_{J_2}^{w'})^{w_1} \leq \bar{P}_K^{w_1}$ and $\bar{P}_K$ is a parabolic subgroup of $\bar{P}_{J_1} = \bar{P}$. Since $\bar{P}_K^{w_1}$ is $\bar{T}\langle\sigma\rangle$-stable we apply the above claim to conclude that $\bar{P}_K^{w_1} = \bar{P}_{J_1} = \bar{P}$. Thus, $w_1 \in \bar{P}_K$, $\bar{P}_K = \bar{P}_{J_1}$, and $J_1 \subseteq J_2^{w'}$. By (2.6) of [8], $L_{J_1} \leq L_{J_2}^w$, where L_{J_1} and $L_{J_2}^w$ are the standard Levi factors of $\bar{P}_{J_1}$ and $\bar{P}_{J_2}^w$, respectively. Reversing the roles of $\bar{P}$ and $\bar{P}_2^w$ we have $L_{J_1} = L_{J_2}^w = L_K$ and $J_1 = J_2^{w'}$.

Fix $\alpha_i \in \pi - J_1$. By (2.4) of [9] we see that W_K is transitive on the set of roots, α, such that α has α_i coefficient equal to 1 and α_j coefficient 0 for each $\alpha_i \neq \alpha_j \in \pi - J_1$. Using this together with the fact that $\pi - J_1$ is a $\langle\sigma\rangle$-orbit we have $W_K\langle\sigma\rangle$ transitive on those roots $\beta \in \bar{\Sigma}$ satisfying $\Sigma_{\alpha_i \notin J_1}(\beta)_i = 1$. For the remainder of the proof, if $\beta \in \bar{\Sigma}$, let $(\beta)_{J_1}$ denote the integer $\Sigma_{\alpha_i \notin J_1}(\beta)_i$. Let $o = \{\beta \mid (\beta)_{J_1} = 1\}$, a $W_K\langle\sigma\rangle$-orbit of roots.

We claim that $\bar{\Sigma}_i \subseteq o$. For suppose $\alpha \in \bar{\Sigma}_i$ and $(\alpha)_{J_1} = c > 1$. Let Γ be the collection of all $\gamma \in \bar{\Sigma}$ with $(\gamma)_{J_1}$ a multiple of c. It is easily checked that Γ is a closed root system of $\bar{\Sigma}$ and this contradicts the fact that $G = \langle \bar{X}_i, \bar{X}_j \rangle$. So $\bar{\Sigma}_i \subseteq o$. Now, $\bar{P}_2^w = \bar{P}_{J_2}^w = \bar{P}_{J_2}^{w'w_1} = \bar{P}_{J_2}^{w'}$, since $w_1 \in L_K = L_{J_2}^{w'}$. So, by symmetry, there is a $W_K\langle\sigma\rangle$-orbit o' such that $\bar{\Sigma}_i \subseteq O'$ and $(\pi - J_2)^{w'} \subseteq o'$. But then, $o = o'$ and we have $\pi^{w'} = J_2^{w'} \cup (\pi - J_2)^{w'} \subseteq J_1 \cup o \subseteq \Sigma^+$. Therefore, $(\bar{\Sigma}^+)^{w'} = \bar{\Sigma}^+$ and $w' = 1$. We now have $\bar{P}_1 = \bar{P}_2^w = \bar{P}_{J_2}^w = \bar{P}_{J_2}^{w'w_1} = P_{J_2^{w'}}^{w_1} = \bar{P}_{J_1}^{w_1} = \bar{P}_{J_1} = \bar{P}$. This proves (iii).

Suppose $X \leq O_p(P_1)$ and $q \geq 4$, where P_1 is a T-stable parabolic subgroup of G. Then $P_1 = N_G(O_p(P_1))$, so by (3.9) of [3], there is a canonical parabolic subgroup $\bar{P}_1$ of $\bar{G}$ such that $P_1 \leq \bar{P}_1$ and $O_p(P_1) \leq R_u(\bar{P}_1)$. Then $T \leq \bar{P}_1$ and we claim $\bar{T} \leq \bar{P}_1$. To see this first note that if $h(\chi) \in T$, then $\chi^\sigma = \chi$ and hence given $\alpha, \beta \in \Delta_i$, $\varphi_\alpha = \varphi_\beta^{q^k}$ for some integer k (a slight change is required for the Suzuki and Ree groups). Therefore, $C_T(\bar{U}_\alpha) = C_T(\bar{U}_\beta)$ for each $\alpha, \beta \in \bar{\Sigma}_i$, and since $C_{\bar{T}}(\bar{X}) = 1$, we necessarily have T cyclic. Write $T = \langle t \rangle$. Then t is semisimple and is contained in a maximal torus $\bar{T}_1$ of $\bar{P}_1$. Then $\bar{T}_1 \leq C_{\bar{G}}(T)^0 = \bar{T}$ by (2.7). So $\bar{T} = \bar{T}_1$ and $\bar{P}_1 = \bar{P}_0$ by (iv). Therefore, $P_1 \leq (\bar{P}_1)_\sigma = (\bar{P}_0)_\sigma = P_0$. If equality failed to hold, then $O_p(P_1) > O_p(P_0)$. However, $O_p(P_1) \leq R_u(\bar{P}_1) = R_u(\bar{P}_0)$, so this is impossible. This proves (v) and (vi) follows from (v) and the above argument.

In §4 we will describe the T-root subgroups of the classical groups and also the groups Y and P_0 of (3.4). One additional result of interest is the following.

(3.5) The set $\{\bar{\Sigma}_1, \ldots, \bar{\Sigma}_t\}$ can be partitioned as $\{\bar{\Sigma}_{i_1}, \ldots, \bar{\Sigma}_{i_k}\} \cup \{\bar{\Sigma}_{j_1}, \ldots, \bar{\Sigma}_{j_k}\}$ in such a way that each of $\langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k} \rangle$ and $\langle \bar{X}_{j_1}, \ldots, \bar{X}_{j_k} \rangle$ is unipotent.

Proof. Let $\bar{P}_1$ be the canonical parabolic subgroup of $\bar{G}$ such that $\bar{X}_1 \leq R_u(\bar{P}_1)$ and $N_{\bar{G}}(\bar{X}_1) \leq \bar{P}_1$ (see (3.9) of [**4**]). Then $\bar{P}_1 = \bar{P}_1^\sigma$ and $\bar{T} \leq \bar{P}_1$. For each $\alpha \in \bar{\Sigma}$, either $\bar{U}_\alpha$ or $\bar{U}_{-\alpha}$ is a subgroup of $\bar{P}_1$. So for each $1 \leq i \leq t$, either $\bar{X}_i$ or $\bar{X}_i^*$ is a subgroup of $R_u(\bar{P}_1)$ or both of $\bar{X}_i$ and $\bar{X}_i^*$ are contained in the Levi factor, $\bar{L}_1$, of $\bar{P}_1$. Inductively, we can partition the roots in the root system for $\bar{L}_1$ so that the result holds in $\bar{L}_1$. Now we obtain $\{\bar{\Sigma}_{i_1}, \ldots, \bar{\Sigma}_{i_k}\}$ by taking the $\langle \sigma \rangle$-orbits in one of the partitioning sets for the root system of $\bar{L}_1$, together with those $\bar{\Sigma}_i$ such that $\bar{X}_i \leq R_U(\bar{P}_1)$. Passing to the opposite parabolic subgroup of $\bar{P}_1$, we see that the result holds.

In the notation of (3.5), the group $\langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k} \rangle_\sigma T$ can be regarded as a replacement for a Borel subgroup. However, there may be several ways to obtain partitions as in (3.5).

(3.6) Assume $q \geq 4$ and that Y is a group acting on G_0 and normalizing T_0. Then Y permutes the set of T-root subgroups of G.

Proof. We may assume $Z(G_0) = 1$, Y acts faithfully on G_0, and $Y = \langle y \rangle$ for some automorphism y of G_0. There is an endomorphism τ of $\bar{G}$ such that $[\tau, \sigma]$ and $\tau|_{G_0} = y$. So τ normalizes T_0 and (2.6) implies that τ normalizes $\bar{T}$. Consequently, τ permutes the set of $\langle \sigma \rangle$-orbits of $\bar{T}$-root subgroups. The result follows.

4. Classical groups. In this section we determine the T-root subgroups $X_1, \ldots, X_t$, when G is a classical group. Choose notation as in §3 and fix $1 \leq i \leq t$. Set $\Delta = \Delta_i$, $\bar{X} = \bar{X}_i$, $X = X_i$, $X^* = X_i^*$. In addition, we set $\bar{Y} = \langle \bar{X}, \bar{X}^* \rangle$, $Y = \bar{Y}_\sigma$, and $\bar{P}$, P the parabolic subgroups of $\bar{Y}$ and Y, respectively, as described in (3.4).

To make the statements and proofs less cumbersome we will assume throughout the section that G and $\bar{G}$ are the appropriate linear groups. However, it is easy to pass from these results to those for other forms of G and $\bar{G}$.

(4.1) If $G_0 \cong \mathrm{SL}(n, q)$, then there exist positive integers r, s, y such that the following hold:

(i) $(r, s) = 1$.

(ii) X is elementary abelian of order $q^{(r-s)sy}$.

(iii) $\langle X, X^* \rangle \cong \mathrm{SL}(r, q^y)$.

(iv) $\langle \overline{X}, \overline{X}^* \rangle = \overline{D}_1 \cdots \overline{D}_y$ a commuting product of copies of $\mathrm{SL}(r, K)$, permuted transitively by $\langle \sigma \rangle$ and each generated by $\overline{T}$-root subgroups of $\overline{G}$.

(v) $X = O_p(P)$ and P is the stabilizer of an s-space of the usual module for $\mathrm{SL}(r, q^y)$.

(vi) The projection of $\overline{X}$ to $\overline{D}_1$ is the unipotent radical of a parabolic subgroup of $\overline{D}_1$ obtained by deleting the (sth) node of the Dynkin diagram for $\overline{D}_1$ (which has type A_{r-1}).

Proof. Write $\langle \overline{X}, \overline{X}^* \rangle = \overline{D}_1 \cdots \overline{D}_y$ with $\langle \sigma \rangle$ transitive on $\{\overline{D}_1, \ldots, \overline{D}_y\}$ (see (3.5)(i)). Let $\gamma = \sigma^y$, so that γ stabilizes each of $\overline{D}_1, \ldots, \overline{D}_y$. Since the root system of each $\overline{D}_j$ is a subsystem of $\overline{\Sigma}$, and since $\overline{\Sigma}$ has type A_{n-1}, there is an integer r such that $\overline{D}_j \cong \mathrm{SL}(r, K)$, for $j = 1, \ldots, t$. So (iv) holds.

We first observe that γ induces a field automorphism on $\overline{D}_1, \ldots, \overline{D}_y$. To see this write $\sigma = \tau q$ as in (2.1). Since σ is a field automorphism of $\overline{G}$, $\tau = w \in \overline{W}$, so $\gamma = w^y q^y$. Now if we set $W_j = \overline{W} \cap \overline{D}_j$, then $N_{\overline{W}}(\overline{W}_j) = \overline{W}_j C_{\overline{W}}(\overline{W}_j)$. It follows that w^y induces an inner automorphism on $\overline{D}_j$; hence $(\overline{D}_j)_\gamma \cong \mathrm{SL}(r, q^y)$. At this point (iii) will follow from (v) and (2.3) of **[23]**.

If we can verify (i) and (vi), then the remaining parts of (4.1) will follow, by projection, from the known structure of parabolic subgroups of type A_{r-1} and the connection between parabolic subgroups of $\overline{D}_1$ and $(\overline{D}_1)_\gamma$. Hence we have reduced the problem to the case of $\overline{G} = \mathrm{SL}(r, K)$, $\overline{G}_\sigma = \mathrm{SL}(r, q^y)$, and $\langle \overline{X}, \overline{X}^* \rangle = \overline{G}$.

By (3.4)(iii), $\overline{X} \leq R_u(\overline{P})$. Regard $\overline{P}$ as the stabilizer in $\overline{G}$ of an s-space of the usual module for $\mathrm{SL}(n, K)$ (here we are using (3.5)(iv)). Then $\overline{P}/R_u(\overline{P})$ is a central product of $\mathrm{SL}(s, K)$, $\mathrm{SL}(r - s, K)$, and a 1-dimensional torus. Each of these groups is stabilized by σ. From the uniqueness of $\overline{P}$ we conclude that T is a minisotropic torus of $P/O_p(P)$, so T contains the central product of cyclic groups of order $(q^{ys} - 1)/(q^y - 1)$, $(q^{y(r-s)} - 1)/(q^y - 1)$, and $q^y - 1$ (see Carter **[5]**).

Let $\alpha \in \Delta_i$. Then T induces a cyclic group on $\overline{U}_\alpha$ and induces algebraic conjugates of φ_α on the other root subgroups in Δ_i. Since $R_u(\overline{P})$ is abelian, $\overline{X}$ is the direct product of the groups $\overline{U}_\alpha$, $\alpha \in \overline{\Sigma}_i$ and T induces a cyclic group on $\overline{X}$. Then $C_{\overline{T}}(\overline{G}) = Z(\overline{G})$ implies $C_{\overline{T}}(\overline{X}) = Z(\overline{G})$ (as $\overline{G} = \langle \overline{X}, \overline{X}^* \rangle$ and $C_{\overline{T}}(\overline{X}^*) = C_{\overline{T}}(\overline{X})$) and so $T/Z(G)$ is cyclic. From this and the above description of T we have $(s, r - s) = 1$. For if $(s, r - s) = d > 1$, then $Z_a \times Z_a \leq T/Z(G)$, where $a = (q^{jd} - 1)/(q^j - 1)$. This proves (i).

The group $R_u(\bar{P})$ is the product of $s(r-s)$ $\bar{T}$-root subgroups of $\bar{G}$. Since T is a minisotropic torus of P, σ acts as w_1w_2q where for $i = 1,2$, w_i is a Coxeter element of the corresponding component, $\mathrm{SL}(s, K)$ or $\mathrm{SL}(r-s, K)$, of the Levi factor of $\bar{P}$. Thus $|w_1| = s$, $|w_2| = r - s$ and $\langle w_1, w_2\rangle$ is transitive on the set of $\bar{T}$-root subgroups in $R_u(\bar{P})$. By (i), $\langle w_1, w_2\rangle = \langle w_1w_2\rangle$, so $\bar{X} = R_u(\bar{P})$ and this proves (vi). This completes the proof of (4.1).

A result quite similar to (4.1) holds when G is replaced by an arbitrary classical group, although there does exist one ambiguity (which is cleared up in §12, but under additional hypotheses).

(4.2) Suppose that G is one of the groups: $\mathrm{Sp}(n, q)$, $\mathrm{SU}(n, q)$, or $\mathrm{SO}^{\pm}(n, q)'$. There exist positive integers r, s, y, such that the following hold:

(i) Either $(r, s) = 1$ or $r = 2s$.

(ii) If $r \neq 2s$, then X/X' is elementary abelian of order $q^{ys(r-2s)}$ ($q^{2ys(r-2s)}$ if Y is a unitary group).

(iii) $\langle \bar{X}, \bar{X}^*\rangle = \bar{D}_1 \cdots \bar{D}_y$, a commuting product of copies of one of the groups $\mathrm{SL}(r, K)$, $\mathrm{Sp}(r, K)$, or $O^+(r, K)'$. Also, $\langle\sigma\rangle$ is transitive on $\{\bar{D}_1, \dots, \bar{D}_y\}$ and each $\bar{D}_i$ is generated by $\bar{T}$-root subgroups of $\bar{G}$.

(iv) If $r \neq 2s$, the $\langle X, X^*\rangle \cong \mathrm{SL}(r, q^y)$, $\mathrm{SU}(r, q^y)$, $\mathrm{Sp}(r, q^y)$, or $O^{\pm}(r, q^y)'$.

(v) If $r \neq 2s$, then $X = O_p(P)$ and P is the stabilizer of a totally isotropic (singular) s-space of the usual module for $\langle X, X^*\rangle$.

(vi) If $r = 2s$ and $q \geq 4$, then X is elementary abelian of order q^{ys} (q^{2ys} if Y is a unitary group), P is the stabilizer of a totally isotropic (singular) s-space of the usual module for Y, and $X \leq O_p(P)$, equality only if $s = 2$ and $\langle X, X^*\rangle$ a unitary group.

Proof. We first make reductions as in the proof of (4.1). Let $\langle \bar{X}, \bar{X}^*\rangle = \bar{D}_1 \cdots \bar{D}_y$, a central product. From (3.5)(i) we get (iii). Then each $\bar{D}_i$ is a classical group, although perhaps of a different type than that of $\bar{G}$. As before, the element σ^y stabilizes each $\bar{D}_i$, but it need not be the case that σ^t induces a field automorphism on each $\bar{D}_i$. Possibly σ^t induces a graph-field automorphism on $\bar{D}_i$, $i = 1, \dots, y$. In any case, we now project to $\bar{D}_1$, as before. That is we assume $\langle \bar{X}, \bar{X}^*\rangle = \bar{G}$ and G is defined over $\mathbf{F}_{q^y}$. If $G = \mathrm{SL}(r, q^y)$, then we are done by (4.1). So suppose this is not the case. Then $G = \mathrm{Sp}(r, q^y)$, $\mathrm{SU}(r, q^y)$, or $O^{\pm}(r, q^y)'$.

Consider the group $\bar{G} = \mathrm{Sp}(r, K)$, $\mathrm{SL}(r, K)$, or $O^+(r, K)'$. Then $\bar{P}$ is the stabilizer in $\bar{G}$ of an s-space, V_1, and $(r-s)$-space, V_2, of the natural module V for $\bar{G}$, satisfying $V_1 \leq V_2$. In the symplectic and orthogonal cases V_1 is totally singular with $V_2 = V_1^{\perp}$. Let $\bar{Q} = R_u(\bar{P})$, so that $Q = \bar{Q}_\sigma = O_p(P)$.

First suppose that $r = 2s$. One checks that $\bar{Q}$ is abelian and that the Levi factor of G contains $\mathrm{SL}(s, q^y)$ ($\mathrm{SL}(s, q^{2y})$ if G is a unitary group). It follows from (3.5)(v) and $q \geq 4$ that T contains a cyclic group of order $q^{ys} - 1/q^y - 1$ ($q^{2ys} - 1/q^y + 1$ if G is unitary). Since $\bar{X} = \oplus_{\alpha \in \Delta} \bar{U}_\alpha$, $C_T(X) \leq C_T(\bar{X}) = C_T(\bar{X}^*)$. So $C_T(X) \leq Z(\bar{G})$ and $|X| \geq q^{ys}$ (q^{2ys} in the unitary case). On the other hand, $\sigma = w\tau q^y$, where w is an s-cycle in the Weyl group, S_s, of a Levi factor of $\bar{P}$ and τ is a graph automorphism. It is easily checked that $\tau = 1$ unless G is unitary, in which case $|\tau| = 2$ (for this use the fact that $G \neq O^-(r, q^t)'$, since V_1 is singular). Therefore $|\Delta| \leq s$ ($2s$ in the unitary case) and so $|X| \leq q^{ys}$ (resp. q^{2ys}). Therefore (vi) holds.

From now on assume $r \neq 2s$. Let $\bar{Z}$ be the subgroup of $\bar{Q}$ that is trivial on V_2 and on V/V_1. Then $\bar{Z} \trianglelefteq \bar{Q}$. If $\bar{G}$ is symplectic or orthogonal then $\bar{Q}$ is the product of those root subgroups $\bar{U}_\beta$ having positive coefficient of α_s (temporarily we label the Dynkin diagram starting at the stalk of the diagram of type A_l) and $\bar{Z}$ is the product of those root subgroups $\bar{U}_\beta$ such that β has α_s-coefficient equal to 2. If G is unitary, $\bar{P}$ is the intersection of two maximal parabolics (conjugate under the graph automorphism of $\bar{G}$) and $\bar{Q}$ is the product of root subgroups corresponding to roots having positive coefficient of α_s or $\alpha_t = \alpha_s^\tau$. Moreover, $\bar{Z}$ is the product of root subgroups for roots having both α_s and α_t coefficient positive.

Now $\bar{X} \leq \bar{Q}$ and since $\bar{G} = \langle \bar{X}, \bar{X}^* \rangle$ we cannot have $\bar{X} \leq \bar{Z}$. Another observation is that $\bar{G} \neq \mathrm{Sp}(r, K)$ with $\mathrm{char}(K) = 2$. Otherwise, $\bar{X}$ is generated by roots with α_s-coefficient equal to 1, these being short roots in the root system of type $C_{r/2}$. But $\mathrm{char}(K) = 2$ implies that the collection of all root subgroups for short roots generates a proper subgroup of $\bar{G}$ having type $D_{r/2}$. Consequently, from the description of $\bar{Z}$ and $\bar{Q}$ we can conclude (using the commutator relations) that $\bar{Z} = \bar{Q}' = Z(\bar{Q})$.

We claim that $\bar{X} = \bar{Q}$. In view of the above it will suffice to show that $\overline{XZ} = \bar{Q}$. Let Γ denote the set of root subgroups $\bar{U}_\beta \leq \bar{Q}$ such that $\bar{U}_\beta \nleq \bar{Z}$. Then $\bar{Q}/\bar{Z} \cong \oplus_{\beta \in \Gamma} \bar{U}_\beta$. Similarly $\overline{XZ}/\bar{Z} \cong \oplus_{\beta \in \Delta} \bar{U}_\beta$, so we must show $\Delta = \Gamma$. That is, we require that $\langle w\tau \rangle$ be transitive on Γ. We illustrate the method with $\bar{G} = \mathrm{Sp}(r, K)$; the other cases follow the same argument with only minor changes. The Levi factor $\bar{L}$ of $\bar{P}$ satisfies $\bar{L}' = \bar{L}_1 \times \bar{L}_2$ with $\bar{L}_1 \cong \mathrm{SL}(s, K)$ and $\bar{L}_2 \cong \mathrm{Sp}(r - 2s, K)$. Write $w = w_1 w_2$ with $w_i \in W_i$, the Weyl group of $\bar{L}_i$, for $i = 1, 2$. As $\bar{T}$ is minisotropic in a Levi factor of $\bar{P}$ (with respect to $\sigma = w_1 w_2 q$), we necessarily have w_1 an s-cycle in $W_1 \cong S_s$. Let $j = |w_2|$, $d = (s, j)$, $s = ds_1$, and $j = dj_1$.

Let $\Gamma_1, \ldots, \Gamma_l$ be the orbits of Γ under $\langle w_1 \rangle$. Each orbit has the form $\Gamma_i = \{\beta_i, \beta_i + \alpha_{s-1}, \ldots, \beta_i + \alpha_{s-1} + \cdots + \alpha_1\}$, where β_i is the element of Γ_i having minimal height, and we order so that $\text{ht}(\beta_1) \leq \cdots \leq \text{ht}(\beta_l)$. Then $\beta_1 = \alpha_s$, $\beta_2 = \alpha_s + \alpha_{s+1}$, etc. and $l = r - 2s$. Let M be the $(l \times s)$-matrix with rows $\Gamma_1, \ldots, \Gamma_l$, and let $C_1, \ldots, C_s$ denote the columns of M. The direct sum of the root subgroups in a given row or column affords the usual representation of $\bar{L}_1$, $\bar{L}_2$, respectively. For the rows this is easy. For a column C_j one checks that for each $\beta \in C_j$ there is a unique $\gamma \in C_j$ such that $\beta + \gamma = \delta$ is a root. Moreover, δ depends only on j. Hence, we obtain the natural module for $\bar{L}_2$ by letting the root subgroups $\bar{U}_\beta$ be singular 1-spaces and realize the form via commutators. (If $\bar{G}$ is an orthogonal group then root subgroups corresponding to a given column commute. However, by taking two adjacent columns we obtain a nondegenerate symplectic form, via commutation, and since $\bar{L}_2$ is represented equivalently in the two column spaces we see that $\bar{L}_2$ necessarily preserves an orthogonal form on each.)

Let $E = \oplus_{j=1}^{l} \bar{U}_{\beta_j}$, viewed as the natural module for $\bar{L}_2$, and let E_0 be the subspace spanned by those $\bar{U}_{\beta_j}$ such that $\Delta \cap \Gamma_j \neq \varnothing$. Then E_0 is $\bar{T}\langle w_2 \rangle$-invariant. As $\bar{T}_2 = \bar{T} \cap \bar{L}_2$ is minisotropic (with respect to w_2), we conclude $\text{rad}(E_0) = 1$. Suppose $E_0 < E$. Then $C_{\bar{T}_2}(E_0)$ has positive dimension, which implies $C_{\bar{T}_2}(\bar{X})$ has positive dimension. However, $C_{\bar{T}_2}(\bar{X}) = C_{\bar{T}_2}(\bar{X}^*)$ and $\bar{G} = \langle \bar{X}, \bar{X}^* \rangle$. This is impossible, hence $E_0 = E$ and $j = l = r - 2s$.

If j_1 is odd then $(\bar{T}_2)_\sigma$ has order divisible by $q^{d/2} + 1$, as does $(\bar{T}_1)_\sigma$. Hence $|T/Z(G)|$ is not cyclic. It follows that there exists $a \in T - Z(G)$ and $\mathfrak{B} \in \Delta$ such that $a \in C(\bar{U}_{\mathfrak{B}})$. But then $a \in C_{\bar{T}}(\bar{X}) = C_{\bar{T}}(\bar{X}^*) = C_{\bar{T}}(\bar{G}) = Z(\bar{G})$, a contradiction. So j_1 is even and $w_2^{j/2} \in \langle w_2^d \rangle$. On the other hand, $\langle w^s \rangle = \langle w_2^{ds_1} \rangle = \langle w_2^d \rangle$, hence $w_2^{j/2} \in \langle w^s \rangle$ the latter group leaving each C_i invariant. The element $w_2^{j/2}$ sends each $\mathfrak{B} \in C_i$ to the unique root $\gamma \in C_i$ with $\mathfrak{B} + \gamma$ a root.

Let $\Delta_1, \ldots, \Delta_d$ be the orbits of Δ under $\langle w^d \rangle = \langle w_1^d w_2^d \rangle = \langle w_1^d, w_2^d \rangle$ (a group of order $s_1 j_1$). Fix $\mathfrak{B} \in \Delta_m$ and suppose $\{\mathfrak{B}\} = \Gamma_i \cap C_j$. Let Γ_k be the image of Γ_i under $w_2^{t/2}$. One checks that if $\gamma \in \Delta$ and $\mathfrak{B} \pm \gamma$ is a root, then $\gamma \in \Gamma_i \cup C_j \cup \Gamma_k$. It follows that $\gamma \in \Delta_m$. Setting $\bar{G}_i = \langle \bar{U}_{\pm\beta}$: $\mathfrak{B} \in \Delta_i \rangle$ we conclude that the groups $\bar{G}_1, \ldots, \bar{G}_d$ commute and generate $\bar{G}$. This forces $d = 1$, $(s, l) = 1$, and $\langle w \rangle$ transitive on Γ, as required ($|\Gamma| = sl = sj$).

As $\bar{X} = \bar{Q}$ we have $X = \bar{X}_\sigma = \bar{Q}_\sigma = Q = O_p(P)$, proving (v). (iv) follows from this and (2.3) of **[23]**. Also $1 = (s, l) = (s, r - 2s) = (s, r)$, proving (i). Finally, one checks that $Q' = \bar{Z}_\sigma$, so $X/X' = \bar{Q}_\sigma/\bar{Z}_\sigma = (\bar{Q}/\bar{Z})_\sigma$ which has order given in (ii). This completes the proof of (4.2).

5. The action of T on root subgroups. In this section we are concerned with the action of T on the nilpotent T-root subgroups $X_1, \ldots, X_t$. The results are fundamental to the rest of the paper and are in the spirit of Lemma 3 of [22].

We adopt the following notation. For $1 \leq i \leq t$, let $\bar{Y}_i = \langle \bar{X}_i, \bar{X}_i^* \rangle$, $Y_i = (\bar{Y}_i)_\sigma$, $\bar{P}_i$ the $\bar{T}\langle \sigma \rangle$-stable parabolic subgroup of $\bar{Y}_i$ satisfying $\bar{X}_i \leq R_u(\bar{P}_i)$ (see (3.4)), and $P_i = (\bar{P}_i)_\sigma$. In addition, set $\bar{V}_i = \bar{X}_i R_u(\bar{P}_i)'/R_u(\bar{P}_i)'$ and $V_i = X_i R_u(\bar{P}_i)'/R_u(\bar{P}_i)'$. Except for the cases where $\bar{\Sigma}_i$ has roots of different lengths (G a Suzuki or Ree group), we see from (2.1) that V_i can be regarded as an $\mathbf{F}_q$-module of dimension $|\bar{\Sigma}_i|$.

(5.1) Let $1 \leq i \leq t$.

(i) $\bar{V}_i$ is $\bar{T}\langle \sigma \rangle$-isomorphic to the external direct sum of the root subgroups $\bar{X}_\alpha$ for $\alpha \in \bar{\Sigma}_i$. Also, $V_i = (\bar{V}_i)_\sigma$.

(ii) If G is not a Suzuki or Ree group, then the representation that T induces on $K \otimes_{\mathbf{F}_q} V_i$ is the direct sum of the representations $\varphi_\alpha|_T$, $\alpha \in \bar{\Sigma}_i$.

(iii) If $q > 3$, the T_0 acts irreducibly on the elementary abelian p-group, V_i.

Proof. Let $\bar{\Sigma}_i = \{\gamma_1, \ldots, \gamma_k\}$. The group $R_u(\bar{P}_i)' = \bar{D}_i$ is the product of certain of the root subgroups for $\bar{T}$ and the proof of (3.4)(iii) shows that $\bar{U}_{\gamma_j} \nleq \bar{D}_i$ for $j = 1, \ldots, k$. Therefore, $\bar{V}_i \cong \bar{X}_{\gamma_1} \times \cdots \times \bar{X}_{\gamma_k}$. Let $\bar{J}_i = \bar{X}_i \cap \bar{D}_i$. Then $\bar{J}_i$ is a product of $\bar{T}$-root subgroups, so $\bar{J}_i$ is connected and Lang's theorem implies that $(\bar{X}_i/\bar{J}_i)_\sigma = X_i\bar{J}_i/\bar{J}_i$. So (i) holds. From here we have (ii). For the Suzuki and Ree groups one can obtain (iii) from a direct check of the possible configurations. So we now exclude these cases.

Let $\hat{}$ denote images in $R_u(\bar{P}_i)/\bar{D}_i$. So $\bar{V}_i = \hat{\bar{X}}_{\gamma_1} \times \cdots \times \hat{\bar{X}}_{\gamma_k}$. As Δ_i is a $\langle \sigma \rangle$-orbit, σ^k stabilizes each of the groups $\hat{\bar{X}}_{\gamma_j}$, $j = 1, \ldots, k$, and $\sigma = \tau q$, where τ is an automorphism of $\bar{G}$. Then τ^k induces scalar multiplication and we see that $(\hat{\bar{X}}_{\gamma_j})_{\sigma^k}$ is elementary abelian of order q^k. By (3.5)(i), $\bar{Y}_i$ is the commuting product of a $\langle \sigma \rangle$-orbit of quasisimple groups, so taking projections we may assume that $\bar{Y}_i$ is quasisimple. By induction on $\dim(\bar{G})$ we may assume $\bar{G} = \bar{Y}_i$. Also, we may assume $Z(\bar{G}) = 1$, so $C_{\bar{T}}(\bar{X}_i) \leq C_{\bar{T}}(\bar{Y}_i) = 1$, and $\bar{T}$ acts faithfully on $\bar{X}_i$. On the other hand, the representations of T on $\hat{\bar{X}}_{\gamma_1}, \ldots, \hat{\bar{X}}_{\gamma_k}$ are algebraic conjugates of each other. We conclude that T acts, faithfully, as a cyclic group on $\hat{\bar{X}}_{\gamma_i}$, for $i = 1, \ldots, k$. Taking projections it is clear that it will suffice to show that (iii) holds for the action of T_0 on $(\hat{\hat{X}}_{\gamma_1})_{\sigma^k} = V$. We have $|V| = q^k$, $|T| = d|T_0| = \Phi_{n_1}(q) \cdots \Phi_{n_s}(q)$, where $\Phi_{n_i}(x)$ is the cyclotomic polynomial of degree $\varphi(n_i)$ and d is the order of the center of the universal covering group of G_0. In particular, $d \mid q \pm 1$. We may assume $\bar{G}$ is an adjoint group.

First suppose $T_0 = T$. By (2.1) $\Phi_{n_1}(x) \cdots \Phi_{n_s}(x)$ is the characteristic polynomial of τ in its action on $\mathbf{R} \otimes X(\bar{T})$. So $|\tau| = l$, where $l = \text{l.c.m.}\{n_1, \ldots, n_s\}$. In particular, $k \mid l$. By Zsigmondy **[28]**, for $j = 1, \ldots, s$, $\varphi_{n_j}(q)$ has as a factor a primitive divisor of $q^{n_j} - 1$, unless $\varphi_{n_j}(q) = p + 1 = 2^c$ or 9. We claim that $n_j \mid k$ for all j. In view of the above, this is clear unless there is a unique j with $\varphi_{n_j}(q) = \varphi_2(q) = 2^c$ or 9. But $\varphi_2(q) \mid q^k - 1$ forces k even. So the claim holds. This implies that $l \mid k$; hence $l = k$. As T acts faithfully on V we also have T acting irreducibly on V (viewed as an $\mathbf{F}_p$-space). So (iii) holds.

For the general case we first note that an easy check gives the result if $|T| = \varphi_{n_1}(q) = q \pm 1$. If $n_j \neq 1, 2$, then each primitive divisor of $q^{n_j} - 1$ divides $|T_0|$. Setting $l_0 = \text{l.c.m.}\{n_j \mid n_j \neq 2\}$ we see that T_0 cannot act on $V_0 < V$ with $|V_0| < q^{l_0}$. So supposing T_0 reducible on V, we have l_0 odd, $l = 2l_0$, $V = V_0 + V_1$, T_0 acts irreducibly on V_0 and V_1, and $|V_0| = |V_1| = q^{l_0}$. Since $\varphi_2(q) \mid |T|$, $d \mid q + 1$ and any primitive divisor of $q + 1$ (if such exists) divides d. If G_0 is a unitary group, then $|T|$ is a product of terms $q^c - (1)^c$ (see Carter **[6]**). So l_0 odd forces $l = 2$, and since T is cyclic $|T|$ divides $q^2 - 1$, and we contradict the fact that $q > 3$. From now on we have $d \leq 4$. Since T is contained in a proper parabolic subgroup of G we necessarily have $q - 1 \mid |T|$ and so $|T_0|$ is divisible by $\frac{1}{d}(q - 1)(q + 1)$. However, $|T_0|$ divides $q^{l_0} - 1 = (q - 1)x$ with $(x, q + 1) = 1$. This forces $d = q + 1 = 4$, a contradiction.

The next two lemmas were communicated to the author by R. Steinberg and lead to a much shorter proof of (5.5) than our original one.

(5.2) If $\alpha \in \bar{\Sigma}$ and if ω is a nonzero weight of $\bar{\Sigma}$, then $|\alpha| \leq 2|\omega|$, with equality precisely when $\bar{\Sigma}$ has type C_n, α is a long root, and ω is $\bar{W}$ conjugate to $\frac{1}{2}\alpha$.

Proof. Since $\bar{W}$ acts irreducibly on $Q \otimes X(\bar{T})$ and preserves the form, we may assume $(\omega, \alpha) > 0$. Combining the fact that ω is a weight with the triangle inequality, we have

$$1 \leq 2(\omega, \alpha)/(\alpha, \alpha) \leq 2|\omega|/|\alpha|.$$

This gives the desired inequality. If $|\alpha| = 2|\omega|$, then the second inequality implies $\omega = c\alpha$ with $c > 0$, while the first inequality shows that $c = \frac{1}{2}$. Finally, for $\frac{1}{2}\alpha$ to be a weight we must have α long and $\overline{\Sigma}$ of type C_n.

(5.3) Let $\alpha \neq \beta \in \overline{\Sigma}$, $q > 5$, and assume $G \neq \mathrm{Sz}(8)$ or ${}^2F_4(8)$. Then $\varphi_\alpha|_{T_0} \neq \varphi_\beta|_{T_0}$. This holds for $q = 5$ unless $\overline{\Sigma}$ is of type C_n, $\beta = -\alpha$ a long root, and $\alpha^\sigma = 5\alpha$.

Proof. We may assume $\overline{G}$ is simply connected. Then $X = X(\overline{T})$ can be identified with the lattice of weights of $\overline{\Sigma}$. Let $q_1 = \sqrt{q}$ if G is a Suzuki or Ree group; otherwise set $q_1 = q$. As in §2 σ acts on X, inducing $q_1\tau$ on $\mathbf{R} \otimes X$, where τ is an isometry. Also the argument of (1.7) of **[25]** shows that $X(q_1\tau - 1)$ is the annihilator of $T_0 = \overline{T}_\sigma$. Then $\alpha - \beta = \omega(q_1\tau - 1)$ for some $0 \neq \omega \in X$. Then

$$|\omega(q_1\tau - 1)| = |\alpha - \beta|$$

(1) $\qquad \leq |\alpha| + |\beta|$ (triangle inequality)

(2) $\qquad \leq 4|\omega|$ (by (5.2))

(3) $\qquad \leq (q_1 - 1)|\omega|$ $(q_1 \geq 5)$

(4) $\qquad \leq |\omega q_1\tau| - |\omega|$ (τ is an isometry)

(5) $\qquad \leq |\omega(q_1\tau - 1)|$ (triangle inequality).

Therefore, we have equality at each stage. From equality in (1) we have α, β dependent. Hence $\beta = -\alpha$. From (2) and (5.2) we conclude $\overline{\Sigma}$ has type C_n with α a long root. Equality in (3) yields $q_1 = q = 5$, while equality in (5) gives $\omega q\tau = c\omega$ with $c > 0$. As τ is an isometry, $c = q$ and $\omega\tau = \omega$. The equation $\alpha - \beta = \omega(q_1\tau - 1)$ now gives $\alpha = 2\omega$, so $\alpha^\sigma = 5\alpha$ and the proof of (5.3) is complete.

(5.4) Assume $G \neq \mathrm{Sz}(q)$, ${}^2F_4(q)$, or ${}^2G_2(q)$ and assume $q \neq 2, 3, 4$, or 9. If $\alpha, \beta \in \overline{\Sigma}$ with $\alpha^{p^i}|_{T_0} = \beta|_{T_0}$ for some $1 \leq p^i < q$, then $\alpha|_{T_0} = \beta|_{T_0}$. This also holds for $q = 9$, unless $p^i = 3$, G is of type C_n, and $\beta = -\alpha$, a long root.

Proof. As in (5.3) we may take $\overline{G}$ to be simply connected and we may write $p^i\alpha - \beta = \omega(q\tau - 1)$ for $0 \neq \omega \in X$ and τ an isometry of X. Set $q = p^j$, so that $j > i$. Then $\omega = \beta + p^i(p^{j-i}\omega\tau - \alpha)$, so write $\omega = \beta + p^i\delta$

with $\delta \in X$. As $j > i$ we have $\delta \neq 0$. Replacing ω by $\beta + p^i\delta$ in the equation $p^i\alpha - \beta = \omega(q\tau - 1)$, we obtain $\alpha + \delta = (q\delta + p^{j-i}\beta)\tau$. Then $|\alpha| + |\delta| \geq |q\delta + p^{j-i}\beta| \geq q|\delta| - p^{j-i}|\beta|$, which together with (5.2) yields $(p^j - 1)|\delta| \leq 2(1 + p^{j-i})|\delta|$. We conclude that $p^{j-i}(p^i - 2) \leq 3$.

For $p \geq 5$, this is impossible. Suppose $p = 3$. Here the only possibility is $q = 9$ and $i = 1$. Moreover, all inequalities must by equalities. Using this and (5.2) one checks that $2\delta = \alpha = -\beta$ a long root and G is of type C_n. Finally, assume $p = 2$. Here $i = 1$ and $(q - 1)|\omega| = |q\omega\tau| - |\omega| \leq |\omega(q\tau - 1)| = |2\alpha - \beta| \leq 2|\alpha| + |\beta| \leq 6|\omega|$ the last equality by (5.2). Hence $q \leq 4$, completing the proof of (5.4).

THEOREM (5.5). *Let $1 \leq i < j \leq t$ and assume $q > 5$.*

(i) *V_i and V_j are inequivalent irreducible $\mathbf{F}_p[T_0]$-modules unless $G =$ Sz(8), ${}^2F_4(8)$, or $q = 9$ and G is of type C_n.*

(ii) *If $\langle X_i, X_j \rangle$ is a p-group, then V_i and V_j are inequivalent irreducible $\mathbf{F}_p[T_0]$-modules.*

Proof. We may assume $\bar{G}$ is simply connected. Write $q = p^a$. By (5.1)(iv) each of V_i and V_j is an irreducible $\mathbf{F}_p[T_0]$-module. For the moment exclude Suzuki and Ree groups. Then $\mathbf{F}_q \otimes_{\mathbf{F}_p} V_i = V_i \oplus V_i^p \oplus \cdots \oplus V_i^{p^{a-1}}$, the direct sum of the Galois conjugates of V_i (which is regarded as an $\mathbf{F}_q[T_0]$-module on the right side of the equation). Similarly for V_j. Assume that V_i and V_j are equivalent $\mathbf{F}_p[T_0]$-modules and tensor the equations with K. Then (5.1)(ii) implies that there exist $1 \leq p^k, p^l < p^a$ and roots $\alpha \in \bar{\Sigma}_i$, $\beta \in \bar{\Sigma}_j$ such that $\alpha^{p^k}|_{T_0} = \beta^{p^l}|_{T_0}$. By (5.4) and (5.3) $q = 9$, G has type C_n and $\beta = -\alpha$ is a long root of $\bar{\Sigma}$. So (i) holds in this situation.

Suppose $q = 9$ with $\beta = -\alpha$ a long root and $\bar{\Sigma}$ of type C_n. Then $\langle \bar{X}_i, \bar{X}_j \rangle = \langle \bar{X}_i, \bar{X}_i^* \rangle = \prod_{\gamma \in \bar{\Sigma}_i} \langle \bar{U}_\gamma, \bar{U}_{-\gamma} \rangle$. It is easy to see that $\langle \bar{X}_i, \bar{X}_j \rangle_\sigma \cong \mathrm{SL}(2, q^s)$, where $s = |\bar{\Sigma}_i|$ and that $\langle X_i, X_j \rangle = \langle \bar{X}_i, \bar{X}_j \rangle_\sigma$. Therefore, we have proved (5.5) for all but the Suzuki and Ree groups.

The Suzuki and Ree groups are handled by direct calculation, which we leave to the reader. We observe that for $G = \mathrm{Sz}(q)$, ${}^2G_2(q)$, $T = T_0$ is necessarily a Cartan subgroup of G. If $G = {}^2F_4(q)$ with T a Cartan subgroup, then as in Lemma 3 of [**22**] $C_T(V_i) \neq C_T(V_j)$ unless $X_j = X_i^*$. In this case $\langle X_i, X_j \rangle = L_2(q)$ or $\mathrm{Sz}(q)$ and we are reduced to the above. Assuming T not a Cartan subgroup it follows that T is necessarily the direct product of Z_{q-1} with a minisotropic torus of $L_2(q)$ or $\mathrm{Sz}(q)$.

We remark that the exceptions in (5.5)(i) are real. If $G = \mathrm{Sz}(8)$ and if T is a Cartan subgroup, then T has equivalent representations on $U/Z(U)$

and on $Z(V)$, where U, V are the unique Sylow 2-subgroups normalized by T. This example carries over to ${}^2F_4(8)$. Similarly, $\mathrm{SL}(2,9)$ is an exception, which carries over to $\mathrm{Sp}(2n,9)$ for all $n \geq 1$.

We conclude this section with the following result.

(5.6) Assume that $q \geq 5$, $G \neq \mathrm{Sz}(q)$ or ${}^2F_4(q)$, T is a Cartan subgroup of G, and that $X_i \neq X_j$ are nilpotent T-root subgroups of G. Then V_i and V_j are inequivalent $\mathbf{F}_p[T_0]$-modules unless $q = 5$ or 9, G is of type C_n and X_i, X_j are opposite long root subgroups.

Proof. The proof is just as in the first paragraph of the proof of (5.5).

II. T_0-Invariant subgroups

This chapter will be concerned with general results concerning T_0-invariant subgroups of G.

6. T_0-invariant solvable groups. In this section we consider T_0-invariant solvable subgroups of G and show that for $q > 7$ each such group is the product of a normal T-invariant p-group and part of the normalizer in G of T. Moreover, we show that each T_0-stable p-subgroup of G is a product of a set of T-root subgroups of G.

We maintain the notation in §2. So $T = \overline{T}_\sigma$, $G_0 = O^{p'}(\overline{G}_\sigma)$, and $T_0 = T \cap G_0$. The main result of this section is the following theorem, although there are several other results that will be useful in other sections.

THEOREM (6.1). *Suppose $q > 7$ and $T_0 \leq S \leq G$, with S solvable. Then*
(i) $S = O_p(S)N_S(T_0)$;
(ii) *$O_p(S)$ is the product of T-root subgroups of G.*

This theorem will follow from the other results of this section, several of which are of independent interest.

(6.2) Suppose $q > 7$ and A is a T_0-invariant, abelian, p'-subgroup of $\mathrm{Aut}(G_0)$. Then $A \leq N(T)$.

Proof. Suppose false and take a counterexample so that $|A| \cdot |G_0|$ is minimal. Then A is an r-group for some prime $r \neq p$, $A > N_A(T_0)$, (by (2.7)), and $A/N_A(T_0)$ is an irreducible $\mathbf{F}_r[T_0]$-module. Also, $Z(G_0) = 1$.

Suppose $C_T(A) \neq 1$ and let $1 \neq t \in C_T(A)$. We consider the groups $C_{\bar{G}}(t)$ and $C_G(t)$. Since $\bar{T} \leq C_{\bar{G}}(t)^0$, we have $T \leq (C_{\bar{G}}(t)^0)_\sigma = Y \trianglelefteq C_G(t)$. Let $Y_0 \trianglelefteq Y$ be as described in (2.9). That is $Y_0 = E(Y)X$ and $X = Z(Y_0) = C_T(E(Y))$. Then by (2.9) $Y = Y_0T$ with only diagonal automorphisms induced on each component of $E(Y)$. Write $E(Y) = D_1 \cdots D_k$, a central product of components.

We claim that $A \leq N(D_i)$ for $i = 1,\ldots,k$. For suppose $a \in A$ and $D_i^a = D_j$ for $j \neq i$. If $x \in T_0 \cap D_i$ then $x^{-1}x^a \in A \leq C(A)$. It follows that $D_j^a = D_i$ and $x^a = (x^{-1})^a$ (mod $Z(D_iD_j)$), which forces $(T_0 \cap D_i)Z(D_i)/Z(D_i)$ to be an elementary abelian 2-group. So D_i is neither a Suzuki or Ree group. Suppose D_i is defined over $\mathbf{F}_{q^b}$ and the overlying algebraic group has Lie rank s. Then by (2.1) $T_0 \cap D_i$ has rank at most s (as an abelian group). On the other hand, (2.4)(iii) shows that

$$|T_0 \cap D_i| \geq \frac{1}{d_i}(q^b - 1)^s \geq \frac{q^b - 1}{q^b + 1}(q^b - 1)^{s-1}$$

$$> \left(1 + \frac{2}{q^b + 1}\right)6^{s-1} > \left(\frac{5}{4}\right)6^{s-1} \qquad (\text{as } q > 7).$$

This forces $s = 1$, so $d_i \leq 2$, and we obtain a contradiction. This proves the claim.

For $i = 1,\ldots,k$, let $K_i = D_1 \cdots \hat{D}_i \cdots D_kX$ and $C_i = YA/K_i$. Then $E(C_i) = D_iK_i/K_i$ and $Z_i = (T_0 \cap D_i)K_i/K_i$ is a maximal torus of $E(C_i)$. By minimality Z_i is A-invariant. So if $T_1 = (T_0 \cap D_1) \cdots (T_0 \cap D_k)$ we have $[A, T_1] \leq \bigcap K_i = X$. Therefore, A normalizes T_1X and hence A normalizes $C_Y(T_1X) = T$ (see (2.8)(i)). This contradiction shows that $C_T(A) = 1$.

Let $A_1 = [A, T_0]$, so $A_1 \leq G_0$. Suppose $A_1 < A$. By induction $A_1 \leq N(T_0)$, so $[A_1, T_0] \leq A_1 \cap T_0 \leq C_T(A) = 1$. By (2.8) $A_1 \leq T_0$, whence $A_1 \leq C_T(A) = 1$. Hence $A = A_1$. The same argument shows $N_A(T_0) = 1$. Therefore, T_0 acts faithfully and irreducibly on A. In particular, AT_0 is a Frobenius group and $A \leq G_0$.

Consider the action of AT_0 on the Lie algebra, M, of $\bar{G}$. Viewing M as a $K[AT_0]$-module and using Clifford's theorem we see that $M|_{T_0}$ contains the regular representation of T_0. So if $\bar{G}$ has Lie rank n we have the inequality $\dim(M) \geq |T_0| \geq d^{-1}(q - 1)^n$ (by (2.4)). Use the fact that $q - 1 > 6$ and the known values of d to obtain a contradiction. This proves (6.2).

(6.3) Assume $q > 7$ and let T_1 be a maximal torus of G. Suppose $T_1 \cap G_0 \leq N_G(T_0)$. Then $T_1 = T$. In particular, T_0 is weakly closed in $N_G(T_0)$.

Proof. We may assume $Z(G_0) = 1$. Suppose $T_0 \neq T_2 = T_1 \cap G_0 \leq N_{G_0}(T_0)$. By (6.2) $T_0 \leq N(T_2)$. Hence $[T_0, T_2] \leq T_0 \cap T_2 \leq Z(T_0T_2)$. If $T_0 \cap T_2 = 1$, then $T_2 \leq C_{G_0}(T_0) = T_0$ by (2.8). Similarly, $T_0 \leq T_2$. So we assume $T_0 \cap T_2 \neq 1$.

Let $C = C_{\bar{G}}(T_0 \cap T_2)^0$. Then $\bar{T}, \bar{T}_1 \leq C$ and $C = \bar{D}Z$, where $\bar{T}_1 = C_{\bar{G}}(T_1)^0$ (a maximal torus), $\bar{D} = E(C)$, and $Z = Z(C) \leq \bar{T} \cap \bar{T}_1$. Note that $\bar{T} \neq \bar{T}_1$ forces $D \neq 1$. Let $D = O^{p'}(\bar{D}_\sigma)$, $T_3 = T_0 \cap D$, and $T_4 = T_2 \cap D$. By (2.5) T_3 and T_4 are maximal tori of D. Since $T_4 \leq N(T_3)$, we conclude, inductively, that $T_4 = T_3$. But then $T_4 = T_3 \leq T_0 \cap T_2 \leq Z(C)$, which contradicts (2.8).

A useful consequence of (6.3) is the following

(6.4) Assume $q > 7$ and let $T_0 \leq P = \bar{P}^\sigma$, where $\bar{P}$ is a parabolic subgroup of $\bar{G}$. Then $\bar{T} \leq \bar{P}$.

Proof. Suppose $T_0 \leq P = \bar{P}^\sigma$. By (5.16) of **[25]** there is a σ-invariant maximal torus $\bar{T}_1$ of $\bar{P}$ such that $T_0 \leq N(\bar{T}_1)$. Then $T_0 \leq N((\bar{T}_1)_\sigma)$ and (6.3) implies $T_0 \leq (\bar{T}_1)_\sigma$. But then $\bar{T}_1 \leq C_{\bar{G}}(T_0)^0 = \bar{T}$. Therefore, $\bar{T} = \bar{T}_1 \leq \bar{P}$.

(6.5) Suppose $q > 7$ and $T_0 \leq N_{G_0}(S)$, where S is a solvable p'-subgroup of G. Then $S \leq N_{G_0}(T_0)$.

Proof. Let S be a minimal counterexample and S/S_0 a chief factor of ST_0. Then $S_0 \leq N(T_0)$. If $[T_0, S] \leq S_0$, then $S_0T_0 \trianglelefteq ST_0$. But $S_0T_0 \leq N(T_0)$, so (6.3) implies that $S \leq N(T_0)$. Thus, we may assume that $[T_0, S/S_0] = S/S_0$, and by minimality, $[T_0, S] = S$. In particular $S \leq G_0$. If $S_0 = 1$ then S is abelian and we are done by (6.2). Suppose then, that $S_0 \neq 1$ and let S_1 be a minimal normal subgroup of ST_0 with $S_1 \leq S_0$. By (6.2) $S_1 \leq N(T_0)$, so $[S_1, T_0, T_0] = 1$. Say $|S_1| = r^a$, with r a prime. Then $O_{r'}(T_0) \leq C(S_1)$.

We claim that $Z(ST_0) \neq 1$. Suppose otherwise. If $O_{r'}(T_0) \nleq C(S/S_0)$, then $[O_{r'}(T_0), S]$ covers S/S_0 and by minimality, $[O_{r'}(T_0), S] = S$. Then $S \leq \langle O_{r'}(T_0)^{ST_0} \rangle \leq C(S_1)$ and T_0 acts irreducibly on S_1. Then $[S_1, T_0, T_0] = 1$ implies $Z(ST_0) \neq 1$. Therefore, $O_{r'}(T_0) \leq C(S/S_0)$. This means that $O_{r'}(T_0)S_0 \trianglelefteq O_{r'}(T_0)S$, and since $O_{r'}(T_0) \leq O_{r'}(S_0T_0)$, we conclude that either $O_{r'}(T_0) = 1$ or $O_{r'}(T_0) \leq O_{r'}(ST_0) \neq 1$. In the latter case, let Y be minimal normal in ST_0 with Y an r'-group contained in $O_{r'}(T_0)S_0$. Then $[Y, T_0, T_0, T_0] \leq [S_0, T_0, T_0] \leq [S_0 \cap T_0, T_0] = 1$. Thus, $C(Y) \geq \langle O_r(T_0)^{ST_0} \rangle$. But $T_0/C_{T_0}(S/S_0)$ is an r-group, and this forces $C(Y) \geq S$. As above, this yields $Z(ST_0) \neq 1$. Therefore, we assume that $O_{r'}(T_0) = 1$ and T_0 induces a cyclic r-group on S/S_0.

Since $[S_1, T_0, T_0] = 1$ we apply Theorem B of Hall-Higman (see p. 359 of **[13]**) and conclude that $T_0/C_{T_0}(S_1)$ has exponent 2 or 3. In particular, $T_0/C_{T_0}(S/S_0) \cong Z_2$ or Z_3. Let $T_1 = C_{T_0}(S/S_0)$. Then $T_1 \leq O_r(T_1S_0)$ and $T_1S_0 \trianglelefteq ST_0$. Since S_1 is minimal normal in ST_0 we have $S_1 \leq Z(O_r(T_1S_0))$ and $[T_1, S_1] = 1$. If $r = 2$ let g be a 2-element in $ST_0 - S_0T_0$. If $r = 3$, then S/S_0 is an elementary abelian 2-group (this follows from the proof of Theorem B of Hall-Higman) and we let $g \in S - S_0$. In either case $\langle S_0, T_0, g\rangle = ST_0$. Therefore, $T_1 \cap T_1^g \trianglelefteq ST_0$. If $T_1 \cap T_1^g \neq 1$, then we may take $S_1 \leq T_1 \cap T_1^g$ and obtain $T_0 \leq C(S_1)$. This would imply $ST_0 \leq \langle T_0^{ST_0}\rangle \leq C(S_1)$, a contradiction. Therefore, $T_1 \cap T_1^g = 1$ and $[T_1, T_1^g] \leq T_1 \cap T_1^g = 1$.

Let $b = | T_1^g |$. Then $b \geq r^{-1} | T_0 | \geq (dr)^{-1}(q-1)^n$, where n is the Lie rank of $\bar{G}$ (here we use (2.4)(iii) and note that the numerical restrictions rule out Suzuki and Ree groups). On the other hand, $T_1^g \leq N(T_0)$, while $T_0 \cap T_1^g = 1$. So by (2.7) we may regard T_1^g as an abelian r-subgroup of $\bar{W}$, the Weyl group of $\bar{G}$. We leave it to the reader to check that the assumption $q > 7$ leads to a contradiction. (In this check the following inequality is useful. For A an abelian r-subgroup of S_{m+1} we have $| A | \leq r^{(m+1)/r}$. To see this let $o_1, \ldots, o_l$ be the orbits of A with $| o_i | = r^{k_i}$. Then $m + 1 = \Sigma\, r^{k_i}$ and $| A | \leq \Pi\, r^{k_i}$ (as A is abelian). Since $r^{k_i} \geq rk_i$ we have $m + 1 \geq \Sigma\, rk_i = r(\Sigma\, k_i)$, and the inequality follows.) This proves the claim, hence $Z(ST_0) \neq 1$.

Choose $1 \neq x \in Z(ST_0)$ and consider ST_0 as a subgroup of $C_G(x)$. By (2.8) $x \in C_{G_0}(T_0) = T_0$, and so $T \leq (C_{\bar{G}}(x)^0)_\sigma = Y$. Also, $Y = Y_0T$, where $Y_0 = E(Y)$. Since $[S, T_0] = S$ we have $S \leq Y_0 = D_1 \cdots D_k$, where $D_1, \ldots, D_k$ are the components of Y_0. Fix $i \in \{1, \ldots, k\}$ and let bars denote images in Y_0T_0 modulo $D_1 \cdots \hat{D}_i \cdots D_k$. Then $\bar{S}$ is normalized by $\bar{T}_0$, hence by $\overline{T_0 \cap Y_0}$. By induction, $\bar{S} \leq N(\overline{T_0 \cap Y_0})$. Therefore, $\bar{S} \leq N(C_{\overline{Y_0T_0}}(\overline{T_0 \cap Y_0})) = N(\bar{T}_0)$ and $[S, T_0] \leq D_1 \cdots \hat{D}_i \cdots D_k$. Repeating this for each i we conclude $S = [S, T_0] \leq Z(Y_0)$, and finally $[S, T_0] = 1$. This is a contradiction proving (6.5).

The next result completes the proof of (6.1)(i).

(6.6) Suppose $q > 7$ and let $T_0 \leq S \leq G$, with S solvable. Then $S = O_p(S)N_S(T_0)$.

Proof. Let S be a minimal counterexample. Suppose $L \triangleleft S$ with $T_0 \leq L$ and let X be a Hall p'-subgroup of L with $T_0 \leq X$. Then $S = LN_S(X) = LN_S(T_0)$ by (6.3) and (6.5). By minimality, $L = O_p(L)N_L(T_0)$, so $S = O_p(L)N_S(T_0) = O_p(S)N_S(T_0)$. We conclude that $S = \langle T_0^S \rangle$. In particular, $S \leq G_0$.

Let $NO_p(S)/O_p(S)$ be minimal normal in $S/O_p(S)$, where N is a p'-group. Then $S = O_p(S)N_S(N)$ and we may assume $T \leq N_S(N)$ (take N in a Hall p'-group containing T_0). By minimality, we conclude that $S = N_S(N)$. Let $C = C_S(N)$. Suppose, $C = S$ and let $1 \neq x \in C$. Then $x \in C_{G_0}(T_0)$ implies $x \in T_0$, so $T \leq (C_{\bar{G}}(x)^0)_\sigma = Y$. We have $Y = Y_0T$, where $Y_0 = D_1 \cdots D_k Z$ as in (2.9). Since $S = \langle T_0^S \rangle$, we conclude $S \leq Y_0T_0$. Fix $1 \leq i \leq k$ and let bars denote images in Y modulo $D_1 \cdots \hat{D}_i \cdots D_k Z$. Set $T_1 = T_0 \cap D_1 \cdots D_k Z$ and $S_1 = \langle T_1^S \rangle$. By minimality and (6.3), $\bar{S}_1 = O_p(\bar{S}_1)N_{\bar{S}_1}(\bar{T}_1) = O_p(\bar{S}_1)(\bar{T}_1)$. It follows that $S_1' \leq J$, where $JZ/Z = O_p(S_1Z/Z)$. Therefore, $S_1 = O_p(S_1)T_1$ and $S = S_1N_S(T_1) \leq O_p(S_1)N_S(C_{Y_0}(T_1))$. Since $N_S(C_{Y_0}(T_1)) = T_0N_{S\cap Y_0}(T_1) = T_0$, we obtain $S = O_p(S_1)T_0 = O_p(S)T_0$, which we are assuming false.

In view of the above, it will suffice to show that $C = S$. So assume $C \lhd S$. Since $T_0 \leq N_{G_0}(N)$, we have $[N, T_0, T_0] = 1$, by (6.2). Therefore, $T_0C \neq S$. Since $S = \langle T_0^S \rangle$ we may choose $C \leq K < L < S$ with $K, L \lhd S$ and such that $LT_0 = S$, L/K is a chief factor of S and $T_0 \cap L \leq K$. (For example, set $L/C = (S/C)'$ and K/C any maximal normal subgroup of L/C). Minimality of S implies $KT_0 = O_p(KT_0)N_{KT_0}(T_0) = O_p(K)N_{KT_0}(T_0)$.

Let X be a p'-Hall subgroup of K with $X^{T_0} = X$ and set $T_1 = T_0 \cap K \leq X$. Let $Y = X \cap O_{pp'}(K)$. Then $S = O_p(K)N_S(Y)$, so minimality of S forces $Y \trianglelefteq S$. Since $KT_0 = O_p(K)N_{KT_0}(T_0)$ we conclude that $T_1 \trianglelefteq K$. At this point we are in a position to use the argument in the proof of (6.5). We have seen that $O_{p'}(Z(S)) = O_{p'}(Z(LT_0)) = 1$. Replace the groups S, S_0, S_1 of (6.5) by L, K, N, respectively. Arguing as in (6.5) we first obtain (via Hall-Higman, Theorem B) that $T_0C/C \cong Z_2$ or Z_3, and then argue that T_0 is an r-group for some prime r. Finally we obtain a numerical contradiction. This completes the proof of (6.6).

To obtain (6.1)(ii) we must consider T_0-invariant unipotent subgroups of G_0. A key result is the following.

(6.7) Let $q > 5$ and assume that $\langle \bar{X}_i, \bar{X}_j \rangle$ is unipotent. Then $[\bar{X}_i, \bar{X}_j]_\sigma = [X_i, X_j]$.

Proof. Let $\bar{L} = [\bar{X}_i, \bar{X}_j]$ and set $L = [X_i, X_j]$. So $L \leq \bar{L}_\sigma$. As $\bar{L}$ is $\bar{T}$-invariant, $\bar{L}$ is a product of $\bar{T}$-root subgroups of $\bar{G}$, and since $\bar{L}$ is σ-invariant these root subgroups fall into $\langle \sigma \rangle$-orbits. The first observation implies $\bar{L} = \bar{L}^0$. If $\bar{L} = 1$, then the result is trivial, so we assume $\dim(\bar{L}) > 0$. Set $\bar{X} = \langle \bar{X}_i, \bar{X}_j \rangle$.

Suppose there is a normal subgroup, $\bar{A}$ of $\bar{X}$, such that $\bar{A} = \bar{A}^\sigma$, $\bar{A}$ is a product of root subgroups of $\bar{T}$, and $1 \neq \bar{L}\bar{A}/\bar{A} = Z(\bar{X}/\bar{A}) = \bar{X}_w\bar{A}/\bar{A}$ for some $w \in \{1,\ldots,t\}$. We claim that $\bar{L}_\sigma\bar{A}/\bar{A} = \bar{L}\bar{A}/\bar{A}$. Let $\hat{}$ denote images in $\bar{X}/\bar{A}$. Then $\hat{\bar{L}} = \hat{\bar{X}}_w$ and $\hat{\bar{X}}_w$ is $\bar{T}$-isomorphic to $\bar{V}_w = \bar{X}_w R_u(\bar{P}_w)'/R_u(P_w)'$. Hence $\hat{L}_\sigma = \hat{X}_w$ (Lang's theorem) is T-isomorphic to V_w. By (5.1) T acts irreducibly on V_w, so it will suffice to show that $\hat{L} \neq 1$. For this it will suffice to check that there exist elements $a \in X_i$ and $b \in X_j$ such that $[\hat{a}, \hat{b}] \neq 1$.

Since $\hat{\bar{L}} = \hat{\bar{X}}_w$, it is not the case that $[\hat{\bar{U}}_\delta, \hat{\bar{U}}_\beta] = 1$ for each $\delta \in \Delta_i$ and $\beta \in \Delta_j$. Therefore, choose $\delta \in \Delta_i$ and $\beta \in \Delta_j$ with $1 \neq [\hat{\bar{U}}_\delta, \hat{\bar{U}}_\beta] \leq \hat{\bar{U}}_w$. Interchanging δ and β, if necessary, we may assume that there is an integer $d = 1, 2$, or 3 such that $\delta + d\beta \in \bar{\Sigma}_w$ ($d = 1$ if δ, β are both long roots) and $[\hat{\bar{U}}_\delta, \hat{\bar{U}}_\beta] \geq \hat{\bar{U}}_{\delta+d\beta}$. Let $|\Delta_i| = l$ and $|\Delta_j| = m$. Then $\Delta_i = \{\delta_1,\ldots,\delta_l\}$ and $\Delta_j = \{\beta_1,\ldots,\beta_m\}$, where $\delta_1 = \delta$, $\beta_1 = \beta$, $\delta_i = \delta_{i-1}^\tau$ for $2 \leq i < l$, and $\beta_i = \beta_{i-1}^\tau$ for $2 \leq i \leq m$ (here τ is the permutation of $\bar{\Sigma}$ associated with σ).

From Lang's theorem we have $(\bar{X}_i/\bar{X}_i')_\sigma = X_i\bar{X}_i'/\bar{X}_i'$ and $(\bar{X}_j/\bar{X}_j')_\sigma = X_j\bar{X}_j'/\bar{X}_j'$. Moreover, the 3-subgroup lemma shows that $1 = [\hat{\bar{X}}_i', \hat{\bar{X}}_j] = [\hat{\bar{X}}_i, \hat{\bar{X}}_j']$. Let $a \in X_i$ and $b \in X_j$. There are elements $x, y \in K$ and $c_2,\ldots,c_l$, $d_2,\ldots,d_m \in K^*$ such that $a = \bar{U}_{\delta_1}(x)\bar{U}_{\delta_2}(c_2x^q) \cdots \bar{U}_{\delta_l}(c_lx^{q^{l-1}}) \pmod{\bar{X}_i'}$ and $b = \bar{U}_{\beta_1}(y)\bar{U}_{\beta_2}(d_2y^q) \cdots \bar{U}_{\beta_m}(d_my^{q^{m-1}}) \pmod{\bar{X}_j'}$ (slightly different for the Suzuki and Ree groups). There are q^l choices for x and q^m choices for y.

For each $\delta_u \in \Delta_i$ there exists at most one $\beta_v \in \Delta_j$ such that $[\bar{U}_{\delta_u}, \bar{U}_{\beta_v}] \geq \bar{U}_{\delta+d\beta}$. The projection of $[\hat{a}, \hat{b}]$ to $\hat{\bar{U}}_{\delta+d\beta}$ is $\hat{\bar{U}}_{\delta+d\beta}(h)$, where $h = \sum zc_ud_v^dx^{q^{u-1}}y^{dq^{v-1}}$, and the sum ranges over the pairs (u, v) for which $[\bar{U}_{\delta_u}, \bar{U}_{\beta_v}] \geq \bar{U}_{\delta+d\beta}$, and z is an integer with $(z, p) = 1$ ($z = \pm 1$ if δ, β and $\delta + \beta$ are all the same length). Fix $y \neq 0$ and for each pair (u, v) let $e_u = zc_ud_v^dy^{dq^{v-1}}$. Then $h = f(x)$, where $f(t) = \sum_u e_ut^{q^{u-1}}$, a non-zero polynomial of degree at most q^{l-1}. There are q^l choices for x, so we may choose x with $h = f(x) \neq 0$. So for suitable choice of x and y we see that $[\hat{a}, \hat{b}]$ has nontrivial projection to $\hat{X}_{j_k}$. We have now proved the claim.

We now claim that if $\bar{A} = \bar{A}^\sigma$ is a product of root subgroups of $\bar{X}$ with $\bar{A} \trianglelefteq \bar{X}$, then $L\bar{A}/\bar{A} = \bar{L}_\sigma\bar{A}/\bar{A}$. This is proved by induction on $\dim(\bar{X}/\bar{A})$. If this dimension is 0 the claim is obvious. So assume the claim holds for all $\bar{A}_1$ with $\bar{A}_1$ satisfying the conditions that $\bar{A}$ satisfies and $\dim(\bar{X}/\bar{A}_1) < \dim(\bar{X}/\bar{A})$.

Suppose that for $i = 1, 2$ there exist $\bar{A} < \bar{A}_i = \bar{A}_i^\sigma \leq \bar{X}$ such that $\bar{A}_i/\bar{A} \leq Z(\bar{X}/\bar{A})$ and $\bar{A}_i = \bar{A} \cdot \bar{X}_{t_i}$ for $t_i \in \{1,\ldots,t\}$. Also, suppose $\bar{A}_1 \neq \bar{A}_2$. By the induction hypotheses $L\bar{A}_i/\bar{A}_i = \bar{L}_\sigma\bar{A}_i/\bar{A}_i$, for $i = 1, 2$. Consider $\bar{A}_1\bar{A}_2/\bar{A}$. Then $\bar{A}_1\bar{A}_2/\bar{A} \cong \bar{V}_{t_1} \times \bar{V}_{t_2}$ (a $\bar{T}\langle\sigma\rangle$-isomorphism). By (5.5) and (5.1) T has inequivalent irreducible representations on V_{t_1} and V_{t_2}. Moreover, for $i = 1,2$ V_{t_i} is T-isomorphic to $(\bar{A}_i/\bar{A})_\sigma = X_{t_i}\bar{A}/\bar{A}$ (for the equality use Lang's theorem and the fact that $\bar{X}_{t_i} \cap \bar{A}$ is a product of $\bar{T}$-root subgroups, hence connected). Therefore,

$$L\bar{A}/\bar{A} \cap \bar{A}_1\bar{A}_2/\bar{A} = L\bar{A}/\bar{A} \cap (\bar{A}_1)_\sigma(\bar{A}_2)_\sigma\bar{A}/\bar{A}$$
$$= (L\bar{A}/\bar{A} \cap X_{t_1}\bar{A}/\bar{A})(L\bar{A}/\bar{A} \cap X_{t_2}\bar{A}/\bar{A}).$$

Now,

$$L\bar{A}_1/\bar{A}_1 = \bar{L}_\sigma\bar{A}_1/\bar{A}_1,$$
$$L \leq \bar{L}_\sigma, \quad |\bar{L}_\sigma\bar{A}/\bar{A}| = |\bar{L}_\sigma\bar{A}_1/\bar{A}_1| \cdot |\bar{L}_\sigma\bar{A}/\bar{A} \cap \bar{A}_1/\bar{A}|,$$

and $\bar{L}_\sigma\bar{A}/\bar{A} \cap \bar{A}_1/\bar{A} = 1$ or $X_{t_1}\bar{A}/\bar{A}$. If the claim is false, we must have the latter case, but $L\bar{A}/\bar{A} \cap X_{t_1}\bar{A}/\bar{A} = 1$. Passing modulo A_2 we have a contradiction. We now suppose that no such groups A_1, A_2 exist.

Let $\bar{Z}/\bar{A} = Z(\bar{X}/\bar{A})$. Then $\bar{Z}^\sigma = \bar{Z}$ and $\bar{Z}$ is a product of root subgroups of $\bar{T}$. By the above, $\bar{Z} = \bar{A} \cdot \bar{X}_w$ for some $w \in \{1,\dots,t\}$. By the first claim we may assume $\bar{L} \cdot \bar{A}/\bar{A} \neq \bar{Z}/\bar{A}$. As $\dim(\bar{X}/\bar{Z}) < \dim(\bar{X}/\bar{A})$ we have $L\bar{Z}/\bar{Z} = \bar{L}_\sigma\bar{Z}/\bar{Z}$. Also, the usual arguments show that $(\bar{Z}/\bar{A})_\sigma = X_w\bar{A}/\bar{A}$. So either $\bar{L}_\sigma\bar{A}/\bar{A} \cap X_w\bar{A}/\bar{A} = 1$, and we are done by order considerations, or $X_w\bar{A}/\bar{A} \leq \bar{L}_\sigma\bar{A}/\bar{A}$. We assume the latter holds. Then $\bar{L} \geq \bar{X}_w$ and so $\bar{L}\bar{A} > \bar{Z}$. Let $\bar{A}_1 \leq \bar{X}\bar{T}\langle\sigma\rangle$ be such that $\hat{\bar{Z}} < \hat{\bar{A}}_1 \leq \bar{L}$ and $\bar{A}_1/\bar{Z}$ is an $\bar{X}\bar{T}\langle\sigma\rangle$-chief factor. Since $\bar{A}_1$ is a product of $\bar{T}$-root subgroups, we must have $\bar{A}_1 = \bar{X}_k\bar{Z}$ for some k. Since $\bar{X}_k\bar{A}/\bar{A} \nleq Z(\bar{X}/\bar{A})$ either $[\bar{X}_i, \bar{X}_k]$ or $[\bar{X}_j, \bar{X}_k]$ is not contained in $\bar{A}$. With no loss of generality we suppose $[\bar{X}_i, \bar{X}_k] \nleq \bar{A}$. Hence $[\bar{X}_i, \bar{X}_k]\bar{A}/\bar{A} = \bar{X}_w\bar{A}/\bar{A}$.

Now $\dim([\bar{X}_i, \bar{X}_k]) < \dim(\bar{L})$, so by induction $[\bar{X}_i, \bar{X}_k]_\sigma = [X_i, X_k]$. Moreover, $\bar{X}_k \leq \bar{L}$ implies that $X_k\bar{Z}/\bar{Z} \leq \bar{L}_\sigma\bar{Z}/\bar{Z} = L\bar{Z}/\bar{Z}$. Therefore,

$$L\bar{A}/\bar{A} \geq [X_i, L]\bar{A}/\bar{A} \geq [X_i, X_k]\bar{A}/\bar{A} = [\bar{X}_i, \bar{X}_k]_\sigma\bar{A}/\bar{A} = X_w\bar{A}/\bar{A}.$$

At this point the equality $L\bar{Z}/\bar{Z} = \bar{L}_\sigma\bar{Z}/\bar{Z}$ and order considerations, imply that $L\bar{A}/\bar{A} = \bar{L}_\sigma\bar{A}/\bar{A}$, proving the claim. The result follows by setting $\bar{A} = 1$.

(6.8) Suppose $q > 5$ and $1 \leq i \leq t$. Then
(i) $X_i' = X_i \cap R_u(\bar{P}_i)'$.
(ii) $\bar{X}_i' = \bar{X}_i \cap R_u(\bar{P}_i)'$.
(iii) $V_i \cong X_i/X_i'$ as $\mathbf{F}_p[T]$-modules.
(iv) $\bar{V}_i \cong \bar{X}_i/\bar{X}_i'$ as $K[\bar{T}]$-modules.

Proof. By (6.7), $[X_i, X_\iota] = [\bar{X}_i, \bar{X}_\iota]_\sigma$, so it will suffice to show that $[\bar{X}_i, \bar{X}_\iota] = \bar{X}_\iota \cap R_u(\bar{P}_\iota)'$. Let $\bar{Y}_\iota = [\bar{X}_i, \bar{X}_\iota]$. Then $\bar{Y}_i$ is $\bar{T}$-invariant, hence a product of $\bar{T}$-root subgroups. The group $\bar{X}_i/\bar{Y}_\iota$ is then $K[\bar{T}]$-isomorphic to the direct product of those root subgroups of $\bar{X}_i$ not contained in $\bar{Y}_\iota$. These root subgroups fall into orbits under $\langle\sigma\rangle$. By definition $\bar{X}_i$ is generated by one such orbit. This $\bar{Y}_\iota = \bar{X}_i \cap R_u(\bar{P}_\iota)'$, proving the result.

The next result will complete the proof of (6.1).

(6.9) Let $A = A^{T_0}$ be a p-subgroup of G_0 and assume $q > 7$. Then

(i) A is a product of T-root subgroups.

(ii) Let $\{C_1,\ldots,C_k\}$ be the composition factors in a fixed AT-composition series for A. For each i there exists a unique $n_i \in \{1,\ldots,t\}$ such that $C_\iota \cong V_{n_\iota}$ as $\mathbf{F}_p[T]$-modules.

(iii) Let $n_1,\ldots,n_k$ be as in (ii). Then $A = X_{n_1} \cdots X_{n_k}$ and if $\bar{A} = \bar{X}_{n_1} \cdots \bar{X}_{n_k}$, then $\bar{A}$ is a subgroup of $\bar{G}$ with $\bar{A}_\sigma = A$.

(iv) Let $n_1,\ldots,n_k$ be as in (ii). Then $\{n_1,\ldots,n_k\} = \{j \mid X_j \leq A\}$.

Proof. Let $1 \neq A$ be a p-subgroup of G_0 with $A = A^{T_0}$. By (3.9) of **[4]** there is a canonical parabolic subgroup $\bar{P}$ of $\bar{G}$ such that $A \leq \bar{Y} = R_u(\bar{P})$ and $N_G(A) \leq \bar{P}$. Then $T_0 \leq \bar{P}$, so by (6.4) $\bar{T} \leq \bar{P}$. Also, $\bar{P} = \bar{P}^0$.

Let $1 = \bar{Y}_0 < \bar{Y}_1 < \cdots < \bar{Y}_k = \bar{Y}$ be a $\bar{T}\langle\sigma\rangle$-composition series for $\bar{Y}$. Then each $\bar{Y}_\iota$ is a product of $\bar{T}$-root subgroups of $\bar{G}$, and for $i = 1,\ldots,k$, $M_i = \bar{Y}_\iota/\bar{Y}_{i-1}$ is $\bar{T}\langle\sigma\rangle$-isomorphic to the external direct product of the root groups in some $\langle\sigma\rangle$-orbit of roots, say Σ_{n_ι}. Hence, $\bar{Y}_\iota = \bar{X}_{n_\iota}\bar{Y}_{\iota-1}$. Recall that for $i = 1,\ldots,k$, $\bar{V}_{n_i} = \bar{X}_{n_\iota}R_u(\bar{P}_{n_\iota})'/R_u(\bar{P}_{n_\iota})'$. Then $M_\iota \cong \bar{V}_{n_\iota}$. Also, Lang's theorem implies that $(M_i)_\sigma = X_\iota\bar{Y}_{\iota-1}/\bar{Y}_{\iota-1} \cong V_{n_\iota}$. By order consideration we have $Y = \bar{Y}_\sigma = X_{n_1} \cdots X_{n_k}$. This shows that $X_{n_1},\ldots,X_{n_k}$ are T-root subgroups satisfying the following conditions: (i) $A \leq X_{n_1} \cdots X_{n_k} = Y$; (ii) $X_{n_1} \cdots X_{n_\iota} \trianglelefteq Y$ and $\bar{X}_{n_1} \cdots \bar{X}_{n_\iota} \trianglelefteq \bar{Y}$ for $i = 1,\ldots,k$; (iii) $\bar{X}_{n_1} \cdots \bar{X}_{n_\iota}/\bar{X}_{n_1} \cdots \bar{X}_{n_{\iota-1}} \cong \bar{V}_{n_\iota}$ and $X_{n_1} \cdots X_{n_\iota}/X_{n_1} \cdots X_{n_{\iota-1}} \cong V_{n_\iota}$ for $i = 1,\ldots,k$. Among all sets of T-root subgroups $\{X_{l_1},\ldots,X_{l_m}\}$ that satisfy (i), (ii), and (iii), choose one such that $|X_{l_1} \cdots X_{l_m}|$ is minimal. We claim that $A = X_{l_1} \cdots X_{l_m}$.

Let $L = X_{l_1} \cdots X_{l_m}$ and $L_\iota = X_{l_1} \cdots X_{l_\iota}$ for $1 \leq i \leq m$. Similarly, set $\bar{L} = \bar{X}_{l_1} \cdots \bar{X}_{l_m}$ and $\bar{L}_\iota = \bar{X}_{l_1} \cdots \bar{X}_{l_\iota}$. Suppose $A < L$. Then for some $iA \cap L_{l_\iota} \leq L_{l_{\iota-1}}$ (i.e. A avoids the LT composition factor L_ι/L_{i-1}). Choose i maximal for this. By minimality of $|L|$, $i < m$. Also, $AL_\iota = L$. Now, $L/L_{\iota-1} = AL_{\iota-1}/L_{\iota-1} \times L_i/L_{\iota-1}$ and this will be a contradiction to minimality if we can show that $AL_{\iota-1}$ is a product of root subgroups satisfying the necessary conditions. To see this consider $\bar{L}/\bar{L}_{i-1}$. Suppose $j, k \geq i$ and consider $[\bar{X}_{l_j}, \bar{X}_{l_k}] \cap \bar{L}_i = \bar{I}$. The group $\bar{I}$ either covers or avoids the $\bar{L}\cdot\bar{T}\langle\sigma\rangle$ composition factor $\bar{L}_i/\bar{L}_{i-1}$ and $\bar{I}$ is a product of $\bar{T}$-root subgroups. So if $\bar{I}$ covers $\bar{L}_i/\bar{L}_{i-1}$, then $\bar{X}_{l_i} \leq [\bar{X}_{l_j}, \bar{X}_{l_k}]$. Consequently, (6.7) implies that $X_{l_i} \leq [X_{l_j}, X_{l_k}]$. But $AL_i = L$ and $L_i/L_{i-1} \leq Z(L/L_{i-1})$. So this forces $L_i/L_{i-1} = X_{l_i}L_{i-1}/L_{i-1} \leq A'L_{i-1}/L_{i-1}$, whereas we have assumed that A avoids L_i/L_{i-1}. Therefore, $\bar{I}$ avoids $\bar{L}_i/\bar{L}_{i-1}$.

Letting $l_j = l_k = l_{i+1}$ we see that $\bar{L}_{i+1}/\bar{L}_{i-1}$ is abelian. Since $\bar{L}_{i+1}$ is a product of root subgroups we have $\bar{L}_{i+1}/\bar{L}_{i-1} = (\bar{X}_{l_{i+1}}\bar{L}_{i-1}/\bar{L}_{i-1}) \times (\bar{X}_{l_i}\bar{L}_{i-1}/\bar{L}_{i-1})$ (consider the action of $\bar{T}\langle\sigma\rangle$). Letting $l_j = l_{i+1}$ and l_k vary, we see that $\bar{L}_{i-1}\bar{X}_{l_{i+1}} \trianglelefteq \bar{L}$. Consequently, the m-tuple $(X_{l_1},\ldots,X_{l_{i-1}}, X_{l_{i+1}}, X_{l_i},\ldots,X_{l_m})$ satisfies conditions (i), (ii), and (iii). Notice, also, that $L_{i+1}/L_{i-1} = (X_{l_{i+1}}L_{i-1}/L_{i-1}) \times (X_{l_i}L_{i-1}/L_{i-1})$, and T_0 acts irreducibly on each factor with inequivalent representations. Since A is T_0-invariant we conclude that $(A \cap L_{i+1})L_{i-1}/L_{i-1} = X_{l_{i+1}}L_{i-1}/L_{i-1}$. Therefore, a rearrangement of $X_{l_1},\ldots,X_{l_m}$ also satisfies conditions (i), (ii), and (iii) with an avoided factor nearer the end of an LT_0 composition series of L. Repeating this a sufficient number of times we obtain a contradiction to the minimality of $|L|$, because at the last step we have A contained in a proper subgroup of L which has the correct form. This proves the claim and the result follows.

We complete this section with one additional result that is useful in computations.

(6.10) Let $\bar{X}_{n_1},\ldots,\bar{X}_{n_k}$ be T-root subgroups. Suppose that either $q > 5$ and $\langle \bar{X}_{n_1},\ldots,\bar{X}_{n_k}\rangle$ is unipotent or $q > 7$ and $\langle X_{n_1},\ldots,X_{n_k}\rangle$ is nilpotent. Then

(a) $\langle \bar{X}_{n_1},\ldots,\bar{X}_{n_k}\rangle_\sigma = \langle X_{n_1},\ldots,X_{n_k}\rangle$.

(b) $[\bar{X}_{n_1},\ldots,\bar{X}_{n_k}]_\sigma = [X_{n_1},\ldots,X_{n_k}]$.

Proof. Suppose $q > 7$ and $\langle X_{n_1},\ldots,X_{n_k}\rangle = D$ is nilpotent. By (3.9) of [4] there is a canonical parabolic subgroup $\bar{P}$ of $\bar{G}$ with $D \leq R_u(\bar{P})$ and $N_{\bar{G}}(D) \leq \bar{P}$. Hence $T_0 \leq \bar{P}$ and, by (6.4), $\bar{T} \leq \bar{P}$. The argument of (6.9) shows that $\bar{X}_{n_i} \leq R_u(\bar{P})$ for $1 \leq i \leq k$. Hence $\langle \bar{X}_{n_1},\ldots,\bar{X}_{n_k}\rangle$ is unipotent. So in either case we have $\bar{X} = \langle \bar{X}_{n_1},\ldots,\bar{X}_{n_k}\rangle$ a unipotent group.

We may assume $k > 1$. Suppose (b) holds. We prove (a) by induction on the number of $\langle\sigma\rangle$-orbits of root subgroups in $\bar{X} = \langle \bar{X}_{n_1},\ldots,\bar{X}_{n_k}\rangle$. We have $\bar{X}' = \langle [\bar{X}_{i_1},\ldots,\bar{X}_{i_l}] : \{i_1,\ldots,i_l\} \subseteq \{n_1,\ldots,n_k\}\rangle$ and since $\bar{X}'$ is invariant under both $\bar{T}$ and $\langle\sigma\rangle$, $\bar{X}' = \langle \bar{X}_{j_1},\ldots,\bar{X}_{j_s}\rangle$ for some $\{j_1,\ldots,j_s\} \subseteq \{1,\ldots,t\}$. Inductively, $(\bar{X}')_\sigma = \langle X_{j_1},\ldots,X_{j_s}\rangle$. Also, $\bar{X}/\bar{X}'$ is the product of the groups $M_i = \bar{X}_{n_i}\bar{X}'/\bar{X}'$. For $i = 1,\ldots,k$, M_i is either trivial or $M_i \cong \bar{V}_i$, so (5.1)(i) implies that $\bar{X}/\bar{X}'$ is the direct product of the nontrivial M_i. Consequently, Lang's theorem implies $\bar{X}_\sigma = \langle X_{n_1},\ldots,X_{n_k}\rangle \bar{X}'_\sigma = \langle X_{n_1},\ldots,X_{n_k}\rangle\langle X_{j_1},\ldots,X_{j_s}\rangle$. We may choose $j_1,\ldots,j_s$ such that for each j_i, $\bar{X}_{j_i} \leq [\bar{X}_{i_1},\ldots,\bar{X}_{i_l}]$ for some $\{i_1,\ldots,i_l\} \subseteq \{n_1,\ldots,n_k\}$. So by (b), $X_{j_i} \leq [X_{i_1},\ldots,X_{i_l}] \leq \langle X_{n_1},\ldots,X_{n_l}\rangle$ and (a) follows. Therefore, it will suffice to prove (b).

To prove (b) argue by induction on k. For $k = 1$ the result is trivial and for $k = 2$ apply (6.7). So suppose $k \geq 3$ and that the result holds for $k - 1$. Set $Y = [X_{n_1},\ldots,X_{n_{k-1}}]$, $\bar{Y} = [\bar{X}_{n_1},\ldots,\bar{X}_{n_{k-1}}]$, $D = [Y, X_{n_k}]$, and $\bar{D} = [\bar{Y}, \bar{X}_{n_k}]$. Let $\Gamma = \{i \mid X_i \leq D\}$. Then (6.9) implies that $D = \prod_{i\in\Gamma} X_i$. Also, (6.9)(iii) shows that $\bar{D}_1 = \prod_{i\in\Gamma}\bar{X}_i$ is a group with $(\bar{D}_1)_\sigma = D$.

Suppose $h \in \{1,\ldots,t\}$ and $\bar{X}_h \leq \bar{Y}$. By (6.7), $[\bar{X}_h, \bar{X}_{n_k}]_\sigma = [X_h, X_{n_k}] \leq [Y, X_{n_k}] = D$. Then $[\bar{X}_h, \bar{X}_{n_k}] \leq \bar{D}_1$. If $\bar{Y}, \bar{X}_{n_k} \leq N_{\bar{G}}(\bar{D}_1)$, then letting h vary we have $\bar{D} = [\bar{Y}, \bar{X}_{n_k}] \leq \bar{D}_1$, whence $\bar{D}_\sigma \leq (\bar{D}_1)_\sigma = D \leq \bar{D}_\sigma$, proving the result. So let $\bar{X}_j \leq \bar{Y}$ or $\bar{X}_j = \bar{X}_{n_k}$. It will suffice to show that $\bar{X}_j \leq N_{\bar{G}}(\bar{D}_1)$. Since $Y, X_{n_k} \leq N(D)$, we have $[X_j, X_i] \leq D$ for each $i \in \Gamma$. Thus, $[X_j, X_i]$ is a product of certain of the groups X_α, for $\alpha \in \Gamma_1 \subseteq \Gamma$. Then (5.5) and (6.8)(iii) imply $[\bar{X}_j, \bar{X}_i] = \prod_{\alpha\in\Gamma_1}\bar{X}_\alpha \leq \bar{D}_1$, as desired. This completes the proof.

7. Nonsolvable T_0-invariant subgroups. In this section we maintain the previous notation. In addition, let Y be a T_0-invariant subgroup of G such that $Y = Y_1 \cdots Y_n$, a central product of groups of Lie type in characteristic p. For $1 \leq i \leq n$ write $Y_i = O^{p'}(Y_i) = Y_i(p^{e_i})$. The goal of this section and the next is to relate Y to the Lie structure of G and to the root system of $\bar{G}$. Throughout this section we assume $p \geq 5$ and $q > 7$.

The main results of this section are as follows:

(7.1) T_0 contains a maximal torus of Y.

(7.2) Suppose $T_0 \leq T_1 \leq T$ and $T_1 \leq N(Y)$. For $1 \leq i \leq n$, let J_i be a Cartan subgroup of Y_i. Then $J = \prod_{i=1}^{n} C_{YT_1}(J_i)$ is a maximal torus of G_0T_1.

These results will be used in later sections to characterize such groups Y. The difficulty is that, at the outset, the groups Y_i are not known to have any connection with the existing Lie structure of G. In particular, p^{e_i} is not known to be a power of q.

We will prove (7.1) and (7.2) together, in a series of steps. Suppose that one of (7.1) or (7.2) is false and choose a counterexample (for some choice of T_0) with $|Y| \cdot |G_0|$ minimal. Then $Z(G_0) = 1$. First assume that (7.1) fails for Y and set $S = N_{T_0}(Y_1)$.

(7.3) (i) T_0 is transitive on $\{Y_1, \ldots, Y_n\}$.
(ii) $S \nleq C(Y)$.

Proof. (i) is trivial from the minimality of Y, since otherwise we could replace Y by the products of the T_0-orbits on $\{Y_1, \ldots, Y_n\}$. For (ii), suppose $S \leq C(Y)$ and let $x \in Y_1 - Z(Y_1)$ be a p'-element. Then $A = \langle x^{T_0} \rangle$ is abelian and T_0-invariant. Therefore, (6.2) implies that $[A, T_0] \leq A \cap T_0 \leq A \cap S \leq C(Y)$. Letting x vary we have $[Y, T_0] \leq C(Y)$, hence $[Y, T_0] \leq Z(Y)$. But this forces $[Y, T_0] = 1$, contradicting (2.8).

(7.4) $Y = Y_1$.

Proof. Suppose $n > 1$ and let $I_1 < Y_1$ be an S-invariant abelian p'-group. Then $I = \langle I_1^{T_0} \rangle$ is a T_0-invariant abelian p'-group and as in the last result we apply (6.2) to obtain $[I, T_0] \leq T_0 \cap I \leq C_Y(T_0)$. It follows that $I_1 Z(Y_1)/Z(Y_1)$ is an elementary abelian 2-group (also $T_0/S \cong Z_2$).

By (2.14) we may take I_1 to be a maximal torus of Y_1. Suppose the overlying algebraic group of Y_1 has Lie rank l and set $q_0 = p^{e_1}$. By (2.1)(iii), I_1 has rank at most l (as an abelian group). So $|I_1/Z(Y_1)| \leq 2^l$. On the other hand $|I_1/Z(Y_1)| = e^{-1}f(q_0)$, where $e \leq l + 1$. By (2.4)(iii) we have $e^{-1}f(q_0) \geq e^{-1}(q_0 - 1)^l \geq e^{-1}4^l$ (since $q_0 \geq p \geq 5$). Therefore, $2^l \geq e^{-1}4^l \geq (l + 1)^{-1}4^l$, forcing $l = 1$. The only possibility is $Y_1/Z(Y_1) \cong \mathrm{PSL}(2, 5)$. However, here one can argue that $T_1 \cap Y_1 \leq Z(Y_1)$ and that I_1 can be chosen as a subgroup of order 3. This is a contradiction.

By (2.14) we may choose a maximal torus I of Y with $I^{T_0} = I$. Let $I_0 = I \cap T_0$.

(7.5) (i) $C_I(T_0) = I_0 \geq [T_0, I]$.
(ii) $Z(Y) \cap T_0 = 1$.
(iii) $[T_0, I]$ is cyclic.

Proof. We first use (6.2) to obtain $[T_0, I] \leq T_0 \cap I = I_0$. Also, $I_0 \leq C_I(T_0) \leq T_0$ by (2.8). Thus (i) holds.

Suppose $1 \neq z \in Z(Y) \cap T_0$ and let $E(C_G(z)) = X_1 \cdots X_s$, a commuting product of groups of Lie type over extension fields of $\mathbf{F}_q$ (see (2.9)). By (2.9)(v), $T_0 \cap X_i$ is a maximal torus of X_i for $i = 1, \ldots, s$. Since $Y = O^{p'}(Y)$ we have $Y \leq X_1 \cdots X_s$, so (2.8) implies that $Y \nleq C(T_0 \cap X_i)$

for some i. Therefore, $Y = [Y, T_0 \cap X_i] \leq X_i$, so minimality of $|Y| \cdot |G_0|$ shows that $T_0 \cap X_i$ contains a maximal torus of Y. This is a contradiction, proving (ii).

Suppose (iii) false and choose $Z_r \times Z_r \cong E \leq [T_0, I]$, where r is prime. Let $1 \neq e \in E$ satisfy $E(C_Y(e)) \neq 1$. (It is not difficult to check the existence of such an e. Consider E contained in a maximal torus $\bar{I}$ of a suitable algebraic group. Then E acts on each $\bar{I}$-root subgroup, inducing a cyclic group.) Now apply (2.9). Write $E(C_Y(e)) = D_1 \cdots D_m$, a commuting product of components. Minimality implies that (7.1) and (7.2) hold for the group $D_1 \cdots D_n$. By (7.1) $T_0 \cap D_i$ contains a maximal torus of D_i for $1 \leq i \leq m$. On the other hand, (2.5)(v) shows that $I \cap D_i$ contains a maximal torus of D_i, and $T_0 \cap D_i$, $I \cap D_i$ normalize each other.

Fix $1 \leq i \leq m$ and let H_i be a Cartan subgroup of D_i. By (7.2) $A_i = C_{T_i T_0}(H_i)$ is a maximal torus of G_0. We claim that each H_i-root subgroup of D_i is also an A_i-root subgroup of G_0. We remark that the argument used here will be quoted in the proofs of (7.8) and (7.9). By (3.6) A_i permutes the H_i-root subgroups of D_i and centralizes H_i. So (5.6) and the assumption $p \geq 5$ implies that either A_i normalizes each H_i-root subgroup of D_i or there exist H_i-root subgroups R_1, R_2 such that A_i normalizes $\langle R_1, R_2 \rangle \cong \mathrm{SL}(2,5)$ or $\mathrm{PSL}(2,5)$. In the first case the claim follows since $A_i \leq C(H_i \cap \langle R_1, R_2 \rangle)$. So suppose the latter case holds and let $\tilde{A}_i = C_{A_i}(\langle R_1, R_2 \rangle)$. Then $|A_i : \tilde{A}_i| \leq 4$ and $C_{\bar{G}}(\tilde{A}_i)^0$ is not a maximal torus. Write $C_{\bar{G}}(\tilde{A}_i)^0 = \bar{X}\bar{Z}$, where $\bar{X}$ is semisimple and $\bar{Z} = Z(\bar{X}\bar{Z})^0$. Let $\bar{A}_i$ be the σ-invariant maximal torus containing A_i. Then $\bar{Z} \leq \bar{A}_i$ and $\bar{A}_i \cap \bar{X}$ is a maximal torus of $\bar{X}$. Now use (2.4) applied to $\bar{X}$ and the fact that $q \geq 25$ (since $p = 5$) to conclude $|A_i \cap \bar{X} : A_i \cap Z(X)| > 4$. This contradicts $\bar{X} \leq C(\tilde{A}_i)$ and proves the claim.

Since each A_i-root subgroup has Frattini quotient on $\mathbf{F}_q$-module, the above claim shows that D_i is defined over a field of size at least q. Thus (6.3) and (2.8) both apply to D_i. From (6.3) we conclude $T_0 \cap D_i = I \cap D_i$ for each i. From (2.8) we see that if $T_{00} = (T_0 \cap D_1) \cdots (T_0 \cap D_m)$ and $C = C(e) \cap C(D_1 \cdots D_m)$, then $S = C_{C_Y(e)}(T_{00}C/C)$ is an abelian subgroup of $C_Y(e)/C$. Since both T_0C/C and IC/C are contained in S, we conclude $[T_0, I] \leq C$.

Then $E \leq C$ and, in particular, E centralizes a proper p-subgroup of Y. By (2.3) of **[23]** this implies that Y is generated by the subgroups $D_1 \cdots D_m$ as E ranges over $E^{\#}$. Hence, $[T_0, I] \leq C(Y) \cap I \cap T_0 \leq Z(Y) \cap T_0 = 1$, by (ii). Then (2.8) gives $I \leq T_0$, which we are assuming false. This proves (iii).

(7.6) Let $[T_0, I] = \langle x \rangle$.
(i) $x \neq 1$.
(ii) $I \leq C_{\bar{G}}(x)^0$.

Proof. If $x = 1$, then $I \leq C_{G_0}(T_0) = T_0$, which we are assuming false. So (i) holds. Let $\bar{C} = C_{\bar{G}}(x)^0$. To prove (ii) we make use of the universal covering group, $\tilde{G}$, of $\bar{G}$. Let π: $\tilde{G} \to \bar{G}$ be the natural surjection and regard σ as acting on $\tilde{G}$ and commuting with π. Then $\tilde{G}_\sigma = G_1$ maps, via π, onto G_0. Now $(Y)\pi^{-1}$ is the central product of part of $Z(\tilde{G})$ with a covering group, Y_1, of Y. Since Y_1 is also a group of Lie type, $(I)\pi^{-1}$ is abelian. Choosing $\tilde{x}$ to be a preimage of x we have $(I)\pi^{-1} \leq C_{\tilde{G}}(\tilde{x}) = C_{\tilde{G}}(\tilde{x})^0$ (see (4.4) of **[25]**). Therefore, $I \leq (C_{\tilde{G}}(\tilde{x}))\pi = C_{\bar{G}}(x)^0$, proving (ii).

At this point we obtain a contradiction. Let $\bar{C} = C_{\bar{G}}(x)^0$ and $C = \bar{C}_\sigma \cap G_0$. By (2.9) $C = E(C)T_0$ and by (2.5)(v) $(T_0 \cap E(C))Z(C)/Z(C)$ is a maximal torus of $E(C)Z(C)/Z(C)$. Moreover, $[I, T_0] \leq I_0 \leq Z(C)$, so $IZ(C)/Z(C)$ centralizes $(T_0 \cap E(C))Z(C)/Z(C)$. It follows from (2.8) that $I \leq T_0 Z(C) = T_0$, a contradiction.

At this point we know that (7.1) holds for Y (and for all smaller groups). Consequently, (7.2) must fail for Y. Recall, that $T_0 \leq T_1 \leq T$.

(7.7) Let $Z = Z(YT_1)$
(i) $Z \leq T_1$.
(ii) $Y = Y_1$, so Y is quasisimple.
(iii) $T_0 \cap Y$ contains a maximal torus, I, of Y.

Proof. Suppose $yt \in Z$, with $y \in Y$ and $t \in T_1$. Then $y \in C_{G_0}(T_0) = T_0$ (by (2.8)), proving (i). (iii) is immediate from (7.1).

Suppose $n > 1$. By minimality of $|Y| \cdot |G_0|$, $P = C_{Y_n T_1}(J_n)$ is a maximal torus of $G_0 T_1$. Also, P normalizes $Y_1 \cdots Y_{n-1}$, so another application of minimality together with (2.3) shows that

$$I = \bigcap_{i=1}^{n-1} C_{Y_1 \cdots Y_{n-1} P}(J_i)$$

is a maximal torus of $G_0 P = G_0 T_1$. Now $C_{YT_1}(J_n) = Y_1 \cdots Y_{n-1} P$, so $\bigcap_{i=1}^{n} C_{YT_1}(J_i) = I$. Therefore $n = 1$, proving (ii).

(7.8) $Z(YT_1) = C_{YT_1}(Y) = 1$.

Proof. Let $Z = Z(YT_1)$ and $C = C_{YT_1}(Y)$. Clearly, $Z \leq C$. Also, CT_1 is a solvable p'-group, so (6.1) implies $C \leq N(T_1)$. On the other hand,

$C \leq YT_1$; we conclude that $[C, T_1] \leq C \cap T_1 \leq Z(YT_1) = Z$. If $[C, T_1] = 1$, then $C \leq C_{G_0T_1}(T_1) = T_1$ and $Z = C$. Suppose $C \neq 1$. Then we conclude $C \cap T_1 \neq 1$.

Set $Z_1 = C \cap T_1$ and $\bar{D} = E(C_{\bar{G}}(Z_1)^0)$. By (7.7) $T_0 \cap Y$ contains a maximal torus of Y, so Y is generated by conjugates of $T_0 \cap Y$. It follows that $Y \leq \bar{D}$. Let $D = O^{p'}(\bar{D}_\sigma)$. Then $Y \leq D$ and (2.5) implies that T_1 contains a maximal torus of each component of D. Let D_1 be a component of D and $T_2 = T_1 \cap D_1$. Then $T_2 \nleq Z(D_1)$, so $T_2 \nleq Z_1$ and $Y = [Y, T_2] \leq D_1$. Letting D_1 vary, we conclude D is quasisimple.

Since $\bar{T}\bar{D}$ is connected we write $\bar{T}\bar{D} = \bar{D}\bar{Z}$, where $\bar{Z}$ is a torus, $\bar{Z} \leq \bar{T}$, and $[\bar{Z}, \bar{D}] = 1$. By induction, $J_2 = C_{YT_2}(J_1)$ is a maximal torus of D. Let $\bar{A}$ be a maximal torus of $\bar{D}\bar{T}$ with $J_2 \leq \bar{A}$. Then $\bar{Z} \leq \bar{A}$ and $\bar{A}^\sigma = \bar{A}$. Set $J_3 = G_0T_1 \cap \bar{A}$. By definition, $\bar{A}_\sigma$ is a maximal torus of $\bar{G}_\sigma$, so J_3 is a maximal torus of G_0T_1.

At this point we apply the argument of (7.5) to show that each nilpotent J_2-root subgroup of D is also a J_3-root subgroup of G_0. Similarly, if we use the groups Y, J_1, J_2, and D we conclude that each J_1-root of Y is a J_2-root subgroup of D, hence a J_3-root subgroup of G. As in the proof of (7.5) we have Y defined over a field of order at least q. Suppose $yt \in C$ with $y \in Y$ and $t \in T_1$. Then $y \in C_Y(T_0 \cap Y)$, so by (2.8) (which now applies to Y) we have $y \in T_0 \cap Y$. This shows that $C \leq T_1$, and so $Z = C = Z_1$.

At this point we invoke Theorem (8.1), the proof of which is independent of (7.1) and (7.2). Let $X_{i_1}, \ldots, X_{i_k}$ be the J_1-root subgroups contained in a fixed J_1-invariant Sylow p-subgroup, U, of Y. Set $\bar{Y} = \langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k}, \bar{X}^*_{i_1}, \ldots, \bar{X}^*_{i_k} \rangle$. Then $Y = O^{p'}(\bar{Y}_\sigma)$ (by (8.1)(iii) applied to $\bar{D}$. If $\bar{D}$ has l simple factors apply (8.1) to a diagonal of $\bar{O}$ normalized by σ^l, then take projections.) Also, $\bar{Y}\bar{A} \leq \bar{D}\bar{A} = \bar{D}\bar{Z}$.

As $T_1 \leq N(Y)$, $YT_1 = Y(YT_1 \cap N(J_1)) \leq Y(N(\bar{D}) \cap N(Y) \cap N(J_2))$. But $N(\bar{D}) \cap N(Y) \cap N(J_2)$ permutes the J_2-root subgroups of Y, so normalizes $\bar{Y}$. Therefore, $T_1 \leq YT_1 \leq YN(\bar{Y}) = N(\bar{Y})$.

Set $\bar{V} = C_{\bar{D}}(\bar{Y})^0$, a T_1-invariant subgroup of $\bar{D}$. By (2.14), T_1 normalizes a σ-invariant maximal torus $\bar{L}$ of $\bar{V}$. We have $\bar{A}\bar{Y} = \bar{Z}_1\bar{Y}$, where $\bar{Z}_1 = Z(\bar{A}\bar{Y})^0$, and $\bar{Z}_1 = \bar{Z}(\bar{Z}_1 \cap \bar{D})$. So $\bar{Z}_1 \cap \bar{D} \leq \bar{V}$ and we see that $\bar{Z}\bar{V}\bar{Y}$ contains a maximal torus of $\bar{G}$. Therefore, $\bar{E} = C_{\bar{Z}\bar{L}\bar{Y}}(T_1 \cap Y)$ is a T_1-invariant maximal torus of $\bar{G}$. Now, $T_0 \leq N(\bar{E}_\sigma \cap G_0)$ and (6.3) implies that $T_0 = \bar{E}_\sigma \cap G_0$, whence $\bar{E} \leq C_{\bar{G}}(T_0) = \bar{T}$. We conclude that $\bar{E} = \bar{T}$ and $\bar{T} < \bar{Y}\bar{V}\bar{Z} \leq N(\bar{Y})$.

By (2.5) $(\bar{Y}\bar{T})_\sigma = \bar{Y}_\sigma\bar{T}_\sigma = YT$. Let $\bar{C}$ be a maximal torus of $\bar{Y}\bar{T}$ with $J_1 \leq \bar{C}$. Then $\bar{Y}\bar{T} = \bar{Y}\bar{C}$ and $YT = (\bar{Y}\bar{T})_\sigma = (\bar{Y}\bar{C})_\sigma = Y\bar{C}_\sigma$. So $YT_1 = YC_1$, where $C_1 = \bar{C}_\sigma \cap G_0T_1$. But C_1 is a maximal torus of $G_0T_1 = G_0C_1$ (see (2.3)) and $C_{YT_1}(J_1) = C_{YC_1}(J_1) = C_1(C_Y(J_1)) = C_1J_1 = C_1$. We are assuming this to be false, so this contradiction proves (7.8).

(7.9) (i) I is minisotropic.

(ii) There does not exist a subgroup $D < Y$ such that $D^{T_1} = D$ and D a group of Lie type in characteristic p.

(iii) For $1 \neq t \in T_1$, $C_Y(t)$ does not contain a component of Lie type in characteristic p.

Proof. For (i), suppose I is contained in a proper parabolic subgroup, K, of Y. The argument of (7.5) shows that each nilpotent I-root subgroup of Y is also a T_1-root subgroup of G. So $O_p(K)$ is a product of T_1-root subgroups and $K^{T_1} = K$ (as $K = N_Y(O_p(K))$). If K^0 is the opposite parabolic then $T_1 \leq N(K^0)$, so T_1 normalizes $K \cap K^0 = L$, a Levi factor of K, containing I. We may assume $J_1 \leq L$. Let $L_1 = L'$, so that $L = L_1J_1 = L_1I$. If $L_1 = 1$, then $J_1 = I \leq T_1$. Since J_1-root subgroups of Y are also T_1-root subgroups we have Y defined over a field with at least q elements. Then (7.8) and (2.8) imply $C_{YT_1}(J_1) = T_1$, a maximal torus of G_0T_1. Suppose then that $L_1 \neq 1$, and let $J_2 = L_1 \cap J_1$, a Cartan subgroup of L_1. Minimality implies that $R = C_{L_1T_1}(J_2)$ is a maximal torus of G_0T_1. As $J_1 \leq C_{L_1J_1}(J_2) = C_{L_1I}(J_2) \leq R$, we also have $R \leq C_{YT_1}(J_1) = J$.

Replacing T_1 by R in the above we have J_1-root subgroups of Y being R-root subgroups of G_0. Again we conclude that the defining field for Y has at least q elements. Then (2.3), (2.8), and (7.8) yield $YT_1 = YJ$ and J Cartan in YJ. So J is abelian, and another application of (2.8) shows that $R = J$, a contradiction. Thus (i) holds.

Suppose $D^{T_1} = D < Y$ and D is a group of Lie type in characteristic p. Let A_1 be a Cartan subgroup of D and $A = C_{DT_1}(A_1)$, a maximal torus of G_0T_1, by minimality. But now consider YA. From (i) we conclude $J_2 = C_{YA}(J_1)$ is a maximal torus of G_0A. Since $YA \leq YT_1$ we also have $J_2 \leq J$. As in the proof of (7.6) the J_1-root groups of Y are also J_2-root subgroups of G_0, so Y is defined over a field of at least q elements. So (2.8) applies to YT_1 and shows that J is a maximal torus of YT_1; in particular an abelian group. But then $J \leq C(J_2)$ and (2.8) forces $J = J_2$, a contradiction to our supposition. This proves (ii) and (iii) follows.

(7.10) Write $Y = Y(q_0)$ and $|T_1| = \frac{1}{d}\Pi\, \Phi_{l_i}(q_0)$.

(i) $d = 1$ or d is prime.

(ii) If $Y \not\cong \mathrm{PSL}(2, q_0)$, then $|T_1|$ is odd.

(iii) T_1 is cyclic, $I = T_1 \cap Y$, and T_1 is a minisotropic torus of YT_1.

Proof. Suppose $t \in T_1$ is an involution and write $Y = O^{p'}(\bar{Y}_\tau)$, where τ is an endomorphism of the algebraic group $\bar{Y}$. If t extends to an involutory automorphism of $\bar{Y}$ commuting with τ, then $\bar{Y}_t^0$ is reductive and (7.9)(iii) implies that $\bar{Y}_t^0$ is a torus. Let $\bar{U}$ be the unipotent radical of a t-invariant Borel subgroup of $\bar{Y}$. Then t inverts $\bar{U}$, $\bar{U}$ is abelian, and $Y \cong \mathrm{PSL}(2, q_0)$. So if $|T_1|$ is even and $Y \ncong \mathrm{PSL}(2, q_0)$, then some involution $t \in T_1$ induces a field or graph-field automorphism of Y, against (7.9)(iii). This establishes (ii). If $Y \cong \mathrm{PSL}(2, q_0)$ then it easily follows that (i) and (iii) also hold. Suppose then that $Y \ncong \mathrm{PSL}(2, q_0)$. Thus $|T_1|$ is odd. (2.8) or its proof in case $q_0 = 5$ shows that elements of odd order in YT_1 centralizing I lie in a maximal torus V of YT_1. Hence $T_1 = V$. (iii) follows from (7.9)(i) and the argument used to prove (7.5)(iii).

For (i) we note that for $Y \ncong \mathrm{PSO}^{\pm}(2k, q_0)$, $\mathrm{PSL}(k, q_0)$, or $\mathrm{PSU}(k, q_0)$, we automatically have $d = 1$, 2, or 3. For the other cases the result follows from (7.9)(ii) and (7.9)(iii). Namely, (7.9)(ii) shows that T_1 must act irreducibly on the underlying vector space so $Y \ncong O^+(2k, q_0)'$. Hence $|T_1| = \frac{1}{d}(q_0^k + 1)$, $\frac{1}{d}(q_0^k - 1)$, or $\frac{1}{d}(q_0^k + 1)$, respectively, with k odd in the unitary case. In the latter two cases (7.9)(iii) forces k to be prime and since $d \mid k$ we are done here. In the remaining case $d \mid 4$ and $d \mid q_0^k + 1$. For k even, $q_0^k + 1 \not\equiv 0 \pmod 4$, so $d = 1$ or 2. If k is odd, T_1 contains an element $1 \neq t$ with $|t| \mid (q_0 + 1)$ and we contradict (7.9)(iii). So (i) holds.

(7.11) Let T_2 be a maximal torus of YT_1 and write $|T_2| = \frac{1}{d}\prod \Phi_{f_i}(q_0) = \frac{1}{d}\prod \Phi_{d_j}(p)$. Assume that $d_{j_1} \neq d_{j_2}$ for $j_1 \neq j_2$ and that $T_2 \leq \hat{T}_2$ is a maximal torus of G_0T_1. Then $T_2 = \hat{T}_2$.

Proof. Suppose $|\hat{T}_2| = \frac{1}{e}\prod \Phi_{r_i}(q) = \frac{1}{e}\prod \Phi_{s_j}(p)$. By (2.10) it will suffice to show that $d = e$ and $\Sigma\,\varphi(d_j) = \Sigma\,\varphi(s_j)$. Write $|T_1| = \frac{1}{d}\prod \Phi_{e_i}(q_0) = \frac{1}{e}\prod \Phi_{c_j}(q)$, viewing T_1 as a maximal torus of YT_1 and G_0T_1, respectively. Set $q_0 = p^a$ and $q = p^b$.

For m and c positive integers $\Phi_m(p^c) = \prod \Phi_{mc_0}(p)$, the product ranging over those divisors c_0 of c such that $(c/c_0, m) = 1$. Using this and the two expressions for $|T_1|$ we have $|T_1| = \frac{1}{d}\prod_{e_i}\prod_{c_0} \Phi_{e_ic_0}(p) = \frac{1}{e}\prod_j\prod_{a_0} \Phi_{c_ja_0}(p)$. Moreover, $\Sigma_{e_i, c_0}\,\varphi(e_ic_0) = a\cdot\mathrm{rank}(\bar{Y}) = \Sigma\,\varphi(d_j)$, while $\Sigma_{c_j, a_0}\,\varphi(c_ja_0) = b\cdot\mathrm{rank}(\bar{G}) = \Sigma\,\varphi(s_j)$. Consequently, it will suffice to show that $d = e$ and $\{e_ic_0\} = \{c_ja_0\}$.

Consider a term $\Phi_{e_ic_0}(p)$. By the primitive divisor theorem (see Zsigmondy **[28]**) and our assumption $p \geq 5$, either $e_ic_0 = 2$ and p is a Mersenne prime or there is a prime divisor r of $\Phi_{e_ic_0}(p)$ with $r \nmid p^x - 1$ for $x < e_ic_0$. Since $d \mid q_0 \pm 1$, $(d, r) = 1$ if $e_i > 2$. For such an r, there is a pair (c_j, a_0) with $r \mid \Phi_{c_j a_0}(p)$. This forces (see the proof of (2.10)) $e_i c_0 \mid c_j a_0$ and either equality holds or r divides $c_j a_0/e_i c_0$. Of course, we can reverse all this, starting with a term $\Phi_{c_j a_0}(p)$.

By (7.10)(i) d is one or prime. Suppose $G_0 \not\cong \mathrm{PSL}(n, q)$ or $\mathrm{PSU}(n, q)$ with $e > 3$. Then $e \leq 3$ and $e \mid p \pm 1$. Using this and the remarks of the previous paragraph, cancel off terms in the two expressions for T_1 where the subscripts $e_i c_0$ and $c_j a_0$ coincide. Starting from the largest $e_i c_0$ and $c_j a_0$ we see that all terms cancel except those where $e_i c_0$ or $c_j a_0$ is 1 or 2 or possibly a single term $\Phi_{2c_0}(p)$, where d is a primitive divisor of the term (note that T_1 minisotropic in YT, forces each $e_i > 1$). So we are left with an expression $\frac{1}{d}(p+1)^x = \frac{1}{e}(p-1)^y(p+1)^z$ or $\frac{1}{d}\Phi_{2c_0}(p)(p+1)^x = \frac{1}{e}(p-1)^y(p+1)^z$. Using the fact that T_1 is obtainable from no proper subsystem of the root system of the overlying algebraic group of Y (see (7.9)(ii)) we use the orders given in Carter [6] and extensions to cover the twisted groups, to conclude $x \leq 1$. In the first case use the facts that $\frac{e}{d} \leq \frac{3}{d}$, $p \geq 5$, and (7.10)(ii) to conclude $e = d$ and $\{e_i c_0\} = \{c_j a_0\}$. In the second case note that $d > 3$ (otherwise, obtain a contradiction using a primitive division of $\Phi_{2c_0}(p)$). This forces $Y \cong \mathrm{PSL}(k, q_0)$ or $\mathrm{PSU}(k, q_0)$ and as in the proof of (7.10)(iii), $d = k$. But then $|T_1| = \frac{1}{d}\Phi_d(q_0)$ or $\frac{1}{d}\Phi_{2d}(q_0)$ and no $e_i = 2$. This is a contradiction. Therefore we may now assume that $G_0 \cong \mathrm{PSL}(n, q)$ or $\mathrm{PSU}(n, q)$ and $e > 3$.

If $G_0 \cong \mathrm{PSL}(n, q)$, then $|T_1| = \frac{1}{e}(1/(q-1))\Pi(q^{n_i} - 1)$, with $\Sigma n_i = n$. For the unitary group, replace q by $-q$, taking absolute values, if necessary. We obtain $|T_1| = \frac{1}{e}(1/(q+1))\Pi(q^{n_i}+1)\Pi(q^{n_i}-1)$, where the first product is over the odd n_i's and the second over the even n_i's. Moreover, e is a divisor of $(n, q-1)$ or $(n, q+1)$, respectively. If $Y \not\cong \mathrm{PSL}(2, q_0)$, then by (7.10)(ii), $|T_1|$ is odd, hence there are at most two terms in the product. In the unitary case, if there are two terms, then both powers of q must be odd.

Let $y \in N_Y(T_1)$ with $|yT_1| = r$, a prime. By (7.9)(iii) and (2.9) $C_{T_1}(y)$ is an r-group with order dividing that of the center of the universal covering group of Y. We also have $N_{G_0T}(T_1)/T_1 \cong \Pi\, Z_{n_i}$, the factors acting on (by raising to powers of q or $-q$) the appropriate factor of T_1, centralizing the rest. By (7.10)(iii), T_1 is cyclic. Therefore, $(n_i, n_j) = 1$ for $n_i \neq n_j$. So r divides n_i for a unique i, centralizing a subgroup of the appropriate factor having order $\frac{1}{f}(q^{n_i/r} \pm 1)/(q \pm 1)$ with f a divisor of e. It follows that $n_i = r$. For each $n_j \neq n_i$, y centralizes a subgroup of T_1 of order $\frac{1}{f}(q^{n_j} \pm 1)/(q \pm 1)$, where $f \mid e$. Suppose there exists an $n_j \neq n_i$ with $n_j > 2$. Then we can choose a primitive divisor, s, of $q^{n_j} \pm 1$, and find an element in $C_{T_1}(y)$ of order s. By the above, $s = r$ and s is a divisor of the universal covering group of Y. But $s > 3$, so this and (7.9)(iii) yield $Y \cong \mathrm{PSL}(r, q_0)$ or $\mathrm{PSU}(r, q_0)$. Accordingly, $|T_1| = \frac{1}{d}(q_0^r \pm 1)/(q_0 \pm 1)$. Using the facts that $n_j \mid s - 1$ and $s = r$ we have $n_j < n_i = r$. Using the earlier primitive divisor argument in the two factorizations of $|T_1|$ (compare largest $e_i c_0$ and $c_j a_0$) we conclude that $ar = br$, hence $q = q_0$. But $r \mid q_0 \pm 1$, so r cannot be primitive for $q^{n_j} \pm 1$ if $n_j > 2$. This is a contradiction, proving that no such n_j exists.

If $Y \cong \mathrm{PSL}(2, q_0)$, then y inverts T_1, $r = 2$, and $n \leq 3$, contradicting $e > 3$. So $Y \not\cong \mathrm{PSL}(2, q_0)$, $|T_1|$ is odd, and by earlier remarks, there are at most two n_i. As $n > 3$, r is an odd prime. At this point the only possibilities are $G_0 \cong \mathrm{PSL}(r, q)$, $\mathrm{PSU}(r, q)$, $\mathrm{PSL}(r + 1, q)$, or $\mathrm{PSU}(r + 1, q)$. We chose r to be an arbitrary prime divisor of $|N_Y(T_1)/T_1|$ and found that $N_{G_0}(T_1)/T_1 \cong Z_r$. Checking Carter [**6**] we see that this forces $Y \cong \mathrm{PSL}(r, q_0)$ or $\mathrm{PSU}(r, q_0)$, thus $|T_1| = \frac{1}{d}\Phi_r(q_0)$ or $\frac{1}{d}\Phi_{2r}(q_0)$. As above, a primitive divisor argument yields $q = q_0$ and $(G_0, Y) = (\mathrm{PSL}(r + 1, q), \mathrm{PSL}(r, q))$ or $(\mathrm{PSU}(r + 1, q), \mathrm{PSU}(r, q))$.

This leads to $|T_1| = \frac{1}{d}((q^r \pm 1)/(q \pm 1)) = \frac{1}{e}(q^r \pm 1)$, where we always take the plus sign in the unitary case and the minus sign otherwise. Therefore, $e = d(q \pm 1)$ and this forces $d = 1$ and $e = q \pm 1 = (r + 1, q \pm 1)$. In particular, Y has trivial multiplier, so the preimage, D, of J_1 in the corresponding linear group is abelian. Order considerations show that D is a diagonalizable subgroup of the appropriate linear group, from which it follows that $J = J_1$ is contained in a maximal torus of $G_0 = G_0 T_1$ (a Cartan subgroup if $G_0 \cong \mathrm{PSL}(r + 1, q)$). Comparing orders we conclude that J is a maximal torus of G_0, contradicting the original assumption. This proves (7.11).

(7.12) Write $Y = Y(q_0)$ with $q_0 = p^a$, and let T_2 be a cyclic subgroup of YT_1 with $|T_2| = \frac{1}{d}\prod \Phi_{d_i}(p)$. Suppose that $\sum \varphi(d_i) = a \cdot \mathrm{rank}(\bar{Y})$ (e.g. T_2 a maximal torus of YT_1) and that $d_i \neq d_j$ for $i \neq j$. Then

(i) T_2 is a maximal torus of YT_1 and of $G_0 T_1$.

(ii) $YT_1 = YT_2$.

(iii) T_2 is a minisotropic torus of YT_1.

(iv) $T_2^{\#}$ consists of regular elements of YT_1 (in the sense of (7.9)(iii)).

Proof. Since T_2 is cyclic, T_2 is contained in a maximal torus $\hat{T}_2$ of YT_1. By hypothesis and (2.10) we have $T_2 = \hat{T}_2$. Now embed T_2 in a maximal torus $\hat{\hat{T}}_2$ of $G_0 T_1$. Then (7.11) shows $T_2 = \hat{\hat{T}}_2$. This proves (i) and (ii) follows from (2.3). Also, $G_0 T_1 = G_0 T_2$. We can now replace T_1 by T_2, and obtain (iii) and (iv) from (7.9).

The remainder of the proof consists of obtaining a contradiction by constructing a certain maximal torus T_2 of YT_1 that contradicts (7.12).

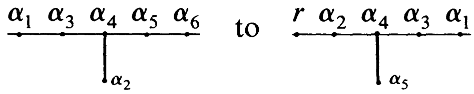
First suppose $Y \cong \mathrm{PSL}(n, q_0)$. Then $YT_1 \leq \mathrm{PGL}(n, q_0)$ and $\mathrm{PGL}(n, q_0)$ contains an isomorphic copy of $\mathrm{GL}(n-1, q_0)$ stabilizing a 1-space of the usual module. So YT_1 contains a cyclic maximal torus, T_2, of order $\frac{1}{d}(q_0^{n-1} - 1)$, with T_2 contradicting (7.12)(iii). So $Y \not\cong \mathrm{PSL}(n, q_0)$. We remark that (7.10)(ii) shows that $|T_1|$ is odd. In particular, $|YT_1 : Y|$ is odd, so if Y is an orthogonal or symplectic group, then $T_1 \leq Y$.

If Y is a classical group of dimension $2n$ in which the natural module has a singular n-space, then the above remarks show that $YT_1 \leq \mathrm{PSp}(2n, q_0)$, $\mathrm{PSO}^+(2n, q_0)'$, or $\mathrm{PGU}(2n, q_0)$. We may then choose T_2 to be a maximal torus of order $\frac{1}{d}(q_0^n - 1)$ ($\frac{1}{d}(q_0^{2n} - 1)$ in the unitary case) with T_2 stabilizing a singular n-space. Again we have a contradiction. If $Y \cong \mathrm{PSO}^-(2n, q_0)'$, then $T_1 \leq Y$ and we consider cases. If n is odd, then $Y = YT_1$ contains a cyclic subgroup, T_2, of order $\frac{1}{d}(q_0^n + 1)$ and T_2 contains a subgroup of order divisible by $(q_0 + 1)/(4, q_0 + 1)$ none of whose nonidentity elements is regular. This contradicts (7.12)(iv). If n is even, $o^-(2n, q_0)$ contains $o^+(2n-2, q_0) \times Z_{q_0+1}$. Here T_2 can be taken as a cyclic group of order $\frac{1}{d}(q_0^{n-1} - 1)(q_0 + 1)$ and contradicting (7.12)(iv). The remaining classical groups are $Y = YT_1 \cong \mathrm{PSO}(2n+1, q_0)'$ and $\mathrm{PSU}(2n+1, q_0)$. Here, use the containments $\mathrm{GL}(n, q_0) \leq \mathrm{PSO}(2n+1, q_0)$ and $\mathrm{GL}(n, q_0^2) \leq \mathrm{PGU}(2n+1, q_0)$ to get a maximal torus T_2 of order $\frac{1}{d}(q_0^n - 1)$ or $\frac{1}{d}(q_0^{2n} - 1)$, respectively. Again we contradict (7.12)(iv). At this stage we take Y to be an exceptional group.

If $Y \cong G_2(q_0)$, then $Y \geq \mathrm{SU}(3, q_0)$ and $\mathrm{SL}(3, q_0)$. Since $p \geq 5$ one of these has center of order 3, and we choose a cyclic group T_2 of order $q_0^2 - q_0 + 1$ or $q_0^2 + q_0 + 1$, accordingly. This violates (7.12)(iv). If $Y = E_7(q_0)$, then $|T_1|$ odd gives $T_1 \leq Y$. By Table (3.3) of **[23]** $Y \geq {}^3D_4(q_0) \times \mathrm{PSL}(2, q_0^3)$ and we take T_2 as the direct product of cyclic groups of order $q_0^4 - q_0^2 + 1$ and $\frac{1}{2}(q_0^3 - 1)$. Again this contradicts (7.12)(iv). If $Y = E_8(q_0)$ then Table (3.3) of **[23]** shows that $Y \geq \mathrm{PSL}(9, q_0)$ or $\mathrm{PSU}(9, q_0)$, according to $3 \mid q_0 + 1$ or $3 \mid q_0 - 1$. Here take T_2 to be cyclic of order $(q_0^9 - 1)/(q_0 - 1)$ or $(q_0^9 + 1)/(q_0 + 1)$, respectively, and contradict (7.12)(iv).

Suppose $Y = F_4(q_0)$. By Carter **[6]**, Table (3.3) of **[23]** and (7.9)(ii) it follows that T_1 is the Coxeter torus of $YT_1 = Y$. Now $F_4(q_0)$ contains ${}^3D_4(q_0)$. To see this use the argument of **[23]** in the verification of Table (3.3) (note that the subgroup of $F_4(K)$ spanned by all long root subgroups in a fixed system has type $D_4(K)$ and the triality graph automorphism is induced by a Weyl group element). Now ${}^3D_4(q_0)$ contains a cyclic maximal torus, T_2, of order $q_0^4 - q_0^2 + 1$. By (7.12), T_2 is a maximal torus of Y, so we may assume $T_2 = T_1$. But this contradicts (7.9)(ii).

Suppose $Y \cong E_6(q_0)$. We claim that $T_1 Y \geq {}^3D_4(q_0) \times T_3$, where T_3 is cyclic of order $\frac{1}{d}(q_0^2 + q_0 + 1)$. Given this, we take $T_2 = T_4 \times T_3$, where T_4 is a cyclic torus of ${}^3D_4(q_0)$ of order $q_0^4 - q_0^2 + 1$. For existence of ${}^3D_4(q_0) \times T_3$, argue as in (3.3) of [23]. Namely, we first argue that there is an element of the Weyl group of Y mapping the diagram



where r is the negative of the root of highest height. Since the Weyl group is transitive on fundamental systems, we can either do this or map the first system to the reverse of the second. In the latter case multiply by the graph automorphism of Y to get a map as desired. However, the resulting map induces an element of order 3 on $Z\overline{\Sigma}$ so cannot involve a graph automorphism. Now, complete the construction as in [23].

Next, suppose $Y \cong {}^2E_6(q_0)$. Here, we note that if $\hat{W}$ is the Weyl group of $E_6(K) = \hat{Y}$, then for τ the graph automorphism, $\hat{W}\langle\tau\rangle \cong \hat{W} \times Z_2$, where the nonidentity central element sends all roots to their negatives. It follows from the previous case that ${}^2E_6(q_0)$ contains ${}^3D_4(q_0) \times T_3$ where T_3 is cyclic of order $\frac{1}{d}(q_0^2 - q_0 + 1)$. So we again get a maximal torus T_2 of YT_1 that contradicts (7.12)(iv).

The final case to consider is $Y \cong {}^3D_4(q_0)$. Then Table (3.3) of [23] shows that Y contains $X \cong \mathrm{PSL}(3, q_0)$ or $\mathrm{PSU}(3, q_0)$ according to whether $3 \mid q_0 + 1$ or $3 \mid q_0 - 1$. Accordingly, $C_Y(X)$ is cyclic of order $q_0^2 + q_0 + 1$ or $q_0^2 - q_0 + 1$. Therefore, we let T_2 be cyclic of order $(q_0^2 - 1)(q_0^2 \pm q_0 + 1)$ resp. and contradict (7.12)(iv). We have now considered all cases and the proof of (7.1) and (7.2) is complete.

8. T_0-invariant groups of Lie type. In this section we continue the analysis of §7. Let Y be a T_0-invariant subgroup of G_0 such that Y is a commuting product of groups of Lie type in characteristic p. Assume that $p \geq 5$ and $q > 7$. In (8.1) we assume $T_0 \cap Y$ is a Cartan subgroup of Y and show that Y is related to the root system of $\overline{G}$. In later sections we will apply (8.1) and the results of §7 to determine Y in the general case. Write $Y = Y_1 \cdots Y_k$ a commuting product of groups of Lie type in characteristic p.

THEOREM (8.1). *Suppose $T_0 \cap Y_i$ is a Cartan subgroup of Y_i for $i = 1,\ldots,k$, and let $U_i \in \mathrm{Syl}_p(Y_i)$ with U_i invariant under $T_0 \cap Y_i$. For each $1 \le i \le k$, there exist T_0-root subgroups $X^i_{j_1},\ldots,X^i_{j_l}$ of G such that the following hold*:

(i) $U_i = X^i_{j_1} \cdots X^i_{j_l}$.

(ii) $Y_i = Y_i(q^{e_i})$ *for some* $e_i \ge 1$.

(iii) $Y_i = O^{p'}((\overline{Y}_i)_\sigma)$, *for* $\overline{Y}_i = \langle \overline{X^i_{j_1}},\ldots,\overline{X^i_{j_l}}, \overline{(X^i_{j_1})^*},\ldots,\overline{(X^i_{j_l})^*}\rangle$.

(iv) $\overline{Y}_i$ *is the commuting product of a* $\langle\sigma\rangle$*-orbit of* e_i *semisimple subgroups of* $\overline{G}$, *each generated by* $\overline{T}$*-root subgroups of* $\overline{G}$.

(v) $\langle \overline{Y}_1,\ldots,\overline{Y}_k\rangle = \overline{Y}_1 \cdots \overline{Y}_k$, *a commuting product.*

By way of example, say $Y \cong {}^2D_4(q^j)$. Then $\overline{Y}$ will be the commuting product of j copies of $D_4(K)$, the components of $\overline{Y}$ corresponding to a subsystem of $\overline{\Sigma}$ having the structure of j orthogonal copies of D_4.

The proof of (8.1) will be carried out in a series of steps. Assume the hypothesis of (8.1). The idea of the proof is this. First we reduce to the case where Y has just one factor. Next we consider the case $Y \cong \mathrm{SL}(2, p^e)$ or $\mathrm{PSL}(2, p^e)$. This is the hardest case. After that we work through the various rank 2 possibilities for Y as well as the 3-dimensional unitary group. The general case follows by induction and an application of (2.13).

(8.3) (i) Each $(T_0 \cap Y)$-root subgroup of Y is a T-root subgroup of G.

(ii) $U_i = X^i_{j_1} \cdots X^i_{j_l}$ for T-root subgroups $X^i_{j_1},\ldots,X^i_{j_l}$ of G.

(iii) $Y_i = Y_i(q^{e_i})$ for some $e_i \ge 1$.

(iv) Let $\overline{Y}_i$ be as in (8.1)(iii). Then $\langle \overline{Y}_1,\ldots,\overline{Y}_k\rangle = \overline{Y}_1 \cdots \overline{Y}_k$ is a commuting product.

Proof. Since $T_0 \le C(T_0 \cap Y_i)$ for $i = 1,\ldots,k$, T_0 normalizes each Y_i. The argument in the proof of (7.8) shows that each $(T_0 \cap Y_i)$-root subgroup is also a T_0-root subgroup of G_0. This proves (i) and (ii) follows from this and (6.9). For (iii) note that the defining field of Y_i has order equal to the minimum of the orders of the root subgroups of Y_i.

Fix i and a $(T_0 \cap Y_i)$-root subgroup, D. Let E be the opposite $(T_0 \cap Y_i)$-root subgroup of Y_i. We claim that $E = D^*$, the opposite T_0-root subgroup in G_0. By (5.5) and (6.8)(iii) it will suffice to show that the representation of T_0 on the Frattini quotient of E is inverse to the representation on the Frattini quotient of D. To see this set $Z = Z(Y_iT_0)$. Then $(T_0 \cap Y_i)Z/Z$ is a Cartan subgroup of Y_iT_0/Z and (2.8) shows that T_0 induces diagonal automorphisms on Y_i. As $T_0 \le C(T_0 \cap Y_i)$, (2.3) shows that T_0/Z is a Cartan subgroup of Y_iT_0/Z, and the claim follows.

So for each i, $Y_i = \langle X^i_{j_1},\ldots,X^i_{j_l},(X^i_{j_1})^*,\ldots,(X^i_{j_l})^*\rangle$, where $\{X^i_{j_1},\ldots,X^i_{j_l}\}$ are the T-root subgroups contained in U_i. To obtain (iv) we need only apply (6.10).

In view of (8.3), we now assume that $Y = Y_1$. Write $X_{j_r} = X^1_{j_r}$ for each j_r.

(8.4) Let $j_m \neq j_n \in \{j_1,\ldots,j_l\}$, with X_{j_m}, X_{j_n} root subgroups of Y corresponding to fundamental roots.

(i) If V is the $(T_0 \cap Y)$-root subgroup of Y opposite to X_{j_m}, then $V = X^*_{j_m}$.

(ii) $[\bar{X}^*_{j_m}, \bar{X}_{j_n}] = 1$.

Proof. (i) was established at the end of the proof of (8.3). It follows from (i) that $X^*_{j_m}$ is a root group of Y corresponding to the negative of a fundamental root. Since the difference of fundamental roots is never a root we conclude that $[X^*_{j_m}, X_{j_n}] = 1$. So (ii) follows from (6.10).

(8.5) Suppose G_0 is a classical group and $Y \cong \mathrm{SL}(2, q^j)$ or $\mathrm{PSL}(2, q^j)$. Then (8.1) holds.

Proof. Here $U = X_i$ for some $1 \leq i \leq t$ and by (8.4), $Y = \langle X_i, X_i^*\rangle$. Let $\bar{D} = \langle \bar{X}_i, \bar{X}_i^*\rangle$. By (3.5), $\bar{D}$ is a reductive group and $\bar{D} = \bar{D}_1 \cdots \bar{D}_m$ a commuting product of a $\langle\sigma\rangle$-orbit of reductive quasisimple groups, each generated by $\bar{T}$-root subgroups of $\bar{G}$. We must show that $m = j$ and that $\bar{D}_1 \cong \mathrm{SL}(2, K)$ or $\mathrm{PSL}(2, K)$. Suppose $m > 1$. Then $O^{p'}(\bar{D}_\sigma)$ is isomorphic to $O^{p'}((\bar{D}_1)_{\sigma^m})$, modulo centers. Also, $\bar{X}_i = \Pi_l(\bar{X}_i \cap \bar{D}_l)$. Replacing G by D_1, σ by σ^m, $\bar{X}_i$ by $\bar{X}_i \cap \bar{D}_1$, $\bar{T}$ by $(\bar{T} \cap D_1)_{\sigma^m}$, and Y by the projection of Y to $\bar{D}_1$, we may assume that $\bar{G} = \langle \bar{X}_i, \bar{X}_i^*\rangle$. Then $Y < G_0$.

By (4.1) and (4.2) $X_i \leq O_p(P)$ for P a parabolic subgroup of G_0 corresponding to the stabilizer of a singular l-space of the usual module, M, of the appropriate classical group. In view of (4.1) and (4.2), we may assume that $G_0 \cong \mathrm{PSp}(2s, q)$, $\mathrm{PSU}(2s, q)$ or $\mathrm{PSO}^+(2s, q)'$. In all cases $s > 1$.

It will be more convenient to deal with the appropriate linear group $G_1 = \mathrm{Sp}(2s, q)$, $\mathrm{SU}(2s, q)$, $\mathrm{SO}^+(2s, q)'$, respectively. Accordingly, we set $\bar{G}_1 = \mathrm{Sp}(2s, K)$, $\mathrm{SL}(2s, K)$, or $\mathrm{SO}(2s, K)$. Then $\bar{G}_1$ is a covering group of $\bar{G}$ and universal except for the orthogonal group. We replace G by G_1 and $\bar{G}$ by $\bar{G}_1$, in order to consider module actions. We retain the other notation, viewing X_i and T_0 as subgroups of G_1, $\bar{X}_i$ and $\bar{T}$ as subgroups of $\bar{G}_1$. Let $\bar{M} = K \otimes M$, the natural module for $\bar{G}_1$, where in the symplectic and orthogonal cases the form is extended naturally.

From (4.2)(vi) and (3.4)(vi), we see that under the action of T_0, M decomposes into the direct sum of the two inequivalent, irreducible, T_0-submodules, M_1 and M_2, each of dimension s. Moreover, the stabilizer in G_1 of M_i induces on M_i either $\mathrm{GL}(s, q)$, a subgroup of index $q+1$ in $\mathrm{GL}(s, q^2)$, or a subgroup of index 2 in $\mathrm{GL}(s, q)$, according to $G_1 \cong \mathrm{Sp}(2s, q)$, $\mathrm{SU}(2s, q)$, or $\mathrm{SO}^+(2s, q)'$.

We claim that $Y = \langle X_i, X_i^* \rangle$ acts irreducibly on M. First note that $T_0 \leq N(Y)$ and by (5.1) $C_{T_0}(Y) = C_{T_0}(X_i) \leq C_{\overline{T}}(\overline{X}_i) = C_{\overline{T}}(\overline{G}_1) = Z(\overline{G}_1)$. Therefore, $(T_0 \cap Y)Z(G_1)$ has index at most 2 in T_0. Using primitive divisors we see that $T_0 \cap Y$ acts irreducibly on M_1 and on M_2. Also, the assumptions $p \geq 5$ and $q > 5$ imply that $T_0 \cap Y$ contains an element, t, inducing scalar action for different scalars on M_1 and M_2. So if Y acts reducibly on M, then Y stabilizes either M_1 or M_2. But this is inconsistent with $t \in T_0 \cap Y \leq Y$ (as $t \notin Z(Y)$), and the claim holds.

View $\overline{M}$ as a $K[T_0]$-module. Since M_1 and M_2 are inequivalent, and irreducible as $(T_0 \cap Y)$-spaces, $\overline{M}$ is the direct sum of 1-dimensional $K[T_0 \cap Y]$-modules affording distinct linear representations of $T_0 \cap Y$. As $\overline{T} \leq C(T_0)$, each $K[T_0 \cap Y]$-submodule is also a $K[\overline{T}]$-submodule of $\overline{M}$.

Since $[M, X_i]$ and $[M, X_i^*]$ are $(T_0 \cap Y)$-invariant, it follows that $\overline{M} = [M, X_i] \oplus [M, X_i^*]$. Write $\overline{M}|_Y = V_1 \oplus \cdots \oplus V_r$, with each V_i an absolutely irreducible $K[Y]$-module. Then, for $1 \leq k \leq r$, $V_k = [V_k, X_i] \oplus [V_k, X_i^*]$. It follows (see (13.1) of **[26]**) that V_k is isomorphic to the extension (to K) of an algebraic conjugate of the usual module for $\mathrm{SL}(2, q^j)$. By the previous paragraph, each V_k is $\overline{T}$-invariant. Therefore, V_k is invariant under $\langle Y, \overline{T} \rangle$. But $\langle X_i, \overline{T} \rangle = \overline{X}_i\overline{T}$ and $\langle X_i^*, \overline{T} \rangle = \overline{X}_i^*\overline{T}$. Hence, $\langle Y, \overline{T} \rangle \geq \langle \overline{X}_i, \overline{X}_i^* \rangle = \overline{G}_1$, and this shows that $r = 1$ and $\overline{G}_1 \cong \mathrm{SL}(2, K)$. But this contradicts $s > 1$, proving (8.5).

(8.6) Suppose $Y \cong \mathrm{SL}(2, q^j)$ or $\mathrm{PSL}(2, q^j)$. Then (8.1) holds.

Proof. In view of (8.5) we may assume that G_0 is an exceptional group. As in the proof of (8.5) we reduce to the case $\overline{G} = \langle \overline{X}_i, \overline{X}_i^* \rangle$, where $Y = \langle X_i, X_i^* \rangle$. Also $C_{T_0}(X_i) = C_{T_0}(X_i^*) = C_{T_0}(\langle X_i, X_i^* \rangle) \leq C_{T_0}(\langle \overline{X}_i, \overline{X}_i^* \rangle) \leq Z(\overline{G})$, so replacing $\overline{G}$ by $\overline{G}/Z(\overline{G})$ we may assume that $C_{T_0}(X_i) = 1$. Thus $Y \cong \mathrm{PSL}(2, q^j)$ and T_0 is cyclic of order $q^j - 1$ or $\frac{1}{2}(q^j - 1)$. An argument with primitive divisors shows that $\overline{G}$ has Lie rank j (observe that the assumption $p \geq 5$ excludes the cases $G_0 \cong \mathrm{Sz}(q)$, ${}^2G_2(q)$, or ${}^2F_4(q)$).

Let P be the unique parabolic subgroup of G satisfying $T \leq P$ and $X_i \leq O_p(P)$ (see (3.5)(v)), and let $P = \overline{P}_\sigma$ for $\overline{P} = \overline{P}^\sigma$, a parabolic subgroup of $\overline{G}$. By (6.4) $\overline{T} \leq \overline{P}$ and we may assume $\overline{B} \leq \overline{P}$. We will consider possibilities for P, locate T in P and $\bar{T}$ in $\bar{P}$, and indicate the element of the Weyl group of $\bar{P}$ that does the twisting. That is we present $\sigma = \tau q$ and determine the orbits of τ on root subgroups in $R_u(\bar{P})$. We can then determine X_i, $\bar{X}_i$, and $\langle \bar{X}_i, \bar{X}_i^* \rangle$. Order considerations show $G_0 \not\cong {}^3D_4(q)$.

First suppose $G_0 \cong G_2(q)$. Then $\bar{P} = \langle \bar{B}, s_1 \rangle$ or $\langle \bar{B}, s_2 \rangle$, and we may take $\tau = s_1$, or s_2, accordingly. Since $|\Delta_i| = 2$, $\bar{X}_i = \langle U_\alpha, U_\beta \rangle$, where $\{\alpha, \beta\}$ is one of $\{\alpha_2, \alpha_1 + \alpha_2\}$, $\{\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$, $\{\alpha_1, \alpha_1 + 3\alpha_2\}$, or $\{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. But $\bar{X}_i$ is abelian, and the commutator relations show this to be false in the first, third, and fourth cases. In the second case $\langle \bar{X}_i, \bar{X}_i^* \rangle = \langle \bar{U}_{\pm\alpha}, \bar{U}_{\pm\beta} \rangle \cong \mathrm{SL}(3, q)$ (the group generated by all long root subgroups), contradicting $\langle \bar{X}_i, \bar{X}_i^* \rangle = \bar{G}$. So $G = G_2(q)$ is not possible.

For the rest of the proof of (8.6) and for the proof of (8.8) it will be convenient to introduce the following table, which indicates possible choices for $\bar{G}$, $\bar{P}$, τ, and $\bar{\Sigma}_i$. In each case the containment $T_0 \leq P$ limits the choices for $\bar{P}$ (usually just one possibility) and we choose an appropriate representative for $\tau = w$ or $\tau = w\delta$ with $w \in W(\bar{P})$ and δ a graph automorphism (only relevant in the case $G = {}^2E_6(q)$). The choices for τ are based on the facts: T_0 is cyclic, minisotropic in P of order divisible by $\frac{1}{2}(q^2 - 1)$, τ has an orbit of length j on $\bar{\Sigma}_i$ (recall that $|\bar{\Sigma}_i| = j$) and $\langle \bar{X}_i, \bar{X}_i^* \rangle = \bar{G}$. The latter fact implies that if $\bar{\Sigma}$ has roots of different lengths, then $\bar{\Sigma}_i$ is an orbit of short roots. Otherwise, $\langle \bar{X}_i, \bar{X}_i^* \rangle$ would be contained in the proper subgroup of $\bar{G}$ generated by all root subgroups, $\bar{U}_\alpha$, with α long. Similarly, if $\bar{P} = \langle \bar{B}, s_i \mid i \neq i_0 \rangle$ ($\bar{P}$ has this form, even if $G = {}^2E_6(q)$), then each $\alpha \in \bar{\Sigma}_i$ must have α_{i_0}-coefficient equal to 1. These conditions eliminate many possibilities for $\bar{\Sigma}_i$.

TABLE (8.7)

G	i_0	τ	orbit representative of $\bar{\Sigma}_i$
$F_4(q)$	1	$(s_2s_3)(s_2^{s_3s_4})$	(none possible)
	4	$(s_3s_2)(s_3^{s_2s_1})$	(none possible)
${}^2E_6(q)$	4	$s_2s_1s_3\delta$	(none possible)
$E_6(q)$	2	$s_1s_4s_6s_3s_5$	(none possible)
$E_7(q)$	2	$s_1s_4s_6s_3s_5s_7$	$\begin{smallmatrix}000000\\1\end{smallmatrix}$, $\begin{smallmatrix}001111\\1\end{smallmatrix}$, $\begin{smallmatrix}111111\\1\end{smallmatrix}$
$E_8(q)$	2	$s_1s_4s_6s_8s_3s_5s_7$	$\begin{smallmatrix}0000000\\1\end{smallmatrix}$, $\begin{smallmatrix}1111111\\1\end{smallmatrix}$, $\begin{smallmatrix}1121000\\1\end{smallmatrix}$, $\begin{smallmatrix}0011111\\1\end{smallmatrix}$, $\begin{smallmatrix}1232111\\1\end{smallmatrix}$

To complete the proof of (8.6) one simply checks (with a bit of calculation) that in none of the cases is $\overline{X}_i$ an abelian group.

(8.8) If $Y \cong \mathrm{SU}(3, q^j)$ or $\mathrm{PSU}(3, q^j)$, then (8.1) holds.

Proof. Write $U = X_i$ and $Z(U) = X_k$, for $i, k \in \{1, \dots, t\}$. Then $|\overline{\Sigma}_i| = 2j$ and $|\overline{\Sigma}_k| = j$. Set $\overline{P} = \langle \overline{X}_i, \overline{X}_i^* \rangle$ and $P = \overline{P}_\sigma \cap G_0$. First assume that G_0 is a classical group. Then the structure of $\overline{P}$ is given in (4.2). Using the notation of (4.2) we first note that X_i nonabelian implies $r \neq 2s$. The result then follows from (4.2)(iv) and (4.2)(v). We may now assume G_0 to be an exceptional group.

Arguing as in the proof of (8.6) we may assume $\overline{G} = \langle \overline{X}_i, \overline{X}_i^* \rangle$ and passing to quotient groups, if necessary, we may assume $Z(\overline{G}) = Z(G_0) = 1$. Also, $Z(YT_0) \leq Z(\overline{G}) \cap T_0 = 1$, so $YT_0 \cong \mathrm{PSU}(3, q^j)$ or $\mathrm{PGU}(3, q^j)$, and T_0 is cyclic of order $q^{2j} - 1$ or $\frac{1}{3}(q^{2j} - 1)$. As in (8.6) we conclude that $\overline{G}$ has Lie rank $2j$. This immediately rules out the case $G_0 \cong E_7(q)$ and ${}^3D_4(q)$ is ruled out by order considerations (namely, $|T_0|$ divides $|P|$).

Suppose $G \cong G_2(q)$. Then $Y \cong \mathrm{PSU}(3, q)$. The remarks preceding Table (8.7) show that $\overline{\Sigma}_i = \{\beta_1, \beta_2\}$ for short roots $\beta_1, \beta_2 \in \overline{\Sigma}$ and (6.7) shows that $[\overline{X}_i, \overline{X}_i] = [\overline{U}_{\beta_1}, \overline{U}_{\beta_2}] = \overline{U}_\gamma$ for $\gamma \in \overline{\Sigma}$. The only possibility is $\overline{P} = N_{\overline{G}}(\overline{U}_\gamma)$, with γ a long root. We may take $\tau = sq$, where $s \in N_{G_0}(T_0)$ and s is in the derived group of the Levi factor of $\overline{P}$. Then $s \in C(\langle \overline{U}_\gamma, \overline{U}_\gamma^* \rangle)$, which gives $s \in C_{G_0}(\langle X_k, X_k^* \rangle)$. Hence $C_{G_0}(s) \geq \mathrm{SL}(2, q)$. On the other hand, s normalizes $\langle X_i, X_i^* \rangle = Y$, since s normalizes T_0, X_i, and X_i^*. So s induces a graph automorphism of $Y \cong \mathrm{PSU}(3, q)$, forcing $C_Y(s)' \cong \mathrm{PSL}(2, q)$. This is a contradiction. Therefore $G_0 \not\cong G_2(q)$.

The remaining cases are $G_0 \cong F_4(q)$, ${}^2E_6(q)$, $E_6(q)$, and $E_8(q)$, where we refer to Table (8.7). The first three are ruled out immediately. Suppose $G_0 \cong E_8(q)$. Here $|\Sigma_k| = j = 4$, so $[\overline{X}_i, \overline{X}_i]$ is the product of 4 $\overline{T}$-root subgroups of $\overline{G}$. However, for each of the possible orbits listed in (8.7) a direct check with the commutator relations shows that $[\overline{X}_i, \overline{X}_i]$ is the product of more than 4 $\overline{T}$-root subgroups. This proves (8.8).

We have now proved (8.1) when Y has Lie rank 1 (noting that $p \geq 5$ excludes Suzuki groups and Ree groups). Next, we establish (8.1) for groups of Lie rank 2. We will use the following notation. For O a representation of the abelian group A and $n \in Z$, O^n is the representation given by $O^n(a) = O(a^n)$.

(8.9) Suppose $Y \cong \mathrm{SL}(3, q^j)$ or $\mathrm{PSL}(3, q^j)$. Then (8.1) holds.

Proof. Write $U = X_i X_k X_l$ with $X_l = [X_i, X_k]$, and regard these subgroups as irreducible $\mathbf{F}_p[T_0]$-modules. Set $q = p^a$. There are linear $\mathbf{F}_{q^j}$-representations $\varphi_i, \varphi_k, \varphi_l$ of T_0 such that $\mathbf{F}_{q^j} \otimes_{\mathbf{F}_p} X_i = \varphi_i^p \oplus \cdots \oplus \varphi_i^{p^{aj}}$. Similarly for X_k, X_l. We have $\langle X_i, X_i^* \rangle \cong \langle X_k, X_k^* \rangle \cong \langle X_l, X_l^* \rangle \cong \mathrm{SL}(2, q^j)$, so by (8.6) each of $\langle \bar{X}_i, \bar{X}_i^* \rangle$, $\langle \bar{X}_k, \bar{X}_k^* \rangle$, and $\langle \bar{X}_l, \bar{X}_l^* \rangle$ is the commuting product of a $\langle \sigma \rangle$-orbit of j copies of SL(2, K), each generated by a $\bar{T}$-root subgroup of $\bar{G}$ and its opposite. Write $\bar{X}_i = \bar{U}_{\beta_1} \times \cdots \times \bar{U}_{\beta_j}$, $\bar{X}_k = \bar{U}_{\gamma_1} \times \cdots \times \bar{U}_{\gamma_j}$, and $\bar{X}_l = \bar{U}_{\delta_1} \times \cdots \times \bar{U}_{\delta_j}$. By (6.7) we have $[\bar{X}_i, \bar{X}_k] = \bar{X}_l$.

Let $Z = Z(YT_0)$ and let $z = yt \in Z$, with $y \in Y$ and $t \in T_0$. Then $y \in C_Y(Y \cap T_0) = Y \cap T_0$ by (2.8). So $Z \leq T_0$. Passing to YT_0/Z and applying (2.7), we conclude that $N_Y(Y \cap T_0) \leq N_Y(T_0) \leq N_{\bar{G}}(\bar{T})$. Since $N_Y(Y \cap T_0)$ is transitive on the $(T_0 \cap Y)$-root subgroups of Y, we see that $\bar{\Sigma}_i$, $\bar{\Sigma}_k$, and $\bar{\Sigma}_l$ are conjugate under $N_{\bar{G}}(\bar{T})$. In particular, the roots in $\bar{\Sigma}_i \cup \bar{\Sigma}_k \cup \bar{\Sigma}_l$ are all of the same length. Consequently, we may choose notation so that $[\bar{U}_{\beta_1}, \bar{U}_{\gamma_1}] = \bar{U}_{\delta_1}$.

Let φ, ψ, θ be the K-representations of $\bar{T}$ afforded by $\bar{U}_{\beta_1}$, $\bar{U}_{\gamma_1}$, $\bar{U}_{\delta_1}$, respectively restricted to T_0. The commutator relations show that $\varphi\psi = \theta$. It follows from (5.1) that we may assume $\varphi = \varphi_i^K$, $\psi = \varphi_k^K$, $\theta = \varphi_l^K$. We claim that $[\bar{U}_{\beta_1}, \bar{U}_{\gamma_r}] = 1$, for $r > 1$. Otherwise, $[\bar{U}_{\beta_1}, \bar{U}_{\gamma_r}] = \bar{U}_{\delta_s}$ for $\delta_s = \beta_1 + \gamma_r$. Then $\theta^{q^{s-1}} = \varphi\psi^{q^{r-1}}$ and since $\varphi\psi = \theta$ we obtain $\varphi^u = \psi^v$, for $u = q^{s-1} - 1$ and $v = q^{r-1} - q^{s-1}$. This implies $(\varphi_i^u)^K = (\varphi_k^v)^K$. Let $T_1 = T_0 \cap \langle X_i, X_i^* \rangle$, a Cartan subgroup of $\langle X_i, X_i^* \rangle \cong \mathrm{SL}(2, q^j)$. If $\tilde{\varphi}_i$ and $\tilde{\varphi}_k$ denote $\varphi_i \mid T_1$ and $\varphi_k \mid_{T_1}$, respectively, then computation within Y yields $\tilde{\varphi}_i = (\tilde{\varphi}_k^{-2})^{p^c}$ for some $0 \leq c < aj$. Therefore, $(\tilde{\varphi}_k^v)^K = (\tilde{\varphi}_k^{-2p^c u})^K$ and so $\tilde{\varphi}_k^{v+2p^c u} = 1$. But this contradicts $|\tilde{\varphi}_k| = q^j - 1$, proving the claim.

Transforming the commutator relation of the previous paragraph by powers of σ and using (8.4)(ii) we obtain the following commutator relations:

(i) $[\bar{U}_{\beta_r}, \bar{U}_{\gamma_r}] = \bar{U}_{\delta_r}$, for $1 \leq r \leq j$.

(ii) $[\bar{U}_{\pm\beta_r}, \bar{U}_{\pm\gamma_s}] = 1$, for $r \neq s$ in $\{1, \ldots, j\}$.

For $1 \leq r \leq j$ let $\bar{D}_r = \langle \bar{U}_{\pm\beta_r}, \bar{U}_{\pm\gamma_r} \rangle$. Then $\bar{D}_r \cong \mathrm{SL}(3, K)$ or $\mathrm{PSL}(3, K)$, and the above relations give $[\bar{D}_r, \bar{D}_s] = 1$ for $r \neq s$. Since $\langle \bar{X}_i, \bar{X}_i^* \rangle = \bar{D}_1 \cdots \bar{D}_j$, the proof of (8.9) is complete.

(8.10) Suppose Y is a non-trivial image of $\mathrm{Sp}(4, q^j)$ or $G_2(q^j)$. Then (8.1) holds.

Proof. The arguments are similar to those in (8.9), although slightly more complicated. We consider only the (more difficult) case of $G_2(q^j)$.

Let α, β be fundamental long and short roots of the root system of Y, so that the complete set of positive roots is $\{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$. Say X_i and X_k are the $(T_0 \cap Y)$-root subgroups of Y corresponding to α and β, respectively. Let X_r, X_s, X_t, and X_w correspond to the compound roots $\alpha+\beta$, $\alpha+2\beta$, $\alpha+3\beta$, $2\alpha+3\beta$, respectively. Then $[X_i, X_k] = X_r X_s X_t X_w$, so (6.7) implies that $[\overline{X}_i, \overline{X}_k] = \overline{X}_r \overline{X}_s \overline{X}_t \overline{X}_w$.

Write $\Delta_i = \{\overline{U}_{\beta_1}, \ldots, \overline{U}_{\beta_j}\}$, where $\overline{U}_{\beta_2} = \overline{U}^{\sigma}_{\beta_1}, \ldots, \overline{U}_{\beta_j} = \overline{U}^{\sigma}_{\beta_{j-1}}$. Similarly, write $\Delta_k = \{\overline{U}_{\gamma_1}, \ldots, \overline{U}_{\gamma_j}\}$. Set $q = p^a$ and regard each of X_i, X_k, X_r, X_s, X_t, and X_w as aj-dimensional $\mathbf{F}_p[T_0]$-modules. As in (8.9) we choose linear $\mathbf{F}_{q^j}[T_0]$ representations o_i, o_k, o_r, o_s, o_t, o_w so that $\mathbf{F}_{q^j} \otimes_{\mathbf{F}_p} X_l = o_l^p \oplus \cdots \oplus o_l^{p^{aj}}$ for $l \in \{i, k, r, s, t, w\}$. We may assume that o_i^K, o_k^K are the K-representations that T_0 induces on $\overline{U}_{\beta_1}$, $\overline{U}_{\gamma_1}$, respectively, and we may assume $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_1}] \neq 1$.

From the commutator relations for $G_2(q^j)$ it follows that there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \mu, \nu \in \{1, \ldots, aj\}$ and that $o_r = o_i^{p^\alpha} o_k^{p^\beta}$, $o_s = o_i^{p^\gamma} o_k^{2p^\delta}$, $o_t = o_i^{p^\varepsilon} o_k^{3p^\eta}$, and $o_w = o_i^{2p^\mu} o_k^{3p^\nu}$.

Suppose $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_l}] \neq 1$ for $1 \le l \le j$. Then $o_i^K (o_k^{q^{l-1}})^K = ((o_i^{p^\alpha} o_k^{p^\beta})^{q^m})^K$, $((o_i^{p^\gamma} o_k^{2p^\delta})^{q^m})^K$, $((o_i^{p^\varepsilon} o_k^{3p^\eta})^{q^m})^K$, or $((o_i^{2p^\mu} o_k^{3p^\nu})^{q^m})^K$, for some $0 \le m \le j-1$. There are elements $t, v \in T_0 \cap Y$ such that $o_i(t) = o_k(v) = 1$ (i.e., $t \in C(X_i)$, $v \in C(X_k)$) and $|o_i(v)| = |o_k(t)| = q^j - 1$. If $l = 1$, evaluate at t and v and conclude that the first possibility must hold and $\alpha = \beta$. Now suppose $l > 1$. Evaluating at t we again see that the first possibility must occur and we obtain the congruence $q^{l-1} \equiv p^\alpha q^m \pmod{p^{aj} - 1}$. Evaluating at v we have $1 \equiv p^\alpha q^m \pmod{p^{aj} - 1}$, contradicting the other congruence. Therefore, $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_l}] = 1$ for each $l > 1$, and transforming by powers of σ we have $\langle \overline{X}_i, \overline{X}_k \rangle$ a central product of the groups $\langle \overline{U}_{\beta_1}, \overline{U}_{\gamma_1} \rangle, \ldots, \langle \overline{U}_{\beta_j}, \overline{U}_{\gamma_j} \rangle$. By (8.4)(ii), $[\overline{U}_{\beta_g}, \overline{U}_{-\gamma_h}] = 1$ for $1 \le g, h \le j$, and by (8.6) $[\overline{U}_{\beta_g}, \overline{U}_{-\beta_h}] = 1 [\overline{U}_{\gamma_g}, \overline{U}_{-\gamma_h}]$ if $1 \le g \neq h \le j$. So letting $\overline{D}_g = \langle \overline{U}_{\pm\beta_g}, \overline{U}_{\pm\gamma_g} \rangle$ for $1 \le g \le j$, we have $\langle \overline{X}_i, \overline{X}_i^*, \overline{X}_j, \overline{X}_j^* \rangle$ equal to the central product of the semisimple groups $\overline{D}_1, \ldots, \overline{D}_j$.

The group $\overline{D}_1$ has as its root system a rank 2 subsystem of $\overline{\Sigma}$. On the other hand, $\langle \overline{X}_i, \overline{X}_h \rangle = \langle \overline{U}_{\beta_1}, \overline{U}_{\gamma_1} \rangle \times \cdots \times \langle \overline{U}_{\beta_j}, \overline{U}_{\gamma_j} \rangle$ and $\langle X_i, X_h \rangle$ has nilpotence class 5. This forces $\overline{D}_1$ to be of type $G_2(K)$ (it also forces $j = 1$, since G_2 is not a sub-root system of any other indecomposable system). Since $\{\overline{D}_1, \ldots, \overline{D}_j\}$ is an orbit under $\langle \sigma \rangle$, we have proved (8.10).

(8.11) Suppose Y is a non-trivial image of $SU(4, q^j)$, ${}^3D_4(q^j)$, or $SU(5, q^j)$. Then (8.1) holds.

Proof. We will discuss the most difficult case where Y is an image of $SU(5, q^j)$. Here U is the product of four root subgroups, X_i, X_k, X_l, X_m, where X_i and X_k are fundamental, $\langle X_i, X_i^* \rangle \cong SL(2, q^{2j})$, $\langle X_k, X_k^* \rangle \cong SU(3, q^j)$, X_l is a conjugate of X_i, and X_m a conjugate of X_k. In addition, each of X_k' and X_m' is a $(T_0 \cap Y)$-root subgroup. Say $X_k' = X_r$ and $X_m' = X_s$. Then each of X_i, X_k, X_l, X_m, X_r, and X_s is a T-root subgroup of G.

View X_i and X_k/X_k' as $2aj$-dimensional $\mathbf{F}_p[T_0]$-modules, where $q = p^a$. There are linear $\mathbf{F}_{q^{2j}}$-representations φ and ψ of T_0 such that $\mathbf{F}_{q^{2j}} \otimes X_i = \varphi^p \oplus \cdots \oplus \varphi^{p^{2aj}}$ and $\mathbf{F}_{q^{2j}} \otimes (X_k/X_k') = \psi^p \oplus \cdots \oplus \psi^{p^{2aj}}$. Let $\Delta_i = \{\overline{U}_{\beta_1}, \ldots, \overline{U}_{\beta_{2j}}\}$ and $\Delta_k = \{\overline{U}_{\gamma_1}, \ldots, \overline{U}_{\gamma_{2j}}\}$. We may assume that φ^K, ψ^K are the K-representations of T_0 induced on $\overline{U}_{\beta_1}$, $\overline{U}_{\gamma_1}$, respectively. We have $[X_i, X_k] = X_l X_m$ (computation in Y) and so (6.10) implies $[\overline{X}_i, \overline{X}_k] = \overline{X}_l \overline{X}_m$. We relabel if necessary so that $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_1}] \neq 1$ and for $1 \leq l \leq j$, $\overline{U}_{\beta_l} = \overline{U}_{\beta_1}^{\sigma^{l-1}}$ and $\overline{U}_{\gamma_l} = \overline{U}_{\gamma_1}^{\sigma^{l-1}}$.

Let δ, ω be linear $\mathbf{F}_{q^{2j}}$-representations of T_0 such that $\mathbf{F}_{q^{2j}} \otimes X_l$, $\mathbf{F}_{q^{2j}} \otimes (X_m/X_m')$ are the sums of the Galois conjugates of δ, ω respectively. Computations in Y imply that there exists α such that we may take $\omega = \varphi\psi^{p^\alpha}$. From the relation $[X_i, X_l] = X_m'$ we see that for some β, γ we must have $\delta\varphi^{p^\beta} = (\omega^{1+q^j})^{p^\gamma} = (\varphi\psi^{p^\alpha})^{(1+p^j)p^\gamma}$.

Suppose $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_l}] \neq 1$. Then by the above $\varphi\psi^{q^{l-1}}$ is a Galois conjugate of one of δ, ω, ω^{1+q^j}. By the previous paragraph each of δ, ω, ω^{1+q^j} can be expressed in terms of φ and ψ. Make this substitution and consider the resulting relation between powers of φ and ψ. There exist elements t_1, t_2 of T_0 such that

$$\varphi(t_1) = \psi(t_2) = 1 \quad \text{and} \quad |\varphi(t_2)| = |\psi(t_1)| = (q^2 - 1)/(5, q + 1).$$

Substituting t_1, t_2 into the above relations we see that such a relation can hold only if the obvious equalities hold between powers of φ and powers of ψ. First substitute $l = 1$ and obtain $\varphi\psi = \omega$ and $\alpha = 0$. Now let $l > 1$ and obtain a contradiction. Consequently $[\overline{U}_{\beta_1}, \overline{U}_{\gamma_l}] = 1$ for $l > 1$, and transforming by powers of σ, we conclude $[\overline{U}_{\beta_u}, \overline{U}_{\gamma_v}] = 1$ for any $u \neq v$. From (6.10) and the fact $[X_i, X_k] \neq 1$, we conclude that $[\overline{X}_i, \overline{X}_k] \neq 1$. Therefore, $[\overline{U}_{\beta_v}, \overline{U}_{\gamma_v}] \neq 1$, for $1 \leq v \leq 2j$.

Consider the group $\overline{D}_1 = \langle \overline{U}_{\pm\beta_1}, \overline{U}_{\pm\gamma_1}, \overline{U}_{\pm\gamma_{j+1}}, \overline{U}_{\pm\beta_{j+1}} \rangle$. The argument of (8.9) shows that $\langle \overline{U}_{\pm\gamma_1}, \overline{U}_{\pm\gamma_{j+1}} \rangle \cong SL(3, K)$, with $\{\gamma_1, \gamma_{j+1}\}$ a fundamental system. By (8.6), $[\overline{U}_{\pm\beta_1}, \overline{U}_{\pm\beta_{j+1}}] = 1$, and by (8.4)(ii) $[\overline{U}_{\beta_u}, \overline{U}_{-\gamma_v}] = [\overline{U}_{-\beta_u}, \overline{U}_{\gamma_v}] = 1$ for $1 \leq u, v \leq 2j$.

We claim that β_1 and γ_1 are roots of the same length. Otherwise, the commutator relations applied to $[\overline{X}_i, \overline{X}_k] = \overline{X}_l \overline{X}_m$ shows that either $(\varphi\psi^2)^K$ or $(\varphi^2\psi)^K$ is a component of one of the representations $\mathbf{F}_{q^j} \otimes (X_m/X_m')$,

$\mathbf{F}_{q^j} \otimes X'_m$, or $\mathbf{F}_{q^j} \otimes X_l$. The previous computations show that this is impossible, proving the claim. There are three classes of $(T_0 \cap Y)$-root subgroups of Y (under $N_Y(T_0 \cap Y)$), with representatives, X_i, X_k, X_r. Since $\langle \bar{U}_{\pm\gamma_1}, \bar{U}_{\pm\gamma_{j\pm1}} \rangle \cong \mathrm{SL}(3, K)$ we conclude that the roots in $\bar{\Sigma}_i \cup \bar{\Sigma}_k \cup \bar{\Sigma}_l \cup \bar{\Sigma}_m \cup \bar{\Sigma}_r \cup \bar{\Sigma}_s$ are all long roots. Hence $\langle \bar{U}_{\pm\beta_1}, \bar{U}_{\pm\gamma_1} \rangle \cong \mathrm{SL}(3, K) \cong \langle \bar{U}_{\pm\gamma_{j+1}}, \bar{U}_{\pm\beta_{j+1}} \rangle$. We can now apply (2.13) and conclude that $\bar{D}_1$ is an image of $A_4(K)$ with fundamental set $\{\beta_1, \gamma_1, \gamma_{j+1}, \beta_{j+1}\}$.

Let $\bar{D}_i = \bar{D}_1^{\sigma^{i-1}}$, for $1 \le i \le j$. Using the aforementioned commutator information together with (8.6) and (8.8) we have $\langle \bar{D}_1, \dots, \bar{D}_j \rangle = D_1 \cdots D_j$, a central product. Thus, (8.1) holds, completing the proof of (8.9).

(8.12) If Y has Lie rank at least 3, then (8.1) holds.

Proof. Let $\beta_1, \dots, \beta_n$ be a fundamental system for the root system of Y, with $U_{\beta_1}, \dots, U_{\beta_n}$ the corresponding $(T_0 \cap Y)$-root subgroups, corresponding to the labeling of the Dynkin diagram of Y (see §1). For $i = 1, \dots, n$, let $U_{\beta_i} = X_{l_i}$. Then $[X_{l_1}, X_{l_n}] = 1$, while $[X_{l_{n-1}}, X_{l_n}] \neq 1$.

Fix $1 \le i \le n$ and write $\Delta_{l_i} = \{\bar{U}_{\beta_{i,1}}, \dots, \bar{U}_{\beta_{i,k_i}}\}$, where $k_i = j$ or $2j$. Arrange notation so that $\bar{U}^{\sigma}_{\beta_{i,k}} = \bar{U}_{\beta_{i,k+1}}$, for each $1 \le k < k_i$. Set $\bar{Z} = \langle \bar{X}_{l_1}, \dots, \bar{X}_{l_{n-1}}, \bar{X}^*_{l_1}, \dots, \bar{X}^*_{l_{n-1}} \rangle$. Inductively, we know that $\bar{Z} = \bar{Z}_1 \cdots \bar{Z}_j$, a commuting product of a $\langle \sigma \rangle$-orbit of Chevalley groups and $O^{p'}(\bar{Z}_\sigma) = \langle U_{\pm\beta_1}, \dots, U_{\pm\beta_{n-1}} \rangle$.

First suppose that Y is an untwisted group. Then $k_i = j$ for $1 \le i \le n$ and we may reorder, if necessary, so that $\bar{Z}_1 = \langle \bar{U}_{\pm\beta_{1,1}}, \dots, \bar{U}_{\pm\beta_{n-1,1}} \rangle$. By (8.9) there exists a unique k such that $\bar{U}_{\beta_{n-1,1}}$ does not commute with $\bar{U}_{\beta_{n,k}}$, and we may reorder Δ_n, if necessary, so that $k = 1$. Set $\bar{D}_1 = \langle \bar{Z}_1, \bar{U}_{\pm\beta_{n,1}} \rangle$. For $i < n-1$, $[U_{\beta_i}, U_{\beta_n}] = 1$, so by (6.10) $[\bar{X}_{l_i}, \bar{X}_{l_n}] = 1$, hence $[\bar{U}_{\beta_{i,1}}, \bar{U}_{\beta_{n,1}}] = 1$. This together with (8.4)(ii) yields $[\bar{U}_{\pm\beta_{i,1}}, \bar{U}_{\pm\beta_{n,1}}] = 1$ for each $i < n - 1$. By (8.9), $\langle \bar{U}_{\pm\beta_{n-1,1}}, \bar{U}_{\pm\beta_{n,1}} \rangle \cong \mathrm{SL}(3, K)$, so we can apply (2.13) and conclude that $\bar{D}_1$ has the same Dynkin diagram as does Y. Moreover, (8.6), (8.4), and induction show that $\bar{D}_1 \cdots \bar{D}_j$ is a commuting product. The result follows.

Now suppose Y is a twisted group. The argument is essentially the same as above, although slightly more complicated. We have $\bar{Z} = \bar{Z}_1 \cdots \bar{Z}_j$ and $Z = O^{p'}(\bar{Z}_\sigma) = \langle U_{\pm\beta_1}, \dots, U_{\pm\beta_{n-1}} \rangle$. To illustrate the charges we consider the case $Y \cong {}^2E_6(q^j)$, leaving the remaining cases to the reader. Here, $n = 4$, $Z \cong O^-(8, q^j)'$ and so $\bar{Z}_1 \cong \cdots \cong \bar{Z}_j \cong D_4(K)$. Each $(T_0 \cap Y)$-root group of Y is abelian, so $\bar{X}_{l_1}, \dots, \bar{X}_{l_4}$ are each the direct product of the root subgroups in $\Delta_1,\ldots,\Delta_4$, respectively. Moreover, $|\Delta_1|=|\Delta_2|=j$, while $|\Delta_3|=|\Delta_4|=2j$.

Labeling roots as before, we may assume $\bar{Z}_1 = \langle \bar{U}_{\pm\beta_{3,1}}, \bar{U}_{\pm\beta_{2,1}}, \bar{U}_{\beta_{1,1}}, \bar{U}_{\pm\beta_{3,j+1}}\rangle$. Here, σ^j interchanges the groups $\langle \bar{U}_{\pm\beta_{3,1}}\rangle$, $\langle \bar{U}_{\pm\beta_{3,j+1}}\rangle$, stabilizes the other two groups, and induces a graph automorphism on the Dynkin diagram of $\bar{Z}_1$. Now, $\bar{X}_{l_4} = \bar{U}_{\beta_{4,1}} \times \cdots \times \bar{U}_{\beta_{4,2j}}$, and by induction we have the structure of $\langle \bar{X}_{l_2}, \bar{X}_{l_3}, \bar{X}_{l_4}, \bar{X}^*_{l_2}, \bar{X}^*_{l_3}, \bar{X}^*_{l_4}\rangle$. We may choose our notation so that $[\bar{U}_{\beta_{4,1}}, \bar{U}_{\beta_{3,1}}] \neq 1$. Set $\bar{D}_1 = \langle \bar{U}_{\pm\beta_{4,1}}, \bar{Z}_1, \bar{U}_{\pm\beta_{4,j+1}}\rangle$ and argue, using (2.13) and commutator information, that $\bar{D}_1 \cong E_6(K)$. The argument is then completed by setting $\bar{D}_i = \bar{D}_1^{\sigma^{i-1}}$ for $1 \le i \le j-1$ and observing that $\langle \bar{D}_1,\ldots,\bar{D}_j\rangle = \bar{D}_1 \cdots \bar{D}_j$, a central product. This completes the proof of (8.10).

Theorem (8.1) follows from (8.6)–(8.12).

9. A technical result. In this section we apply the results of §7 and §8 and establish a technical result that will be useful in §10. Continue the assumptions $p \ge 5$ and $q > 7$.

We introduce the following notation which will be used throughout the rest of the paper. If Y is a T_0-invariant subgroup of G, set $Y(T_0) = \langle X_i \mid X_i \le Y\rangle$ and $\bar{Y}(T_0) = \langle \bar{X}_i \mid X_i \le Y\rangle$.

Throughout the section we let $Y \le G$ with $Y^{T_0} = Y$ and $Y/O_p(Y)$ a central product of groups of Lie type in characteristic p. Let T_1 be a p'-Hall subgroup of a Sylow p-normalizer of Y. Then $T_1O_p(Y)/O_p(Y)$ is a Cartan subgroup of $Y/O_p(Y)$. This forces $C_{YT_0}(T_1)$ to be solvable and we choose a p'-Hall subgroup, T_2, of $C_{YT_0}(T_1)$. Then $T_2 \ge T_1$. We call T_2 a *Cartan* subgroup of YT_0.

(9.1) (i) T_2 is a maximal torus of G_0;

(ii) $Y = Y(T_2) = O^{p'}(\bar{Y}(T_2))_\sigma$ and $YT_0 = YT_2$;

(iii) $\bar{Y}(T_2) = \bar{Y}(T_0)$ and $\bar{Y}(T_2)\bar{T}_2 = \bar{Y}(T_2)\bar{T}$, where $\bar{T}_2 = C_{\bar{G}}(T_2)^0$ (the maximal torus of $\bar{G}$ containing T_2);

(iv) T_2 can be chosen so that there is a T_2-invariant subgroup $J \le Y$ and Y is the semidirect product $Y = O_p(Y)J$ and $JT_2 = JT_0$;

(v) T_2 is a p'-Hall subgroup of a Sylow p-normalizer of YT_0.

We remark that a missing item in (9.1) is the assertion $Y = Y(T_0)$. At this stage we do not even have $G_0 = G_0(T_0)$; that is we have yet to establish the fact that G_0 is generated by its T_0-root subgroups.

The rest of this section concerns the proof of (9.1). We first show that (i) holds, noting that this is just (7.2) in case $O_p(Y) = 1$. So suppose $O_p(Y) \neq 1$ and let $\bar{P}$ be the canonical parabolic subgroup of $\bar{G}$ with $O_p(Y) \leq R_u(\bar{P})$ and $N_{\bar{G}}(O_p(Y)) \leq \bar{P}$. Then $\bar{P}$ is σ-invariant and (6.4) shows $\bar{T} \leq \bar{P}$. Let $\bar{L}$ be the Levi factor of $\bar{P}$, with respect to the maximal torus $\bar{T}$. Set $J = \bar{L} \cap YR_u(\bar{P})$. Then $J \leq O^{p'}(\bar{L}_\sigma) = L$ and J is a T_0-invariant commuting product of groups of Lie type in characteristic p. Now let $\hat{T}_1$ be a Cartan subgroup of J and $\hat{T}_2 = C_{JT_0}(\hat{T}_1)$. By (7.2), $\hat{T}_2$ is a maximal torus of G_0. But $JR_u(\bar{P})_\sigma = YR_u(\bar{P})_\sigma$ implies that $\hat{T}_2$ and T_2 are conjugate by an element in $R_u(\bar{P})_\sigma$. Hence (i) holds.

Let $\bar{T}_2 = C_{\bar{G}}(T_2)^0$, a maximal torus. We claim $Y = Y(T_2) = O^{p'}((\bar{Y}(T_2))_\sigma)$. Let V_1 and V_2 be Sylow p-subgroups of Y normalized by T_1 and such that $Y = \langle V_1, V_2 \rangle$. The argument of (7.5) applied either to YT_0 or to $YT_0R_u(\bar{P})_\sigma/R_u(\bar{P})_\sigma$ (according to whether or not $O_p(Y) = 1$) shows that $T_2 \leq N(V_1) \cap N(V_2)$. By (6.9), V_1 and V_2 are each products of T_2-root subgroups of G_0 and $V_i = \bar{V}_i(T_2)_\sigma$, for $i = 1, 2$. As $Y = \langle V_1, V_2 \rangle$ we have $Y = Y(T_2)$. If $O_p(Y) = 1$, let $\bar{K} = \bar{G}$, and if $O_p(Y) \neq 1$ let $\bar{P}$ be as above and $\bar{K}$ the Levi factor of $\bar{P}$ containing $\bar{T}_2$. So $\bar{K} = \bar{K}^\sigma$.

If $O_p(Y) \neq 1$, embed each V_i in the unipotent radical of a $\langle\sigma\rangle T_2$-invariant parabolic subgroup of $\bar{P}$ and use (5.1) and (5.5) to conclude that $\bar{V}_i(T_2) \leq \bar{P}$, for $i = 1, 2$. Moreover, in this situation, each $\bar{T}_2$-root subgroup of $\bar{P}$ is contained either in $R_u(\bar{P})$ or in $\bar{K}$. It follows that $\bar{V}_i(T_2) = \overline{O_p(Y)}(T_2)(\bar{V}_i(T_2) \cap \bar{K})$ for $i = 1, 2$. By (6.10) $\overline{O_p(Y)}(T_2)$ is normalized by $\bar{V}_i(T_2) \cap \bar{K}$ for $i = 1, 2$. Therefore, $\bar{Y}(T_2) = \overline{O_p(Y)}(T_2)(\bar{Y}(T_2) \cap \bar{K})$, and this also holds if $O_p(Y) = 1$. Since $V_i = O_p(Y)(V_i \cap \bar{K})$, for $i = 1, 2$, $Y \cap \bar{K}$ complements $O_p(Y)$. Hence, (8.1)(iii) shows that $Y(T_2) = O^{p'}(\bar{Y}(T_2)_\sigma)$. This proves the claim.

Set $\bar{J} = \bar{Y}(T_2) \cap \bar{K}$, $J = O^{p'}(\bar{J}_\sigma)$, and $X = YT_0$. Then $Y = Y(T_2) = O_p(Y)J$, a semidirect product. The Frattini argument gives $X = YN_X(T_2) = O_p(Y)JN_X(T_2)$. Now, $J = J(T_2)$ is the group generated by all T_2-root subgroups of Y whose opposite root group is also in Y. Since $N_X(T_2) \leq N(\bar{T}_2)$, we conclude that $N_X(T_2)$ normalizes both J and $\bar{J} = \bar{J}(T_2)$. Thus, $JN_X(T_2)$ is a group and normalizes $\bar{J}\bar{T}_2$. As $JN_X(T_2)$ complements $O_p(Y)$, we may replace T_0 by a Y-conjugate that lies in $JN_X(T_2)$. In particular, T_0 normalizes a maxial torus, $\bar{T}_3 = \bar{T}_3^\sigma$, of $\bar{J}\bar{T}_2$. Then $T_0 \leq N(G_0 \cap (\bar{T}_3)_\sigma)$, so (6.3) implies $T_0 = G_0 \cap (T_3)_\sigma$. But then (2.8) implies $\bar{T}_3 \leq C_{\bar{G}}(T_0)^0 = \bar{T}$. Hence, $\bar{T} = \bar{T}_3$ and $\bar{J}\bar{T}_2 = \bar{J}\bar{T}$. Then (2.5) implies $JT_2 = JT_0$. Now $Y = Y(T_2) = O^{p'}(\bar{Y}(T_2)_\sigma) = O_p(Y)J$, therefore $YT_2 = YT_0$, completing (ii).

From the above, $Y = O_p(Y)J$, a semidirect product, and $JT_0 = JT_2$. This proves (iv). From $\bar{Y}(T_2) = \overline{O_p(Y)}(T_2)\bar{J}$ and the fact that $\bar{J}\bar{T}_2 = \bar{J}\bar{T}$, we conclude $\bar{Y}(T_2)\bar{T}_2 = \bar{Y}(T_2)\bar{T}$. We next prove $\bar{Y}(T_2) = \bar{Y}(T_0)$. Let $\bar{D} = \overline{O_p(Y)}(T_2)$, so that $\bar{D}_\sigma = O_p(Y)$, by (6.9). Earlier arguments imply $\bar{D} \trianglelefteq \bar{Y}(T_2)$, so $\bar{D} \trianglelefteq \bar{Y}(T_2)\bar{T}_2 = \bar{Y}(T_2)\bar{T}$. Thus, $\bar{T}\langle\sigma\rangle$ normalizes $\bar{D}$, and since $\bar{D}_\sigma = O_p(Y)$ we again apply (6.9) to conclude $\bar{D} = \overline{O_p(Y)}(T_0)$. So it will suffice to show $\bar{J} = \bar{J}(T_2) = \bar{J}(T_0)$. We have $\bar{J}\bar{T}_2 = \bar{J}\bar{T}$, so each component of $\bar{J}$ is $\bar{T}$-stable and generated by certain $\bar{T}$-root subgroups of $\bar{G}$. If $\alpha \in \bar{\Sigma}_l$ and if the $\bar{T}$-root subgroup $\bar{U}_\alpha$ is contained in $\bar{J}$, then $\bar{X}_l \leq \bar{J}$. Hence $X_l \leq O^{p'}(\bar{J}_\sigma) = J$. This shows $\bar{J} \leq \bar{J}(T_0)$.

For the other containment, suppose $X_i \leq J$. If X_t is a p-group, then $\bar{J}$ contains a $\bar{T}\langle\sigma\rangle$-invariant parabolic subgroup $\bar{P}$ with $X_i \leq R_u(\bar{P})$. Then $R_u(\bar{P})$ is the product of certain $\bar{X}_l$, $1 \leq l \leq t$, and (6.9) implies $\bar{X}_i \leq R_u(\bar{P})$. Hence $\bar{X}_i \leq \bar{J}$.

Now suppose X_i is of Lie type and defined over $\mathbf{F}_{q^k}$. Let $\bar{C}$ be a σ-invariant maximal torus of $\bar{X}_i\bar{T}$ contained in a σ-stable Borel subgroup of $\bar{X}_i\bar{T}$ (see (2.9) of [25]). Set $C = G_0 \cap \bar{C}_\sigma$. Then $C \cap X_i$ is a Cartan subgroup of X_i, while C is a maximal torus of G_0. Moreover, $X_iT_0 = X_iC$ by (2.5). Therefore, $C \leq X_iT_0 \leq JT_0$ and replacing T_0 by C in the above we conclude $\bar{C} \leq \bar{J}\bar{T}_2 = \bar{J}\bar{T}$. Each C-root subgroup of X_i is a p-group, so the argument of the last paragraph gives $\bar{X}_i \leq \bar{J}$. Therefore, $\bar{J}(T_0) \leq \bar{J}$, and (iii) holds.

It remains to prove (v). We have seen that $T_2 \leq N(V_1)$, so (v) follows from this and (ii). This completes the proof of (9.1).

III. The main theorems

10. Classification. In this section and the next we complete our analysis of subgroups of G invariant under a maximal torus. We show that any such subgroup arises from a subset of $\bar{\Sigma}$. However, to carry out the proof, we must invoke the classification of finite simple groups. Let $\mathcal{K}$ denote the list of simple groups; the alternating groups, groups of Lie type, and the 26 sporadic groups (see (11.1)). At one point we will need the fact that any T_0-invariant simple section of G is isomorphic to a group in $\mathcal{K}$. Very little information about groups in $\mathcal{K}$ is actually required for the proofs of the main results, but the author sees no way to avoid an application of the classification theorem.

The fundamental result is (10.1) below, while (10.2) provides extra information, which follows fairly easily from (10.1) and previous results. To state (10.2) we require the following terminology and notation. For a subset $\Delta \subseteq \bar{\Sigma}$ let $\bar{D} = \langle \bar{U}_\alpha \mid \alpha \in \Delta\rangle$ and $\hat{\Delta} = \{\beta \in \bar{\Sigma} \mid \bar{U}_\beta \leq \bar{D}\}$. We say Δ is *closed* if $\Delta = \hat{\Delta}$, in which case we write $\bar{D} = \bar{G}(\Delta, \bar{T})$. If Δ is a closed, $\langle\sigma\rangle$-invariant, subset of $\bar{\Sigma}$, we set $G(\Delta, \bar{T}) = O^{p'}(\bar{G}(\Delta, \bar{T})_\sigma)$.

Throughout this section and the next we make the standing hypothesis that $p > 3$ and $q > 11$.

THEOREM (10.1). *Let Y be a T_0-invariant subgroup of G.*
(i) $Y(T_0) \trianglelefteq Y$ *and* $\langle T_0^Y \rangle = Y(T_0)T_0$.
(ii) $Y(T_0) = O^{p'}(\bar{Y}(T_0)_\sigma)$.

THEOREM (10.2). *Let $T_0 \leq Y \leq G$. Then*
(i) $O_p(Y) \trianglelefteq Y(T_0)$ *and* $Y(T_0)/O_p(Y) = E(Y/O_p(Y))$.
(ii) *$Y(T_0)$ is the semidirect product of $O_p(Y)$ and a T_0-invariant subgroup $J = J(T_0)$, and J is a central product of groups of Lie type over extension fields of $\mathbf{F}_q$. Also, $O_p(Y)$ is a product of T-root subgroups of G.*
(iii) *If T_1 is any maximal torus of G_0 with $T_1 \leq Y$, then $Y(T_0) = Y(T_1)$, $\bar{Y}(T_0) = \bar{Y}(T_1)$, and $Y(T_0)T_0 = Y(T_1)T_1$.*
(iv) *There is a maximal torus T_1 of G_0 such that $T_1 \leq Y$ and $Y = Y(T_1)N_Y(T_1)$.*
(v) *There is a unique $\langle \sigma \rangle$-invariant, closed, subset Δ of $\bar{\Sigma}$, such that $\bar{Y}(T_0) = \bar{G}(\Delta, \bar{T})$ and $Y(T_0) = G(\Delta, \bar{T})$.*

We will first show how to derive (10.2) from (10.1). So suppose $T_0 \leq Y$ and that the hypotheses and conclusions of (10.1) are satisfied. Set $\Delta = \{\alpha \mid \bar{U}_\alpha \leq \bar{Y}(T_0)\}$. Clearly Δ is $\langle \sigma \rangle$-invariant, closed, and $\bar{Y}(T_0) = \bar{G}(\Delta, \bar{T})$. Hence, $Y(T_0) = G(\Delta, \bar{T})$. This gives (v).

By (2.5) $\bar{Y}(T_0) = R_u(\bar{Y}(T_0))\bar{L}$, where $\bar{L} = \bar{L}^\sigma$ is $\bar{T}$-invariant and $\bar{L}$ is semi-simple. Then $(R_u(\bar{Y}(T_0)))_\sigma = O_p(Y(T_0))$, $J = O^{p'}(\bar{L}_\sigma)$ is a central product of groups of Lie type over extension fields of $\mathbf{F}_q$, and $Y(T_0) = O_p(Y(T_0))O^{p'}(\bar{L}_\sigma)$. By (6.1) $O_p(Y) \leq Y(T_0)$, so $O_p(Y(T_0)) = O_p(Y)$ and (ii) holds.

Since $q > 11$, $J = E(J)$ and so (10.1)(i) implies $Y(T_0)/O_p(Y) \trianglelefteq E(Y/O_p(Y))$. Suppose equality fails to hold and let $X/O_p(Y) = C(Y(T_0)/O_p(Y)) \cap E(Y/O_p(Y))$. Then $[X, T_0] \leq X$, while (10.1)(i) implies $[X, T_0] \leq Y(T_0)T_0$. This forces $[X, T_0] \leq O_p(Y)$, whence $O_p(Y)T_0 \trianglelefteq I$, where $I = XT_0$. The Frattini argument implies $I = O_p(Y)N_I(T_0)$. Now, $T_0O_p(Y)/O_p(Y) \leq Z(I/O_p(Y))$ and $T_0 \cap O_p(Y) = 1$. Therefore, $N_I(T_0) \leq C_I(T_0) = T_0$ by (2.8). But then $I = O_p(Y)T_0$, a contradiction. This proves (i).

Let T_1 be a maximal torus of G_0 with $T_1 \leq Y$. Replacing T_0 by T_1 in the above we have $Y(T_1)/O_p(Y) = E(Y/O_p(Y))$, so $Y(T_1) = Y(T_0)$. Let T_3 be a Cartan subgroup of J and $T_2 = C_{JT_0}(T_3)$. By (9.1) T_2 is a maximal torus of G_0 with $\bar{L} = \bar{J}(T_2) = \bar{J}(T_0)$. Also, the proof of (9.1) showed $\bar{J}(T_2)\bar{T}_2 = \bar{J}(T_0)\bar{T}$, where $\bar{T}_2 = C_{\bar{G}}(T_2)^0$, a maximal torus of $\bar{G}$. By (9.1) and (2.5)(iv), $JT_2 = G_0 \cap (\bar{J}(T_2)\bar{T}_2)_\sigma = G_0 \cap (\bar{J}(T_0)\bar{T})_\sigma = JT_0$. Since $\bar{T}_2 \leq \bar{J}(T_0)\bar{T} \leq \bar{Y}(T_0)\bar{T}$, we have $\bar{T}_2 \leq N(R_u(\bar{Y}(T_0)))$. From (6.9) and the fact $R_u(\bar{Y}(T_0))_\sigma = O_p(Y)$ we conclude $R_u(\bar{Y}(T_0)) = \overline{O_p(Y)}(T_2)$, and this proves $\bar{Y}(T_0) = \bar{Y}(T_2)$.

The results of the last paragraph yield $Y(T_0)T_0 = Y(T_2)T_2$. Since T_3 is a Cartan subgroup of J we have $Y = Y(T_2)N_Y(T_3)$. Now $N_Y(T_3)$ normalizes $C(T_3) \cap Y(T_2)T_2$ and this group is solvable with T_2 as a Hall p'-subgroup. So the Frattini argument yields $Y = Y(T_2)N_Y(T_2)$. Notice that this gives (iv). In addition, the factorization shows $Y \leq N_{\bar{G}}(\bar{Y}(T_2)\bar{T}_2)$ (since $Y(T_2) \leq \bar{Y}(T_2)$). In particular, $T_1 \leq N(\bar{Y}(T_2)\bar{T}_2)$ so by (5.16) of [25] T_1 normalizes a maximal torus $\bar{T}_4$ of $\bar{Y}(T_2)\bar{T}_2$. By (6.3) we have $T_1 = (\bar{T}_4)_\sigma \cap G_0$, so (2.8) implies $\bar{T}_4$ is the unique maximal torus of $\bar{G}$ containing T_1. Therefore, $\bar{Y}(T_2)\bar{T}_2 = \bar{Y}(T_2)\bar{T}_4$ so (2.5) implies $Y(T_2)T_2 = Y(T_2)T_1$. By the above, $Y(T_2)T_2 = Y(T_0)T_0$ and $Y(T_0) = Y(T_2) = Y(T_1)$. Hence, $Y(T_1)T_1 = Y(T_0)T_0$. Replacing T_1 by a $Y(T_2)$-conjugate we may assume $JT_2 = JT_1$. Now replace T_0 by T_1 in the previous argument to get $\bar{Y}(T_2) = \bar{Y}(T_1)$. Hence $\bar{Y}(T_1) = \bar{Y}(T_0)$ and this establishes (iii), completing the proof of (10.2).

The rest of §10 and all of §11 concerns the proof of (10.1). Toward this end suppose the result false and let G_0 be a counterexample of least order for which (10.1) fails for some pair (T_0, Y). Then $Z(G_0) = 1$. We may assume $T_0 \leq Y$ (otherwise replace Y by YT_0), and among all such groups Y choose one with $|Y|$ minimal. In other words, if T_1 is a maximal torus of G_0 and $T_1 \leq Y_1$ where $|Y_1| < |Y|$, then (10.1) holds for the pair (T_1, Y_1).

For X a finite group, let $E(X)_p$ denote the product of all components of X that are of Lie type in characteristic p.

(10.3) Suppose X is a proper, T_0-invariant, subgroup of G_0 and $X/O_p(X) = E(X/O_p(X))_p$. Then (10.1) holds for the group XT_0.

Proof. Let T_2 be a p'-Hall subgroup of a Sylow normalizer of XT_0. By (9.1)(i) T_2 is a maximal torus of G_0. By (9.1)(iii) $\bar{X}(T_0) = \bar{X}(T_2)$ and by (9.1)(ii) $X = O^{p'}(\bar{X}(T_2)_\sigma) = O^{p'}(\bar{X}(T_0)_\sigma)$. Since $[X_i, T_0] = X_i$ for any T-root subgroup of G we have $\langle T_0^X \rangle = XT_0$. Therefore it remains to show $X = X(T_0)$.

By (9.1)(iv) we have $X = O_p(X)J$, a semidirect product, where J is T_0-invariant, $JT_0 = JT_2$ and $J = E(J)_p$. By (6.1) $O_p(X)$ is a product of T_0-root subgroups, so we may assume $X = J$. By (7.1), T_0 contains a maximal torus of each component of X, so we may also assume X to be quasisimple. Then (8.1) shows that X is defined over $\mathbf{F}_{q^j}$ for some j and $\bar{X}(T_2)$ is the commuting product of a $\langle\sigma\rangle$-orbit of j semi-simple groups, say $\bar{X}(T_2) = \bar{D}_1 \cdots \bar{D}_j$. Let $X_0 = O^{p'}((\bar{D}_1)_{\sigma^j})$, a group of Lie type over $\mathbf{F}_{q^j}$ having $A = X_0 \cap (\bar{T} \cap \bar{D}_1)_{\sigma^j}$ as maximal torus. From the minimality of $|G_0|$, (10.1) holds within X_0 and applying the result to the A-invariant subgroup X_0 of X_0 we have $X_0 = X_0(A)$. So X_0 is generated by its A-root subgroups (corresponding to $\langle\sigma^j\rangle$-orbits of $(\bar{T} \cap \bar{D}_1)$-root subgroups of $\bar{D}_1$). Using the fact that the map $x_0 \to x_0 x_0^{\sigma} \cdots x_0^{\sigma^{j-1}}$ is a surjection from X_0 to X and mapping A-root subgroups of X_0 to T_0-root subgroups of X we have $X = X(T_0)$, as required.

(10.4) (i) $Y \ngeq G_0$.
(ii) If $O_p(Y) = 1$, then $E(Y)_p = 1$.

Proof. Suppose $Y \geq G_0$. Then (10.1) holds if $Y(T_0) = G_0$. So $Y(T_0) < G_0$, and we set $Y_1 = Y(T_0)T_0 < Y$. Minimality implies that (10.1) holds for Y_1. However, $\bar{Y}_1(T_0) = \bar{G}$ and $Y_1(T_0) = Y(T_0) \neq O^{p'}(\bar{G}_\sigma)$. This is a contradiction, proving (i).

For (ii), suppose that $O_p(Y) = 1$ and $X = E(Y)_p \neq 1$. By (i), $X < G_0$. If X_i is any T_0-root subgroup of G, then $[X_i, T_0] = X_i$ (use (7.1), (5.5), and (6.8)), hence $X_i \leq \langle T_0^{X_i} \rangle$. So if we can show $\langle T_0^Y \rangle \leq XT_0$, then (7.1) implies $\langle T_0^Y \rangle = XT_0 = \langle T_0^X \rangle$, and (10.3) shows that (10.1) holds for XT_0, and hence for Y. Since we are assuming this to be false, it will suffice to show $\langle T_0^Y \rangle \leq XT_0$.

Let T_1 be a Cartan subgroup of X and let $D = C_Y(X)$. The Frattini argument shows $Y = XN_Y(T_1)$ and, of course $D \leq C_Y(T_1) \trianglelefteq N_Y(T_1)$. By (7.2), $T_2 = C_{XT_0}(T_1)$ is a maximal torus of G_0. By minimality of $|Y|$, (10.1) holds for the group $N_Y(T_1)$, which contains the torus T_2. Also, (10.1) holds for the groups DT_2 and $C_Y(T_1)$, each of which contains T_2. So applying (10.1) to DT_2 we have one of $O_p(D) \neq 1$, $E(D)_p \neq 1$, or $T_2 \trianglelefteq DT_2$. As $D \trianglelefteq Y$, the first two situations are out. Hence $T_2 \trianglelefteq DT_2$. Considering Aut(X), we see that $C_Y(T_1)/D$ is a solvable p'-group. Therefore, applying (10.1) to $C_Y(T_1)$ we conclude $T_2 \trianglelefteq C_Y(T_1)$. From (6.3) we have $T_2 \trianglelefteq N_Y(T_1)$, and this proves $XT_2 \trianglelefteq NX_Y(T_1) = Y$. Now, (9.1) shows $XT_2 = XT_0$, so $\langle T_0^Y \rangle \leq XT_0$, as required.

(10.5) $O_p(Y) = 1$.

Proof. Suppose $T_0 \leq Y \leq G$ and $O_p(Y) \neq 1$. By (3.9) of [4] there is a canonical parabolic subgroup $P < G_0 Y$ with $O_p(Y) \leq O_p(P)$ and

$N_G(O_p(Y)) \leq P$. In particular $T_0 \leq P$. By (6.4) $P = G_0 Y \cap \bar{P}_\sigma$, for $\bar{P}$ a σ-invariant parabolic subgroup of $\bar{G}$ containing $\bar{T}$. Write $P = VL(T \cap P)$, where $V = O_p(P)$ and L is the derived group of the Levi factor, $L(T \cap P)$, of P. If $L = 1$, then $Y = O_p(Y)(T \cap Y)$ and (10.1) follows from (6.9). Since Y is a counterexample to (10.1) we have $L \neq 1$.

Set $T_{00} = T_0 \cap L$, a maximal torus of L. One checks that the T_{00}-root subgroups of L are just the T-root subgroups of G that are contained in L. Let $\tilde{}$ denote images in LV/V and set $Y_1 = Y \cap VL$. Then $\tilde{Y}_1(\tilde{T}_{00}) \trianglelefteq \tilde{Y}_1$ and $\langle \tilde{T}_{00}^{\tilde{Y}_1} \rangle = \tilde{Y}_1(\tilde{T}_{00})\tilde{T}_{00}$ (by minimality of $|G_0|$). Set $A/V = \tilde{Y}_1(\tilde{T}_{00})$ and $X = Y \cap A$. Then $X \trianglelefteq Y_1$ and $X/O_p(X) = E(X/O_p(X))_p$. Also, $XT_{00} \trianglelefteq Y_1$.

Let T_2 be a p'-Hall subgroup of a Sylow p-normalizer of XT_0. By (9.1)(i) T_2 is a maximal torus of G_0, and (9.1)(ii) implies $XT_0 = XT_2$.

We claim $XT_0 \trianglelefteq Y$. First note that $O_p(X) = O_p(Y) = O_p(Y_1)$, then argue as in the proof of (10.2) that $X/O_p(Y) = E(Y_1/O_p(Y))$. This shows $X \trianglelefteq Y$. Let V_1 be a Sylow p-subgroup of X with $T_2 \leq N_Y(V_1) = D$. Suppose $D < Y$. Minimality of $|Y|$ implies $D(T_2) \trianglelefteq D$ and $\langle T_2^D \rangle = D(T_2)T_2 \trianglelefteq D$. As Y/Y_1 is an abelian p'-group, $D(T_2) \leq Y_1$. Also, $Y = XN_Y(V_1) = XD$ implies $XD(T_2)T_2 \trianglelefteq Y$. Now $D(T_2) \leq LV$ and $D(T_2)\tilde{}$ is generated by $(T_2 \cap L)\tilde{}$-root subgroups of $\tilde{L}$. Hence $[D(T_2)\tilde{}, (T_2 \cap L)\tilde{}] = D(T_2)\tilde{}$. On the other hand, $X(T_2 \cap L) = XT_{00} \trianglelefteq Y_1$. Therefore, $D(T_2) \leq X$ and so $XT_0 = XT_2 = XD(T_2)T_2 \trianglelefteq Y$, as required. Suppose then that $D = Y$; that is $X = V$.

Let $y \in Y$. Then V_1T_{00} and $V_1T_{00}^y$ are normal in Y_1, and so $\langle V_1T_{00}, V_1T_{00}^y \rangle = V_1F$, where F is a p'-group normalizing T_{00}. Also, F is generated by T_{00} and T_{00}^{yv} for some $v \in V_1$. Applying (6.3) to the maximal torus T_{00} of L, we conclude $F = T_{00}$. As y was arbitrary, $V_1T_{00} \trianglelefteq Y$. So $Y = V_1N_Y(T_{00})$ which shows $V_1C_Y(T_{00}) \trianglelefteq Y$. But (2.8) implies $C_Y(T_{00}) \leq V_1(T \cap Y)$, and $V_1(T \cap Y) \trianglelefteq Y$. Since $XT_0 = V_1T_0 = V_1(T \cap Y) \cap G_0 \trianglelefteq Y$, the claim is proved.

By the claim $XT_2 = XT_0 \trianglelefteq Y$, so $Y = XN_Y(T_2)$. By (10.3) $X = X(T_0)$ and by (6.1) $X = X(T_2)$. If E is a T_2-root subgroup of G or a T_0-root subgroup of G, then $[E, T_2] = E$ or $[E, T_0] = E$, respectively. Hence $X = Y(T_2) = Y(T_0)$. These remarks and (10.3) show that (10.1) holds, which we have assumed false. The proof of (10.5) is now complete.

(10.6) Suppose $F^*(Y) = \mathrm{Fit}(Y)$. Then
(i) $F^*(Y)$ is a p'-group;
(ii) If $Y_1 \trianglelefteq Y$ and $Y_1T_0 < Y$, then $Y_1 \leq N(T_0)$.
(iii) T_0 normalizes no non-trivial p-subgroup of Y.
(iv) $Y = \langle T_0^Y \rangle$.

Proof. (i) is immediate from (10.5). Suppose $Y_1 \trianglelefteq Y$ and $Y_1T_0 < Y$. Minimality of $|Y|$ implies that (10.1) holds for Y_1T_0. Therefore, either $O_p(Y_1) \neq 1$, $E(Y_1)_p \neq 1$, or $Y_1 \leq N(T_0)$. By (i) the first possibility is out, and our hypothesis rules out the second possibility. Therefore $Y_1 \leq N(T_0)$, establishing (ii).

For (iii), suppose T_0 normalizes the non-identity p-subgroup, D, of Y. Then $R = F^*(Y)DT_0$ is solvable. By (6.1), $R = O_p(R)N_R(T_0)$. Since $C_Y(F^*(Y)) \leq F^*(Y) = O_{p'}(F^*(Y))$, we have $O_p(R) = 1$ and $D \leq N(T_0)$. But then $[D, T_0] = 1$, against (2.8). This gives (iii).

Finally, let $Y_1 = \langle T_0^Y \rangle$ and suppose $Y_1 < Y$. Then $Y_1 \leq N(T_0)$ by (ii). Therefore, $T_0 \leq \mathrm{Fit}(Y_1) \leq \mathrm{Fit}(Y)$. By (6.1) $T_0 \trianglelefteq \mathrm{Fit}(Y)$, and (6.3) implies $T_0 \trianglelefteq Y$. However, with $T_0 \trianglelefteq Y$, (10.1) is a triviality, whereas we are assuming it false. This is a contradiction. So $Y_1 = Y$ and (iv) holds.

(10.7) Suppose $F^*(Y) = \mathrm{Fit}(Y)$. Then $F^*(Y) = O_r(Y)$ for some prime $r \neq p$.

Proof. Suppose $X = F^*(Y) = \mathrm{Fit}(Y)$ and $X = O_{p_1}(X) \times \cdots \times O_{p_l}(X)$, where $p_1,\ldots,p_l$ are distinct prime divisors of $|X|$ and $l > 1$. By (6.1) $X \leq N(T_0)$, so $[X, T_0] \leq T_0 \cap X$. If $T_0 \cap O_{p_i}(X) = 1$ for some i, then $[T_0, O_{p_i}(X)] \leq T_0 \cap O_{p_i}(X) = 1$ and $O_{p_i}(X) \leq C_G(T_0) = T$ (by (2.8)). But (10.6)(iv) shows $Y \leq G_0$, whence $O_{p_i}(X) \leq T \cap G_0 = T_0$, a contradiction. Therefore, $T_0 \cap O_{p_i}(X) \neq 1$ for $i = 1,\ldots,l$. Suppose $T_0 \leq X$. Then for $y \in Y$, $T_0^y \leq X \leq N(T_0)$, so $T_0^y = T_0$ by (6.3). Hence $T_0 \trianglelefteq Y$ and (10.6)(iv) gives the contradiction $Y = T_0$. Therefore, $1 < T \cap X < T$. Also, X is a p'-group by (10.6)(i).

For $i = 1,\ldots,l$ the groups $O_{p_i'}(T_0)$ and $O_{p_i}(X)$ normalize each other. Hence, they commute. Set $Y_i = \langle O_{p_i'}(T_0)^Y \rangle$. Then $Y_i \trianglelefteq Y$ and $Y_i \leq C(O_{p_i}(X))$. Suppose that for some $1 \leq i \leq l$ we have $Y_iT_0 < Y$. By (10.6)(ii) $Y_i \leq N(T_0)$, hence $O_{p_i'}(T_0) \leq \mathrm{Fit}(Y_i) \leq \mathrm{Fit}(Y) = X$. Let $C_i = C_Y(O_{p_i'}(X))$. Then $O_{p_i}(X) \leq C_i \trianglelefteq Y$ and $T_0 \leq Y_iC_i$. By (10.6)(iv), $Y = Y_iC_i$ and since $Y_i \leq N(T_0)$ and $Y = \langle T_0^Y \rangle$ we have $Y = C_iT_0 = C_i(O_{p_i'}(T_0))$. Since both C_i and $O_{p_i'}(T_0)$ centralize $Z = O_{p_i'}(T_0)$ we conclude $Z \leq Z(Y)$.

Since $O_{p_i'}(T_0) \neq 1$ we choose $1 \neq z \in Z$ and consider the group $C_{G_0}(z) \geq Y$. Let $D_1,\ldots,D_k$ be the components of $C_G(z)$. Then $D_1 \cdots D_kT_0 \trianglelefteq C_{G_0}(z)$ with quotient group isomorphic to a subgroup of the center of the universal covering group of G_0 (see (2.9)). Since $Y = \langle T_0^Y \rangle$, $Y \leq D_1 \cdots D_kT_0$ and we have $Y = T_0\hat{Y}$, where $\hat{Y} = Y \cap D_1 \cdots D_k$. If $1 \leq j \leq k$, let $\hat{Y}_j = \{d_j \in D_j \mid d_jg \in \hat{Y}$ for some $g \in D_1 \cdots \hat{D}_j \cdots D_k\}$. Then $\hat{Y}_j$ is a group and essentially the projection of $\hat{Y}$ to D_j (note that the projection is not defined since the product may not be direct). Then

$\hat{Y} \leq \hat{Y}_1 \cdots \hat{Y}_k$ and $T_0 \cap D_j \leq \hat{Y}_j$ for $1 \leq j \leq k$. Also, $T_0 \cap D_j$ is a maximal torus of D_j, for $1 \leq j \leq k$. Minimality of $|G_0|$ implies that (10.1) holds for each of the containments $T_0 \cap D_j \leq \hat{Y}_j \leq D_j$.

We conclude that for $1 \leq j \leq k$ one of the following hold: $O_p(\hat{Y}_j) \neq 1$, $E(\hat{Y}_j)_p \neq 1$, or $T_0 \cap D_j \trianglelefteq \hat{Y}_j$. Since $O_p(Y) = 1 = E(Y)_p$, we necessarily have $T_0 \cap D_j \trianglelefteq \hat{Y}_j$ for $1 \leq j \leq k$, hence $I = (T_0 \cap D_1) \cdots (T_0 \cap D_k) \trianglelefteq \hat{Y}$. By (2.8) $T_0 = D_1 \cdots D_k T_0 \cap C(I)$, so $\hat{Y} \leq N(T_0)$. We then have $T_0 \trianglelefteq Y = \langle T_0^Y \rangle$, so $T_0 = Y$, a contradiction. We have proved that $Y_i T_0 = Y$ for $1 \leq i \leq l$. In particular, Y/Y_i is abelian for $1 \leq i \leq l$. Set $\tilde{Y} = \bigcap_{i=1}^{l} Y_i$. Then $Y/\tilde{Y}$ is abelian, so $Y = \langle T_0^Y \rangle = \tilde{Y} T_0$. However, $\tilde{Y} \leq C_Y(X)$ (as $Y_i \leq C(O_{p_i}(X))$). Therefore, $\tilde{Y} \leq X$, $Y = XT_0 \leq N(T_0)$, and as $Y = \langle T_0^Y \rangle = T_0$, this is a final contradiction.

(10.8) $F^*(Y) \neq \mathrm{Fit}(Y)$.

Proof. By way of contradiction, suppose $F^*(Y) = \mathrm{Fit}(Y)$. By (10.7), $X = \mathrm{Fit}(Y) = O_r(Y)$ for some prime r and $r \neq p$ by (10.6). As in the first paragraph of the proof of (10.7) we have $1 < T_0 \cap X < T_0$. Also, $X \leq N(T_0)$, so $O_{r'}(T_0) \leq C_Y(X) \leq X$, and we conclude that T_0 is an r-group.

Fix $y \in Y$ with $T_0^y \neq T_0$ and set $\tilde{Y} = O_r(Y)\langle T_0, T_0^y \rangle$. If $\tilde{Y} < Y$, then minimality of $|Y|$ shows that (10.1) holds for $\tilde{Y}$. However, $C_Y(X) \leq X$ implies that $E(\tilde{Y})_p = O_p(\tilde{Y}) = 1$. Therefore, $T_0 \trianglelefteq \tilde{Y}$. But (6.3) implies T_0 is weakly closed in its normalizer, whence $T_0^y = T_0$, a contradiction. Therefore, $Y = \tilde{Y} = O_r(Y)\langle T_0, T_0^y \rangle$.

Let $A = T_0 \cap X$, $B = T_0^y \cap X$, and $V = \langle A, B \rangle$. Both of A, B are normal in X. If $A \cap B \neq 1$, then $\langle T_0, T_0^y \rangle \leq C(A \cap B)$ so we can choose $1 \neq z \in A \cap B \cap Z(Y)$. Then $Y \leq C_G(z)$ and the argument of (10.7) gives a contradiction. Therefore, $A \cap B = 1$. This shows that $V = \langle A, B \rangle = A \times B$. Also, $[V, T_0] \leq [X, T_0] \leq X \cap T_0 = A \leq V$, so $T_0 \leq N_Y(V)$ and similarly $T_0^y \leq N_Y(V)$. Hence $V \trianglelefteq Y$.

If $t \in T_0$, then $[X, t, t] \leq [A, t] = 1$. Therefore, t^r centralizes each Y-chief factor contained within X. But the intersection of the centralizers of such chief factors is an r-group and hence X. Therefore, $t^r \in X$ and $T_0^r \leq X \cap T_0 = A$. If $(T_0^y)^r$ contains an element, j, of order r^2, then for $t \in T_0$ we have $[t^r, j] \in [A, B] = 1$. As VT_0 has nilpotence class 2 we conclude $1 = [t^r, j] = [t, j]^r = [t, j^r]$. That is, $j^r \in C_Y(T_0) = T_0$, contradicting $A \cap B = 1$. Therefore, T_0^y, and hence T_0, has exponent at most r^2.

Say $\bar{G}$ has Lie rank n, so that $|T_0| = \frac{1}{d} f(q)$, where $f(t) = \prod \Phi_{d_i}(t)$ and $\sum \varphi(d_i) = n$. By (2.4)(iii) $|T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{d}(12)^n$. As an abelian group

T_0 has rank at most n (see (2.3)). Therefore, $|T_0| \leq (r^2)^n$ and $d \geq (12/r^2)^n$. As $d \leq n+1$, we have $r > 2$. Also, X/A is isomorphic to a subgroup of $\overline{W}$, so $n \geq 2$. If $r = 3$, then the inequality forces $q = 13$ and consideration of primitive divisors leads to a contradiction. Therefore, $r \geq 5$.

Suppose $\overline{G}$ is an exceptional group. Then $r \mid |\overline{W}|$ implies $\overline{W}$ is of type E_6, E_7, or E_8. If $r = 5$, then $|\overline{W}|_r = r, r, r^2$, respectively, while if $r = 7$, $|\overline{W}|_r = 1, r, r$, respectively. In any case, $r \leq 7$. If $|\overline{W}|_r = r$, then $X/A \cong Z_r$, so $X = V \cong Z_r \times Z_r$ and $Y/X \cong \mathrm{SL}(2, r)$. But then $|T_0| = r^2$, against $|T_0| \geq \frac{1}{d}(q-1)^n$. Therefore, $r = 5$, $\overline{W}$ is of type E_8, and $|X/A|$ divides r^2. If $|A| = r^2$, then $|B| = r^2$ and $X = A \times B$ is elementary of order r^4. So $Y/X \leq \mathrm{SL}(4, r)$, $|T_0| \leq r^2 \cdot r^4 = 5^6$, again contradicting $|T_0| \geq \frac{1}{d}(q-1)^n \geq (12)^8$. If $|A| = r$, then $|X| \leq r^3$ and one argues $|T_0| \leq r^4$, impossible. So $\overline{G}$ is not an exceptional group.

Therefore, G_0 is a classical group (${}^3D_4(q)$ is out as $r \mid |W|$ and $r \geq 5$) and we let M be the natural module for the appropriate covering group, $\hat{G}_0$, of G_0. Let $\hat{T}_0$ be the preimage of T_0 and $\hat{B}$ a Sylow r-subgroup of the preimage of B. Write $M = M_1 \oplus \cdots \oplus M_l$, a decomposition into irreducible $\hat{T}_0$-irreducible submodules. If $\hat{G}_0 \not\cong \mathrm{SL}(m, q)$ then M has a non-degenerate bilinear form and we can arrange $M_1, \ldots, M_l$ so that for some $k \leq l$, $M_1, \ldots, M_k$ are non-degenerate, while each of $M_{k+1}, \ldots, M_l$ is totally singular. Moreover, $M = M_1 \perp \cdots \perp M_k \perp (M_{k+1} \oplus M_{k+2}) \perp \cdots \perp (M_{l-1} \oplus M_l)$, with $\hat{T}_0$ inducing contragredient representations (contragredient followed by a field automorphism, in the unitary case) on the pairs $\{M_{k+1}, M_{k+2}\}, \ldots, \{M_{l-1}, M_l\}$. Set $N_1 = M_{k+1} \oplus M_{k+2}, \ldots, N_j = M_{l-1} \oplus M_l$, where $j = \frac{1}{2}(l-k)$. We now have $M = M_1 \perp \cdots \perp M_k \perp N_1 \perp \cdots \perp N_j$, and for $\hat{G}_0 = \mathrm{SL}(m, q)$, $M = M_1 \oplus \cdots \oplus M_k$, considering $j = 0$.

We claim that T_0 is not cyclic. Otherwise, $A \cong Z_r$ and $T_0 \cong Z_{r^2}$. Then $\mathrm{Aut}(T_0)$ has Sylow r-subgroups of order r, which implies $X/A \cong Z_r$. Thus, $X = V \cong Z_r \times Z_r$ and $Y/X \cong \mathrm{SL}(2, r)$. But then Y splits over V, forcing $T_0 \cong Z_r \times Z_r$, a contradiction. This proves the claim, which implies $l > 1$ and $G_0 \not\cong \mathrm{PSL}(2, q)$.

Next we show that $k + j = 2$. If $k + j = 1$, then $\hat{T}_0$ is cyclic, contradicting the preceding paragraph. So $k + j \geq 2$. If $k + j \geq 3$, then letting $\hat{T}$ be the maximal torus in $\mathrm{GL}(m, q)$, $\mathrm{Sp}(m, q)$, $\mathrm{SO}^{\pm}(m, q)$, or $\mathrm{GU}(m, q)$ containing $\hat{T}_0$, we have the Sylow 2-subgroup of $\hat{T}$ of rank at least 3. Thus, the Sylow 2-subgroup of $\hat{T}_0$ has rank at least 2, which forces $|T_0|$ to be even. However, T_0 is an r-group and $r \geq 5$. So $k + j = 2$, as desired. Accordingly, write $M = M' \oplus M''$, where each of M' and M'' has the form M_i or N_i.

The representations of $\hat{T}_0$ on M' and M'' have different kernels (otherwise $\hat{T}_0$ would be cyclic) and each of M' and M'' is either irreducible or the sum of two inequivalent $\hat{T}_0$-submodules. So M is the sum of at most 4 pairwise inequivalent, irreducible, $\hat{T}_0$-submodules, and since $B \leq N(T_0)$ and $r \geq 5$, B necessarily stabilizes each of these irreducibles.

Let $I \leq M$ be an irreducible $\hat{T}_0$-submodule of M such that $[\hat{T}_0|_I, \hat{B}|_I] \neq 1$. Say $\dim(I) = d$. Then $\hat{T}_0|_I$ can be regarded as a subgroup of the multiplicative group of the field $\mathbf{F}_{q^d}$ ($\mathbf{F}_{(q^2)^d}$ in the unitary case) and $\hat{B}$ induces field automorphisms on $\hat{T}|_I$. Therefore, $r \mid d$. On the other hand, $\Phi_d(q)$ divides the order of $\hat{T}_0|_I$, so let s be a primitive divisor of $\Phi_d(q)$. As $d > 2$, we have $s \mid |T_0|$, whence $s = r$. However, d is the order of q, modulo s, so $d \mid s - 1$. We now have $r \mid d$ and $d \mid r - 1$, which is absurd. This contradiction proves (10.8).

(10.9) If $T_0 \leq X$ and (10.1) holds for X, then $E(X) = E(X)_p$.

Proof. As (10.1) holds for X, $\langle T_0^X \rangle = X(T_0)T_0$. Let J be the product of those components of X not contained in $E(X)_p$. Then $[T_0, J] \leq J \cap X(T_0)T_0 \leq Z(J)$. The 3-subgroup lemma then shows $[T_0, J] = 1$. But this gives $J \leq C_{G_0}(T_0) = T_0$, a contradiction.

(10.10) (i) $F^*(Y) = E(Y)$;
(ii) $Z(E(Y)) = 1$; and
(iii) $E(Y)_p = 1$.

Proof. By (10.5) $O_p(Y) = 1$, so (10.4) shows $E(Y)_p = 1$. Also, (10.8) implies $F^*(Y) \neq \mathrm{Fit}(Y)$, so $X = E(Y) \neq 1$ and is a product of components, none of which is of Lie type in characteristic p. If $XT_0 < Y$, then minimality of $|Y|$ and (10.9) gives a contradiction. Therefore $Y = XT_0$.

To prove the result it will suffice to show $\mathrm{Fit}(Y) = 1$. By (6.1) we have $T_0 \trianglelefteq \mathrm{Fit}(Y)T_0$ (recall $O_p(Y) = 1$), hence $[T_0, \mathrm{Fit}(Y)] \leq T_0 \cap \mathrm{Fit}(Y) = A$. If $A = 1$, then $\mathrm{Fit}(Y) \leq C_{G_0}(T_0) \cap \mathrm{Fit}(Y) = T_0 \cap \mathrm{Fit}(Y) = A = 1$, as required. Suppose $A \neq 1$. Then $A \leq \mathrm{Fit}(Y) \leq C(X)$ and so $A \leq Z(Y)$. Fix $1 \neq a \in A$ and consider the embedding $Y \leq C_{G_0}(a)$.

Since T_0 centralizes no component of Y, $X = [T_0, Y]$ and this implies $X \leq D_1 \cdots D_k$, where $D_1, \ldots, D_k$ are the components of $C_{G_0}(a)$ (see (2.9)). By (7.1), $T_0 \cap D_1$ is a maximal torus of D_1, and we may reorder, if necessary, so that $Y_1 = [T_0 \cap D_1, Y] \neq 1$. Argue in D_1 with the subgroup $Y_1(T_0 \cap D_1)$, using minimality of $|G_0|$ and (10.9) to obtain a contradiction.

(10.11) Let $X = F^*(Y)$. Then $Y = XT_0 \leq \mathrm{Aut}(X)$ and either X is simple or X is the commuting product of two T_0-conjugate simple groups.

Proof. Let X_1 be a component of Y and $\{X_1,\ldots,X_k\}$ the orbit of X_1 under T_0. By (10.10)(iii), $E(Y)_p = 1$, so minimality of $|Y|$ and (10.9) imply $Y = X_1 \cdots X_k T_0$. By (10.10)(ii), each of $X_1,\ldots,X_k$ is a simple group. Suppose $k > 1$.

Let $T_2 = N_{T_0}(X_1)$, so that $T_2 \leq N(X_i)$ for $i = 1,\ldots,k$. Let r be a prime divisor of k and let t be an r-element of T_0 with $|tT_2| = r$. If $t^r = 1$, consider the group $C_X(t)T_0$ and obtain a contradiction (using minimality of $|Y|$). Hence, $t^r \neq 1$. From order consideration we have $r \mid |X_1|$, so it follows that r divides $|A|$, where $A = C_X(t^r)$. Write $A = A_1 \cdots A_k$ with $A_i = A \cap X_i$, $i = 1,\ldots,k$.

Apply (10.1) to the group AT_0. Let $C/O_p(A) = A(T_0)/O_p(A)$. Then $CT_0 \trianglelefteq AT_0$ and $C = C_1 \cdots C_k$, where $C_i = C \cap X_i$, for $i = 1,\ldots,k$. If $C > O_p(A)$, then by (9.1) T_0 contains a maxial torus of C. However, $T_0 \cap C_1 \leq T_0 \cap X_1 = 1$ (otherwise $T_0 \leq N(X_1)$). Therefore, $C = O_p(A)$ and $O_p(A)T_0 \trianglelefteq AT_0$. Let $1 \neq a_1 \in A_1$ with $|a_1| = r$. For $g \in T_0 - T_2$, $a_1^{-1}a_1^g \in X_1X_1^g \cap O_p(A)T_0$. So, modulo $O_p(A)$ this element centralizes T_0, and it follows that $X = X_1X_1^g$. This proves (10.11).

At this stage in the proof of (10.1) we consider the possibilities for $X = E(Y)$. This is where the classification of finite simple groups becomes relevant. Write $X = X_1$ or $X_1 \times X_2$.

(10.12) $X_1 \not\cong A_m$, for $m \geq 5$.

Proof. Suppose $X_1 \cong A_m$. First we rule out the case $X = X_1 \times X_2$. Otherwise, let $t \in T_0 - N(X_1)$ with t a 2-element. Then $X_1^t = X_2$ and $j = t^2 \in N(X_1)$. So j acts as an element of S_m on each of X_1 and X_2 and we set $A_i = C_{X_i}(j)$, for $i = 1,2$. Then $A_2 = A_1^t$. The structure of A_i is determined from the cycle decomposition of j. From (10.1) we conclude $T_0 \trianglelefteq A_1A_2T_0$. So for $1 \neq a_1 \in A_1$, $a_1^{-1}a_1^t \in [A_1A_2, T_0] \leq T_0 \leq C(t)$. This forces $|a_1| = 2$; hence A_1 is elementary abelian. From the known structure of A_1, we conclude $m \leq 5$ and $|T_0| \leq 8$. But $X_1 \times X_2 \leq G_0$ forces $n \geq 2$, and we obtain a contradiction from (2.4). Therefore, $X = X_1$.

Let $\Omega = \{1,\ldots,m\}$. Since $Y \leq \mathrm{Aut}(X)$ and since the order restrictions on $|T_0|$ force $m > 6$, we have $Y \cong A_m$ or S_m, and T_0 acts on Ω. We claim T_0 is transitive on Ω. Otherwise, we can write $\Omega = \Omega_1 \cup \Omega_2$, a disjoint union of T_0-invariant subsets. By minimality of $|Y|$ and (10.9) we have $|\Omega_i| \leq 4$ for $i = 1, 2$, hence $|\Omega| \leq 8$. By considering subgroups of $L_2(q)$ we see that $n \geq 2$, and so order restrictions (see (2.4)) lead to a contradiction. Therefore, T_0 is transitive on Ω, and as T_0 is abelian, T_0 is regular. In particular, $|T_0| = m$. If m is not a prime, write $\Omega = \Omega_1 \cup \cdots \cup \Omega_l$, a disjoint union, corresponding to a system of imprimitivity for T_0 on Ω. Then T_0 stabilizes a subgroup V isomorphic to $A_{\Omega_1} \times \cdots \times A_{\Omega_l}$. If $|\Omega_1| > 4$, then the minimality of $|Y|$ and (10.9) (applied to VT_0) gives a contradiction. Suppose $|\Omega_1| \leq 4$. Then VT_0 is solvable, and since $p \geq 5$, we conclude from (6.1) that $l = 2$ and V is an elementary abelian 2-group. That is, $|\Omega_1| = 2$ and $m = 4$, a contradiction. Therefore, m is prime. Also, $T_0 \cong Z_m$, $T_0 \leq X = F^*(Y)$, and $N_Y(T_0)/T_0$ is cyclic of order $\frac{1}{2}(m-1)$. We use this information in order to get a numerical contradiction.

Suppose G_0 is a classical group. Let $\hat{G}_0$ be the appropriate linear group acting on the natural module M, and let $\hat{T}_0$ be the preimage in $\hat{G}_0$ of T_0. With notation as in the proof of (10.8) we have $M = M_1 \oplus \cdots \oplus M_k$ if $\hat{G}_0 \cong \mathrm{SL}(n, q)$, while $M = M_1 \perp \cdots \perp M_k \perp N_1 \perp \cdots \perp N_j$, otherwise. Here $\hat{T}_0$ stabilizes each of the subspaces, acts irreducibly on each M_i, while each N_i decomposes into two $\hat{T}_0$-invariant totally singular subspaces. If $\hat{G}$ denotes the full linear group ($\mathrm{GL}(n, q)$, $\mathrm{GU}(n, q)$, $\mathrm{Sp}(n, q)$, or $SO^{\pm}(n, q)$) and $\hat{T}$ the maximal torus of $\hat{G}$ containing $\hat{T}_0$, then $\hat{T} = \hat{T}_1 \times \cdots \times \hat{T}_k$ if $\hat{G}_0 \cong \mathrm{SL}(n, q)$, or $\hat{T} = \hat{T}_{11} \times \cdots \times \hat{T}_{1k} \times \hat{T}_{21} \times \cdots \times \hat{T}_{2j}$ otherwise, where the appropriate subgroups act on the corresponding M_i or N_j, and are trivial on all other parts of the decomposition.

Now, T_0 has prime order. If G_0 is a symplectic or even dimensional orthogonal group, say $\dim(M) = 2n$, then $|\hat{T}|/|T_0|$ divides 4, and since $q \geq 13$ we conclude that $k + j = 1$. It follows that $N_{G_0}(T_0)/T_0$ has order at most $2n$. If $\hat{G}_0 \cong O(2n+1, q)'$, we get the same conclusion, although here one of M_i's has dimension 1 and $\hat{T}$ induces Z_2 on this factor. Suppose $\hat{G}_0 \cong \mathrm{SL}(n, q)$. Then $\hat{T}_1$ is cyclic of order $q^{m_i} - 1$, where $m_i = \dim(M_i)$. This forces $k \leq 2$ and if $k = 2$, then one of M_1 and M_2 has dimension 1. So here $N_{G_0}(T_0)/T_0$ has order n or $2n - 1$. Similarly, if $\hat{G}_0 \cong \mathrm{SU}(n, q)$ we have $N_{G_0}(T_0)/T_0$ of order at most $2n$. Thus, in all cases, $|N_{G_0}(T_0)/T_0| \leq 2r$, where r is the Lie rank of $\bar{G}$. This gives the inequality, $2r \geq \frac{1}{2}(m-1)$, or $4r + 1 \geq m$. Also, $m \geq \frac{1}{d}(q-1)^r \geq \frac{1}{d}(12)^r$ (by (2.4)) and $d \leq r + 1$. Hence, $(4r+1)(r+1) \geq 12^r$, a contradiction. This shows that G_0 is not a classical group.

Let G_0 be an exceptional group and $\bar{G}$ of Lie rank r. Then $\frac{1}{2}(m-1)$ divides $|\bar{W}|$, while $m \geq \frac{1}{d}(q-1)^r \geq \frac{1}{d}12^r$. Considering the possibilities for $|\bar{W}|$ we obtain a contradiction.

(10.13) X_1 is not a group of Lie type in any characteristic.

Proof. Suppose that X_1 is of Lie type and defined over $\mathbf{F}_{q_0}$, where $q_0 = r^a$ and r is prime. By (10.10)(iii), $r \neq p$. Minimality of $|Y|$ and (10.9) shows that T_0 stabilizes no subgroup $J < X$ with $E(J)_r \neq 1$.

Suppose r divides $|T_0 \cap X|$. We first claim that T_0 is an r-group. From (3.9) of [4] it follows that there is a canonical parabolic subgroup, D of X, with $O_r(T_0 \cap X) \leq O_r(D)$. Then $T_0 \leq N(D)$ and minimality of $|Y|$ implies that (10.1) holds for DT_0. Since $O_p(DT_0) \leq O_p(D) = 1$ and since $E(DT_0)_p \leq E(D)_p = 1$, we must have $T_0 \trianglelefteq DT_0$. In particular, $[O_{r'}(T_0), O_r(D)] \leq O_{r'}(T_0) \cap O_r(D) = 1$. Then $[D, O_{r'}(T_0)] \leq D \cap C(O_r(D)) \cap O_{r'}(T_0) = 1$. In particular, $O_{r'}(T_0)$ centralizes a Borel subgroup of X, and checking $\mathrm{Aut}(X)$ we see that $O_{r'}(T_0) = 1$. This proves the claim.

Let $U \in \mathrm{Syl}_r(X)$ with $T_0 \leq N(U)$. Then $N_Y(U)T_0$ is solvable and, as above, $T_0 \trianglelefteq N_Y(U)T_0$. Suppose $X = X_1 \times X_2$. Then $r = 2$. Let $t \in T_0 - N(X_1)$ and $a_1 \in N_{X_1}(U \cap X_1)$. Then $a_1^{-1}a_1^t = [a_1, t] \in T_0 = O_2(T_0)$, from which it follows that $q_0 = 2$. Also, $a_1^{-1}a_1^t \in T_0$ and T_0 abelian implies $U \cap X_1$ is abelian. But then, $X_1 \cong \mathrm{SL}(2,2)$, a contradiction. Therefore, X is simple. Since $T_0 \trianglelefteq N_Y(U)T_0$, $[N_Y(U), T_0, T_0] = 1$, and consideration of $T_0 = O_r(T_0) \leq \mathrm{Aut}(X)$ yields $T_0 \leq U$. Let D_1 be any proper parabolic subgroup of X with $U \leq D_1$. Then minimality of $|Y|$ yields $T_0 \trianglelefteq D_1$. If $X \not\cong \mathrm{PSL}(2, q_0)$, then letting D_1 vary, we conclude $T_0 \trianglelefteq X$, a contradiction. Therefore, $X \cong \mathrm{PSL}(2, q_0)$, and $N_Y(U) \leq N(T_0)$ forces $|T_0| = q_0$. Also, $Y = X$ and $N_Y(T_0)/T_0$ is cyclic of order $q_0 - 1/(2, q_0 - 1)$. At this point we have the same situation that existed at the end of the proof of (10.12) (set $m = q_0$ but allow for the fact that T_0 may not have prime order) and this led to a numerical contradiction. We conclude that $T_0 \cap X$ is an r'-group.

Suppose $1 \neq t \in T_0 \cap X$ and consider $C = C_X(t)$. Let $t \in I$, a maximal torus of X. By (2.9), there are commuting groups of Lie type, $D_1, \ldots, D_l$, over extension fields of $\mathbf{F}_{q_0}$ such that $D_1 \cdots D_l I$ is normal in C with quotient isomorphic to a subgroup of the center of the universal cover of Y. If $E(C) \neq 1$, we contradict the minimality of $|Y|$. Hence, $E(C) = 1$. Consequently, either $D_1 \cdots D_l I = I$ or $q_0 = 2$ or 3 and $D_i \cong \mathrm{SL}(2, q_0)$, $\mathrm{PSL}(2, q_0)$, $\mathrm{SU}(3,2)$, or $\mathrm{PSU}(3,2)$, for $i = 1, \ldots, l$.

Suppose one of the latter cases occurs and set $J = [D_1 \cdots D_l, T_0]$. Then $J \leq D_1 \cdots D_l \cap T_0 \leq O_{r'}(D_1 \cdots D_l)$ (as T_0 is an r'-group). Also, $J \trianglelefteq D_1 \cdots D_l T_0$. Since T_0 is abelian, $J \cap D_i \leq Z(D_i)$ for any i with $D_i \cong \mathrm{SL}(2,3)$ or $\mathrm{SU}(3,2)$. For such an i, $[T_0, D_i, D_i] \leq [Z(D_i), D_i] = 1$, and since D_i is generated by r-elements, $[T_0, D_i] = 1$. But then $D_i \leq C_{G_0}(T_0) = T_0$, whereas T_0 is an r'-group. We conclude that $D_i \cong \mathrm{PSL}(2, q_0)$ or $\mathrm{PSU}(3,2)$ for $i = 1, \ldots, l$. Normality of J in $D_1 \cdots D_k T_0$ and the previous commutator argument shows that $J = O_{r'}(D_1 \cdots D_l)$. In particular, there are root subgroups $A_1 \cong A_2 \cong Z_{q_0}$ of Y such that $\mathrm{PSL}(2, q_0) \cong \langle A_1, A_2 \rangle \leq C$ and $E = O_{r'}(\langle A_1, A_2 \rangle) \leq T_0$. Let t_1 be a generator of E. As above, $E(C_X(t_1)) = 1$. This implies that Y has Lie rank at most 2. Since $X \leq G_0$, $n \geq 2$, and $|T_0| \geq \frac{1}{3}12^2 \geq 48$ (see (2.4)). As T_0 is an abelian r'-subgroup of $C_Y(t)$ we obtain a numerical contradiction.

We now have $D_1 \cdots D_l I = I \trianglelefteq C$. Let m be the Lie rank of the overlying algebraic group of X_1. Then $|C|$ is bounded by the order of a maximal torus of the universal cover of X, so (2.4) implies that $|T_0 \cap X| \leq (q_0 + 1)^m$ (replace q_0 by $q_1 = \sqrt{q_0}$ for the Suzuki and Ree groups). Here we use the fact that if $X = X_1 \times X_2$, then $T_0 \cap X_1 = T_0 \cap X_2 = 1$. Regarding $T_0/T_0 \cap X$ as an abelian subgroup of $\mathrm{Out}(X)$ we have $|T_0| \leq (q_0 + 1)^{m+2}$ (again we replace q_0 by $q_1 = \sqrt{q_0}$ in the Suzuki and Ree cases). This inequality also holds in case $T_0 \cap X = 1$.

We cannot have $X_1 \cong \mathrm{Sz}(q)$, $L_3(4)$, or $U_4(3)$. For the first two this follows since T_0 is abelian of order at least 48. If $X_1 \cong U_4(3)$, use (5.16) of **[23]** together with the existence of an extraspecial 3-group in X_1 of order 3^5 to conclude that $n \geq 4$. Then $|T_0| \geq \frac{1}{5}12^4$, contradicting the above inequality.

Suppose M is a faithful module in characteristic p for a covering group of G_0. Using the main theorem of **[18]**, the containment $X < G_0$, and the above paragraph, we can obtain lower bounds on $\dim(M)$. Excluding the Suzuki and Ree groups, we then have $\dim(M) \geq \frac{1}{2}(q_0^m - 1)$ (in most cases this is too low, but for the symplectic groups in odd characteristic, it is exact). For $X \cong \mathrm{Sz}(q_0)$, ${}^2G_2(q_0)$, ${}^2F_4(q_0)$ we have $\dim(M) \geq (q_0/2)^{1/2}(q_0 - 1)$, $q_0(q_0 - 1)$, $(q_0/2)^{1/2}q_0^4(q_0 - 1)$, respectively. In what follows we use these bounds on $\dim(M)$ to obtain contradictory inequalities involving $|T_0|$. The contradiction is most easily obtained for the three exceptional cases, although they must be considered individually. We, therefore, leave these cases to the reader and present a treatment of the remaining cases.

First, suppose G_0 to be a classical group and let M be the usual module for the corresponding linear group. Then $\dim(M) \leq 2n + 1$. By the previous paragraph, $2n + 1 \geq \frac{1}{2}(q_0^m - 1)$, so $n > \frac{1}{4}(q_0^m) - 1$. We then have the inequalities $(q_0 + 1)^{m+2} \geq |T_0| \geq \frac{1}{d}(q - 1)^n \geq \frac{1}{2}(q - 1)^{n-1} > \frac{1}{2}(12)^{(1/4)(q_0^m)-2}$. This yields $288(q_0 + 1)^{m+2} > 12^{(1/4)(q_0^m)}$. When $m \geq 2$ and $q_0 \geq 5$ this is impossible. Moreover, if $q_0 = 3$ or 4, then $m \leq 2$, while if $q_0 = 2$, then $m \leq 4$. Suppose $m = 1$. Then the inequality forces $q_0 \leq 25$. Considering subgroups of $\mathrm{PSL}(2, q)$, we see that $n \geq 2$. But then $|T_0| \geq \frac{1}{d}12^n \geq 48$, contradicting $T_0 \leq \mathrm{Aut}(X)$. Therefore, $m \geq 2$ and it follows that $q_0 = 2$, 3, or 4. Also, $m = 2$ in the latter cases.

We treat these cases separately, using the inequality $(*)$ $(q_0 + 1)^{m+2} \geq |T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{d}12^n$. Suppose $q_0 = 4$. Then $2n + 1 \geq \frac{1}{2}(q_0^m - 1) > 7$, so $n \geq 4$, which contradicts $(*)$. Suppose $q_0 = 3$. Then $m = 2$ and X is simple. For the moment, exclude the case $X \cong \mathrm{PSp}(4,3)$. The result in **[18]** then gives $2n + 1 \geq \dim(M) \geq 6$. So $n \geq 3$ and we contradict $(*)$. If $X \cong \mathrm{PSp}(4,3)$, then X contains the split extension of an elementary abelian 3-group of order 3^3 and S_3 and $O_3(X)$ has class 3. However, G_0 is a classical group with $n \geq 2$, and one checks that this forces $n \geq 3$. This is a contradiction, leaving only the case $q_0 = 2$. Since T_0 is an r'-group we can improve the earlier bound to get $|T_0| \leq (q_0 + 1)^{m+1} = 3^{m+1}$. If $m = 4$, the bound $2n + 1 \geq \frac{1}{2}(2^m - 1)$ forces $n \geq 3$, whence $|T_0| \geq \frac{1}{4}12^3$, contradicting the above. If $m = 2, 3$, then $n \geq 2$ and we again have a contradiction unless $X_1 \cong \mathrm{PSU}(4,2)$ and $n = 2$. But $\mathrm{PSU}(4,2) \cong \mathrm{PSp}(4,3)$ and we have already observed that this forces $n \geq 3$.

At this point G_0 is an exceptional group, and, except for the case $G_0 \cong G_2(q)$, these cases are easier than the above. Suppose $G_0 = G_2(q)$, so that G_0 has a 7-dimensional representation in characteristic p. We then have the inequality $7 = \dim(M) \geq \frac{1}{2}(q_0^m - 1)$, so $q_0^m \leq 13$. If $q_0 = 2$, then as above, $|T_0| \leq (q_0 + 1)^{m+1}$ which forces $m \geq 4$ (as $d = 1$), a contradiction. If $q_0 = 3$, then $(*)$ forces $m = 2$ (X would be solvable if $m = 1$), so $X \cong \mathrm{PSL}(3,3)$, $\mathrm{PSp}(4,3)$, or $\mathrm{PSU}(3,3)$. But then we can improve the earlier bound on $|T_0|$ obtaining $|T_0| \leq 2(q_0 + 1)^{m+1}$, which contradicts $(*)$. Therefore, $q_0 \geq 4$, $m = 1$, and $X \cong \mathrm{PSL}(2, q_0)$. This contradicts $(*)$ (as $q_0 \leq 13$). So $G_0 \ncong G_2(q)$.

For the other exceptional groups argue as follows. In each case X acts on a module M of dimension 27 if $G_0 \cong F_4(q)$, $E_6(q)$, or ${}^2E_6(q)$, dimension 56 if $G_0 \cong E_7(q)$, and dimension 248 if $G_0 = E_8(q)$. These bounds give easy contradictions. Details are omitted, but we illustrate with the case $G_0 = F_4(q)$. Here $\dim(M) = 27 \geq \frac{1}{2}(q_0^m - 1)$, while $n = 4$ and $d = 1$. As above we obtain a contradiction. This completes the proof of (10.13).

11. Classification (continued). In this section we complete the proof of (10.1). In view of the classification theorem and (10.11)–(10.13) we have $X = F^*(Y)$ a sporadic simple group or the direct product of two sporadic groups interchanged by an element of T_0. Our method is to first show that T_0 is T.I. set in Y, of odd order, and to use this together with properties of the individual groups to obtain a contradiction. An effort has been made to keep the number of special properties to a minimum, avoiding an extensive list of references. For the most part we only need the orders of the sporadic groups (Table (11.1), below) and the structure of centralizers of involutions (available in Table 1 of [**2**]). For certain groups we do appeal to the literature for additional information. A somewhat shorter proof could be obtained by citing a much larger number of references, but this did not seem worthwhile.

Let n be the Lie rank of $\bar{G}$. The containment $X \leq G_0$ certainly forces $n \geq 2$, whence $|T_0| \geq \frac{1}{d}(q-1)^n \geq 48$ (as $d \leq n+1$). In the following table we list the sporadic simple groups and their orders.

TABLE (11.1)

X	$\lvert X \rvert$	X	$\lvert X \rvert$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	ON	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 13$
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	F_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	F_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	F_{24}'	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot$
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$		$23 \cdot 29$
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot$	Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
	$31 \cdot 37 \cdot 43$	F_1	$2^{46} \cdot 3^{26} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17$
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$		$19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$
Mc	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	F_2	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot$
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$		$23 \cdot 31 \cdot 47$
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	F_3	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
		F_5	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
		Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$

(11.2). If $t \in T_0$ is an involution, then $T_0 \leq O_2(C_Y(t))$ and $C_Y(t)$ is 2-constrained.

Proof. The possibilities for $C_X(t)$ are presented in Table 1 of [**2**] and in [**17**] for $X = J_4$. Suppose $C_X(t)$ is not 2-constrained. We have $T_0 \leq N(E(C_X(t)))$, so minimality of $|Y|$ implies (10.1) holds for $E(C_X(t))T_0$. The only possibility is $X \cong J_3$ or $J_3 \times J_3$, with t inducing an outer automorphism on each component of X. Hence, $C_X(t) \cong \mathrm{PSL}(2, 17)$ or $\mathrm{PSL}(2, 17) \times \mathrm{PSL}(2, 17)$, and $q = 17$. Also, $T_0 \cap X \leq C_X(t)$, $T_0 \cap X$ intersects each component trivially if X is not simple, and $|T_0 \cap X| \geq 12$ (24 if X is simple). Since T_0 is a p'-group, this is impossible. Therefore, $C_X(t)$ is 2-constrained. From (10.1) and the minimality of $|Y|$ we conclude $C_Y(t) \leq N(T_0)$, which forces $T_0 \leq O_2(C_Y(t))$.

(11.3) (i) $|T_0|$ is odd.
(ii) X is simple.

Proof. Suppose $|T_0|$ is even and apply (11.2) to obtain $T_0 \leq O_2(C_Y(t))$ and $C_Y(t)$ 2-constrained for each involution $t \in T_0$. Suppose $X = X_1 \times X_2$, with $X_2 = X_1^y$, for some $y \in T_0$. The previous remarks show that each involution, t, of T_0 normalizes X_1 and X_2, and since $y \in O_2(C_Y(t))$ we necessarily have $C_X(t)$ a 2-group (consider $[a_1, y]$ for $a_1 \in C_{X_1}(t)$ of odd order). However, Table 1 of [2] and [17] show this to be impossible. Therefore (ii) holds.

We know that T_0 is a 2-group of order at least 48, and this forces $|T_0| \geq 2^6$. By Table 1 of [2] and [17] the only sporadic groups X having an involution t such that $O_2(C_Y(t))$ contains an abelian subgroup of order 2^6 are R_u, Co_1, Co_2, F_{22}, F_{23}, F_{24}', F_2, J_4, or F_1. Moreover, in each case $O_2(C_Y(t))$ has exponent at most 4.

Since $|T_0| \geq 2^6$, T_0 has rank at least 3. From (2.1)(iii) we conclude $n \geq 3$. Hence, $|T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{4} \cdot 12^3$ (recall, $d \leq n+1$), and since T_0 is a 2-group, $|T_0| \geq 2^9$. Repeat the argument. Namely, T_0 of exponent at most 4 implies T_0 of rank at least 5. Therefore, $n \geq 5$ and $|T_0| \geq \frac{1}{6} \cdot 12^5 > 2^{15}$. Eventually, we obtain $|T_0| > |X|$, a contradiction.

(11.4) (i) $X = Y$.
(ii) $N_X(T_0)$ is the unique maximal subgroup of Y containing T_0.
(iii) T_0 is a T.I. set in $X = Y$.

Proof. $X = Y$ by (11.3) and Table 1 of [2]. Suppose (ii) holds. If T_0 is not a T.I. set in Y, then there exists $y \in Y - N(T_0)$ with $1 \neq T_0 \cap T_0^y$. Choose $1 \neq t \in T_0 \cap T_0^y$, so that $T_0, T_0^y \leq C_Y(t)$. By (ii), $C_Y(t) \leq N(T_0)$, so (6.3) forces the contradiction $T_0 = T_0^y$. Therefore, it will suffice to prove (ii).

Let $T_0 \leq M < X$. We must show $N_X(T_0) \geq M$. Suppose false, so that minimality of $|Y|$ implies that (10.1) holds for M. Therefore, either $O_p(M) \neq 1$ or $E(M)_p \neq 1$. Suppose $E(M)_p \neq 1$ and let D be a component of $E(M)_p$. By (7.1) $T_0 \cap D$ is a maximal torus of D, hence $T_0 \leq N(D)$. Then (7.2) shows that $T_2 = C_{T_0D}(T_1)$ is a maximal torus of G_0, where T_1 is a Cartan subgroup of D. Let V be a Sylow p-subgroup of D normalized by T_2. Then V is a product of T_2-root subgroups of G_0 (use (9.1)), each of which has order a power of q. If $D \not\cong L_2(q^a)$, for some integer a, then T_1 is necessarily of even order. But then $|T_2|$ is even, and applying (11.3) to

T_2 rather than to T_0, we have a contradiction. So $D \cong L_2(q^a)$ and $\frac{1}{2}(q^a - 1)$ is odd.

Suppose $|V| = q$ and let $\overline{T}_2 = C_{\overline{G}}(T_2)^0$, a maximal torus of $\overline{G}$. There is a $\overline{T}_2$-root subgroup $\overline{V}$ of $\overline{G}$ such that $V = \overline{V}_\sigma$. Then D is generated by V and the opposite T_2-root subgroup of G_0, and it follows that either $D \cong \mathrm{SL}(2, q)$ or $G_0 \cong \mathrm{PSp}(4, q)$ and V is a root subgroup for a short root. The first case contradicts $|T_2|$ odd. In the second case, consideration of $N_{G_0}(V)$ shows that $|T_2| = \frac{1}{2}(q-1)^2$ or $\frac{1}{2}(q^2 - 1)$, either way we again contradict $|T_2|$ odd. Therefore, $a > 1$, and since $\frac{1}{2}(q^a - 1)$ is odd, so is a. Thus, $q^3 \mid |V|$. Also, $q \equiv -1 \pmod 4$ forces q to be an odd power of p and $p \equiv -1 \pmod 4$. From Table (11.1) we easily rule out all possibilities for X. Hence, $E(M)_p = 1$.

We then have $O_p(M) \neq 1$. Let J be a minimal T_0-root subgroup contained in $O_p(M)$. Then $|J| = q^a$ for some $a \geq 1$. If $a = 1$, argue as in the above paragraph and contradict the fact that $|T_0|$ is odd. Hence $a \geq 2$.

Suppose $a = 2$. Then $J = X_i$ for some $i \in \{1, \ldots, v\}$ and $\overline{X}_i = \overline{U}_\alpha \times \overline{U}_\beta$ for α, β roots of the same length. Let $\overline{D} = \langle \overline{U}_{\pm\alpha}, \overline{U}_{\pm\beta} \rangle$. Then $\overline{D}$ is a $\overline{T}\langle\sigma\rangle$-invariant rank two subgroup of $\overline{G}$, so $D = O^{p'}(\overline{D}_\sigma)$ is a perfect central extension of one of the groups $\mathrm{PSL}(2, q^2)$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$, or $\mathrm{PSp}(4, q)$. By (7.1), $T_0 \cap D$ is a maximal torus of D. Also, $T_0 \leq N(J)$, so $T_0 \cap D$ is contained in a proper parabolic subgroup of D. However, one checks that this forces $|T_0 \cap D|$ even, a contradiction. Therefore, $a \geq 3$.

From Table (11.1) we conclude that X is one of the groups Ly, F_5, F_2, or F_1. In the first two cases, J is necessarily of order 5^6 and a Sylow 5-subgroup of X. However, J is elementary abelian, while Ly contains $G_2(5)$ (Lyons **[19]**) and F_5 contains a HS section (Table 1 of **[2]**), which contains a $U_3(5)$ section. This is impossible, forcing $X \cong F_2$ or F_1. At this point we appeal to (11.8) (which is proved independently of (11.4)) to obtain a contradiction. This completes the proof of (11.4).

(11.5) G_0 has a faithful irreducible projective module M over a field of characteristic p and satisfying

(i) $\dim(M) \leq 2n + 1$ if G_0 is a classical group;
(ii) $\dim(M) \leq 8$ if $G_0 \cong G_2(q)$;
(iii) $\dim(M) \leq 27$ if $G_0 \cong F_4(q)$, $E_6(q)$, or ${}^2E_6(q)$;
(iv) $\dim(M) \leq 56$ if $G_0 \cong E_7(q)$; and
(v) $\dim(M) \leq 248$ if $G_0 \cong E_8(q)$.

Proof. If G_0 is a classical group use the natural module associated with the corresponding linear group. If $G_0 = G_2(q)$, the containment $G_2(q) \leq D_4(q)$ implies the result. Each of the groups in (iii) is contained in $E_6(K)$. To obtain the module M, consider the group $E_7(K)$ and let P be the standard parabolic subgroup whose derived group involves $E_6(K)$. Then $M = R_u(P)$ can be viewed as a 27-dimensional module for the Levi factor of P. A similar procedure establishes (iv), while in (v) we let M be the Lie algebra of $\bar{G} = E_8(K)$.

(11.6) (i) $|T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{14} \cdot 12^n$.
(ii) If $n \geq k$, then $|T_0| \geq 1/(k+1) \cdot 12^k$.

Proof. By (2.3) $|T_0| \geq \frac{1}{d}(q-1)^n$. As $d \leq q+1$ and $q \geq 13$, we have $\frac{1}{d}(q-1)^n \geq ((q-1)/(q+1))(q-1)^{n-1} \geq \frac{6}{7}12^{n-1} = \frac{1}{14}12^n$, which proves (i). To prove (ii) use the inequality $|T_0| \geq \frac{1}{d}(q-1)^n \geq (1/(n+1))12^n$ and repeated use of the inequality $(1/(l+1))(q-1)^l > \frac{1}{l}(q-1)^{l-1}$.

(11.7) X is not a Mathieu group.

Proof. Suppose X is a Mathieu group. Regard X as a subgroup of M_{24}, acting on a set Ω of size 24. Let Δ be an orbit of T_0 of maximal length. Fix $\alpha \in \Delta$ and set $T_1 = (T_0)_\alpha$. By (11.3) $|\Delta|$ is odd and since T_0 is abelian, T_1 fixes each point in Δ. From $|T_0| \geq 48$ we conclude $T_1 \neq 1$, and since $|\Delta| \geq 3$, $T_1 \leq M_{21} \leq \mathrm{Aut}(\mathrm{PSL}(3,4))$.

If $|\Delta| = 3$, then $|T_1| \geq 16$ forces $7 \mid |T_1|$, whence T_0 has an orbit of size *a* multiple of 7. This contradicts the choice of Δ. Therefore $|\Delta| \geq 5$ and T_1 must be contained in a Cartan subgroup of $\mathrm{Aut}(\mathrm{PSL}(3,4)) \geq M_{21}$. Then $|T_1|$ divides 9 and $|\Delta| \geq 7$. This forces $|T_1| = 3$, whence $|\Delta| \geq 16$, impossible.

(11.8) $X \not\cong F_2$ or F_1.

Proof. If $X \cong F_2$ or F_1, then X contains a covering of ${}^2E_6(2)$ (see Table 1 of **[2]**). By **[18]** any faithful projective representation of ${}^2E_6(2)$ in odd characteristic has degree at least $3 \cdot 2^9$. So (11.5) implies G is a classical group and $n \geq \frac{1}{2}(3 \cdot 2^9 - 1)$. Combining this with (11.6) we have $|T_0| > |X|$, a contradiction.

(11.9) $X \not\cong Mc$ or Co_3.

Proof. Suppose $X \cong \mathrm{Mc}$ or Co_3. By (11.4)(ii) $N_X(T_0)$ is a maximal subgroup of X, all such subgroups being determined in Finkelstein **[11]**. Now X acts faithfully on the Lie algebra, M, of G. Also, $\hat{A}_8 \leq \mathrm{Mc} \leq \mathrm{Co}_3$ and $A_8 \cong \mathrm{GL}(4,2)$. Decomposing M into eigenspaces for the central involution of the $\hat{A}_8$ and using **[18]**, we have $\dim(M) \geq 2 \cdot 7 = 14$. This implies $n \geq 4$ and so $|T_0| \geq \frac{1}{5} 12^4$. The results in **[11]** yield a contradiction.

(11.10) $X \ncong \mathrm{HS}$.

Proof. Suppose $X \cong \mathrm{HS}$. Then X is 2-transitive of degree 176, with 1-point stabilizer PSU(3, 5) and 2-point stabilizer an extension of SL(2, 5) by Z_6. Say X acts on Ω and Δ is a non-trivial orbit of T_0. For $\alpha \in \Delta$, $T_1 = (T_0)_\alpha$ stabilizes each point in Δ, hence $T_1 \leq \mathrm{SL}(2,5) \times Z_3$ and $|T_1| \leq 15$. In particular, $|T_0| \leq 15 \cdot 176$.

If $p \neq 5$, then using the containment $\mathrm{PSU}(3,5) < G_0$ together with **[18]** and (11.5) we have $n \geq 4$. Hence, $|T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{5} \cdot 12^4$, contradicting $|T_0| \leq 15 \cdot 176$. So $p = 5$; which forces $q \geq 25$ and $|T_0| \geq \frac{1}{d}(q-1)^n \geq \frac{1}{3} \cdot 24^2 = 2^6 3$. Also, T_0 has order prime to 5, so $|T_1| \leq 3^2$ and $|\Delta| > 21$. The lower bound on $|T_0|$ implies that each orbit of T_0 is non-trivial, of odd length, and of order greater than 21, but dividing $|X|$. Table (11.1) shows this to be impossible.

(11.11) $X \ncong \mathrm{Ru}$ or Suz.

Proof. Suppose $X \cong \mathrm{Ru}$ or Suz then X contains a subgroup $(A \times R)D$, where $Z_2 \times Z_2 \cong R \trianglelefteq RD \cong A_4$, $D \leq N(A)$, and $A \cong \mathrm{Sz}(8)$ or PSL(3, 4), respectively (this follows from Table 1 of **[2]**). Let M be a nontrivial, projective, irreducible module for G_0 in characteristic p, and let $\hat{G}_0$ be the representation group for M.

Write $M = C_M(\hat{R}) \oplus [M, \hat{R}]$. We first show that $[\hat{A}, [M, \hat{R}]] \neq 0$. Suppose otherwise. Using (3.3) and (8.10)(i) of **[2]** we see that there exists $x \in \hat{X}$ with $\hat{R}^x \leq \hat{A}\hat{R}$ and $\hat{R}^x \cap \hat{A} \leq Z(\hat{X}) \geq \hat{R}^x \cap \hat{R}$. Then $\hat{A}\hat{R} = \hat{A}\hat{R}^x$. Since $\hat{A}$ is trivial on $[M, \hat{R}]$ we conclude $[M, \hat{R}^x] \geq [M, \hat{R}]$, hence equality holds. Therefore, $C_M(\hat{R}) = C_M(\hat{R}^x)$, whereas $\hat{R}^x$ induces a subgroup of $\hat{A}$ on $C_M(\hat{R})$ and, surely, $[\hat{A}, C_M(\hat{R})] \neq 0$. Therefore, $[\hat{A}, [M, \hat{R}]] \neq 0$, as asserted.

Suppose $X \cong \mathrm{Ru}$. Then the main theorem of **[18]** implies $\dim(M) \geq \dim([M, \hat{R}]) \geq 8$ and $\dim(M) \geq 14$ in case $Z(\hat{X})$ has odd order. If $\hat{R}$ is not abelian, then Clifford's theorem and Schur's lemma imply that M contains the direct sum of two faithful $K[\hat{A}]$-composition factors, hence $\dim(M) \geq 16$. If $\hat{R}$ is abelian, we can write $[M, \hat{R}] \geq M_1 \oplus M_2 \oplus M_3$, the sum of three faithful irreducibles for $\hat{A}$, permuted transitively by $\hat{D}$. Here $\dim(M) \geq 24$ and $\dim(M) \geq 42$ if $Z(\hat{X})$ has odd order. Apply (11.5) and note that in (11.5)(iii) the module corresponded to a 3-fold cover of $E_6(K)$. We conclude that $n \geq 7$ and if $n = 7$, then $G_0 \cong E_7(q)$ and $d = 2$. Therefore, $|T_0| \geq \frac{1}{2}12^7$, and this contradicts (11.3) and (11.4)(i).

Now suppose $X \cong$ Suz and let M be the Lie algebra of $\bar{G}$. Here $X = \hat{X}$ acts on M and A does not centralize $[M, R]$. Also, $[M, R] = M_1 \oplus M_2 \oplus M_3$, where M_1, M_2, and M_3 are the fixed spaces of the involutions in R. The spaces M_1, M_2, and M_3 are left invariant by A and permuted transitively by D. It follows from **[18]** that $\dim(M) \geq 3 \cdot \dim(M_1) \geq 45$. If $\bar{G} = F_4(K)$, we have a contradiction by replacing M by the module in (10.5)(iii) (a module for the 3-fold cover of $E_6(K)$). Hence $n \geq 5$. Since $|T_0|$ is odd equality holds only if $G_0 = \mathrm{PSp}(10, q)$ or $\mathrm{PSO}^-(10, q)'$ with $|T_0| \geq \frac{1}{2}(q-1)^6$. The only possibilities are $|T_0| = \frac{1}{2}(q^5+1)$ with $q = 13$ or 17, or $n = 6$ and $G_0 = \mathrm{PSU}(7, 13)$. In each case we have a numerical contradiction.

(11.12) $X \not\cong J_1$.

Proof. We use (11.4)(iii) to conclude $|T_0|^2 \leq |X|$. If $n \geq 3$, then $|T_0| > \frac{1}{4}12^3$ and this is impossible. Thus, $n = 2$. A Sylow 2-normalizer of X contains a Frobenius group of order $2^3 7$. Therefore, any projective irreducible for X in odd characteristic has dimension at least 7. By (11.5) $\bar{G} \cong G_2(K)$, hence $d = 1$ and $|T_0| \geq (q-1)^2$. As $|T_0|^2 \leq X$, this forces $q \leq 19$. As $|T_0|$ is odd, $|T_0| = \Phi_3(q)$ or $\Phi_6(q)$, and one checks that $|T_0| \nmid |J_1|$.

(11.13) $X \not\cong J_2$ or J_3.

Proof. Suppose $X \cong J_2$ or J_3 and let t be a 2-central involution in X. By Table 1 of **[2]** we have $C_X(t)$ an extension of an extraspecial group of order 2^5 by A_5.

We first claim $n \geq 3$. To see this let M be a faithful $K[X]$-module, and write $M = [M, t] \oplus C_M(t)$. As M is faithful each of the factors is non-trivial, and they are both $C_X(t)$ invariant. Since $O_2(C_X(t))$ is extraspecial, we have $\dim([M, t]) \geq 4$. Now consider $C_M(t)$. Involutions in $O_2(C_X(t))$ are non-trivial on $C_M(t)$ and it is easy to see that such involutions are conjugates of t (use Table 1 of **[2]**). Thus, $O_2(C_X(t))$ acts in a non-trivial manner on $C_M(t)$ and it follows from Clifford's theorem that $\dim(C_M(t)) \geq 5$. Consesquently, $\dim(M) \geq 9$. Now $\mathrm{SL}(3, K)$ acts on its

Lie algebra of dimension 8; $G_2(K)$ acts in 8 dimensions; while PSp(4, K) $\cong O(5, K)'$ acts in 5 dimensions. This proves the claim, and so $|T_0| \geq \frac{1}{4}12^3$.

From Sylow's theorem, we have the Sylow 7-subgroups of J_2 self-centralizing while the Sylow r-subgroups of J_3 are self-centralizing for $r = 17, 19$. Since $|T_0|$ is odd, the above inequality shows $|T_0| = 3^3 \cdot 5^2$ or $3^5 \cdot 5$, according to $X \cong J_2$ or J_3. However, (11.4) implies that $|X: N_X(T_0)| \equiv 1 \pmod{|T_0|}$ and this is impossible.

(11.14) $X \not\cong J_4$.

Proof. Suppose $X \cong J_4$. We refer the reader to Janko **[17]** for properties of J_4. If t is a 2-central involution then $O_2(C_X(t))$ is extraspecial of order 2^{13}. So if M is the Lie algebra of $\bar{G}$ we have $\dim(M) \geq 2^6$. Thus $n \geq 6$ and $|T_0| \geq \frac{1}{7}12^6$.

If $P \in \mathrm{Syl}_r(X)$, then by **[17]** P is self-centralizing for $r = 23, 29, 31, 37$, and 43. The Sylow 11-subgroups of X are non-abelian, so by Table 1 we have $|T_0|$ dividing $3^3 \cdot 5 \cdot 7 \cdot 11^2$. This contradicts the inequality above.

(11.15) $X \not\cong$ He.

Proof. Suppose $X \cong$ He. From Table 1 of **[2]** it follows that X contains a klein group R such that $E(C_X(R))$ is a covering group of PSL(3, 4) with center R, and $N_X(R)$ is transitive on R. An easy argument using **[18]** shows that a faithful projective $K(E(C_X(R)))$-module has dimension at least $4 \cdot 3 = 12$. So (11.5) yields $n \geq 4$, hence $|T_0| \geq \frac{1}{5} \cdot 12^4$.

Since $N_X(R)$ has a section isomorphic to PGL(3, 4), X has non-abelian Sylow 3-subgroups. Therefore, $3^3 \nmid |T_0|$. The Sylow 17-subgroups of X are self-centralizing, so $|T_0|$ is prime to 17. From Table (11.1) and the above inequality we conclude that $O_5(T_0) \in \mathrm{Syl}_5(X)$ or $O_7(T_0) \in \mathrm{Syl}_7(X)$. Since 5 and 7 are divisors of $|C_X(R)|$, we may assume (by (11.4)(iii)) that $R \leq N(T_0)$. But then $T_0 = \langle C_{T_0}(r) \mid 1 \neq r \in R \rangle$, whereas $C_{T_0}(r) \leq N(R)$ for each $r \in R^{\#}$ and $|N_X(R)|$ is not divisible by $|T_0|$. This is a contradiction.

(11.16) $X \not\cong$ Ly.

Proof. Suppose $X \cong$ Ly. Let M be a module as given in (11.5). If $p \neq 5$, then the containment $G_2(5) \leq X$ (see Lyons **[19]**) and **[18]** implies $\dim(M) \geq 120$. Hence $n \geq 8$, equality possible only if $\bar{G} = E_8(K)$ and $d = 1$. It follows that $|T_0| \geq 12^8$. But (11.4)(iii) implies $|T_0|^2 \leq |X|$, and this is impossible. Therefore, $p = 5$ and $q \geq 25$.

Since $p = 5$, $|T_0|$ is prime to 5. As $|T_0| \geq 48$, and Sylow r-subgroups of X are self-centralizing for $r = 31, 37, 67$ (see (3.3) of **[19]** or use Sylow's theorem), we conclude $|T_0|$ divides $3^7 \cdot 7 \cdot 11$. By **[19]** there is an involution $j \in X$ with $C_X(t) \cong \hat{A}_{11}$, the covering group of A_{11}. We may regard $C_X(t)$ as acting on M, and write $M = M_1 \oplus M_2$, where $M_1 = C_X(t)$ and $M_2 = [M, t]$. Each of M_1 and M_2 is non-trivial and considering the action of a Frobenius subgroup of $C_X(t)$ of order 55 we have $\dim(M_i) \geq 5$ for $i = 1, 2$. Hence $\dim(M) \geq 10$, so (11.5) implies $n \geq 4$, with equality only if $\bar{G} = F_4(K)$ and $d = 2$. Therefore, $|T_0| \geq \frac{1}{2}24^4 = 3^4 \cdot 2^{11}$. From the previous divisibility condition we have $|T_0| = 3^7 \cdot 7 \cdot 11$, whereas X does not have abelian Sylow 3-subgroups. This is a contradiction.

(11.17) $X \not\cong \mathrm{ON}$.

Proof. Suppose false. We quote O'Nan **[20]** for the following fact about X. Namely, a Sylow 3-subgroup, A, of X is elementary abelian of order 3^4, its normalizer being an extension of $Q_8 \circ D_8$ by D_{10}. It follows that each orbit of $N(A)$ on $A^{\#}$ has size a multiple of 40. Let $\hat{X}$ be the covering group of X that acts on M. If $\hat{A}$ is abelian, then the above and Clifford's theorem implies $\dim(M) \geq 40$. Thus, $n \geq 7$ and $|T_0| \geq \frac{1}{8} \cdot 12^7$. Otherwise, $\hat{A}$ contains an extraspecial subgroup of order 3^5. Here, $\dim(M) \geq 9$ and $3 \mid |Z(\hat{G}_0)|$. The proof of (11.5), shows $n = 6$ with equality only if $\bar{G} = E_6(K)$. Hence, $|T_0| \geq \frac{1}{3} \cdot 12^6$. But (11.4)(iii) implies $|T_0|^2 \leq |X|$, and this contradicts Table (11.1).

(11.18) $X \not\cong F_{22}$, F_{23}, or F'_{24}.

Proof. Suppose $X \cong F_{22}$, F_{23}, or F'_{24}. Let M be a module as in (11.5). We first obtain lower bounds for $\dim(M)$. Suppose $X \cong F_{22}$ or F_{23}. By Table 1 of **[2]** there is an involution $j \in X$ such that $C_X(j) = C_X(j)'$ and $C_X(j)/\langle j \rangle \cong U_6(2)$ or F_{22}, respectively. Let $\hat{D}$ be the derived group of the preimage of $C_X(j)$ in a covering group of X that affords M. By Griess **[14]**, **[15]** we see that $\hat{D}$ contains an involution $\hat{j}$ such that $\hat{j}Z(\hat{D}) = j$ (in fact $\hat{D} \cong C_X(j)$ if $X \cong F_{23}$).

Write $M = C_M(\hat{j}) \oplus [M, \hat{j}]$. Clearly $[M, \hat{j}]$ affords a non-trivial module for $\hat{D}$. This is also true of $C_M(\hat{j})$. For if $\hat{D}$ is trivial on $C_M(j)$, then choosing $j \neq j^x \in C_X(j)$ we find that $\hat{j}^x$ can be chosen so that $\hat{j}^x = \hat{j}$, and this is impossible. If $X \cong F_{22}$, then by **[18]** we have $C_M(\hat{j})$ and $[M, \hat{j}]$ each of dimension at least 21, and so $\dim(M) \geq 42$. If $X \cong F_{23}$, we have $C_M(\hat{j})$ and $[M, \hat{j}]$ modules for $\hat{D}$, each non-trivial as before. Hence $\dim(M) \geq 84$. This also holds if $X \cong F'_{24}$, since $F_{23} \leq F'_{24}$.

If $X \cong F_{22}$, then by (11.5) $n \geq 7$, with equality possible only if $\bar{G} = E_7(K)$. Therefore, $|T_0| \geq \frac{1}{2} \cdot (12)^7$. On the other hand, (11.4)(iii) implies $|T_0|^2 \leq X$, and this is a contradiction.

Now suppose $X \cong F_{23}$ or F_{24}. If X is a classical group then by (11.5) $n \geq \frac{1}{2}(84 - 1) > 41$. Hence $|T_0| > \frac{1}{42} \cdot (12)^{41}$, contradicting $|T_0|^2 \leq |X|$. So X is not a classical group, which forces $\bar{G} = E_8(K)$. From Table (11.1) we have $23 \mid |X|$. However, 23 does not divide $|E_8(q)|$ for $13 \leq q < 47$ with $q \neq 23$ (one checks this by noting that neither $\Phi_{11}(q)$ nor $\Phi_{22}(q)$ divides $|E_8(q)|$). On the other hand 17 does not divide $|E_8(23)|$. Therefore, $|T_0| \geq \frac{1}{2}(46)^8$, contradicting $|T_0|^2 \leq |X|$.

(11.19) $X \ncong \mathrm{Co}_2$.

Suppose $X \cong \mathrm{Co}_2$. Table 1 of **[2]** shows that some involution in X has centralizer containing an extension of an elementary abelian group, A, of order 2^4 by GL(4, 2) (natural action). It follows from Clifford's theorem that $\dim(M) \geq 15$, where M is as in (11.5) (an easy argument shows that we can regard A as acting on M). By (11.5), $n \geq 7$ if $\bar{G}$ is a classical group; otherwise $n \geq 4$. Suppose $\bar{G} \cong F_4(K)$. By Table 1 of **[2]** we see that X contains an elementary abelian group, E, of order 2^{10}. By (5.16) of **[25]**, E normalizes a maximal torus of $\bar{G}$, and this forces the Weyl group of $F_4(K)$ to have an elementary abelian subgroup of order at least 2^6. This is false. So $\bar{G} \ncong F_4(K)$ and we conclude from (11.5) that $n \geq 6$, equality only if $\bar{G} = E_6(K)$. It follows that $|T_0| \geq \frac{1}{3} \cdot 12^6$. Sylow's theorem implies that subgroups of X having order 11 or 23 are self-centralizing, so Table (11.1) implies $|T_0| \mid 3^6 5^3 7$. This is a contradiction.

(11.20) $X \ncong \mathrm{Co}_1$.

Proof. Suppose $X \cong \mathrm{Co}_1$. We first claim that $n \geq 8$, equality possible only for $\bar{G} = E_8(K)$. Table 1 of **[2]** shows that there is an involution $j \in X$ such that $C_X(j) \geq A \times R \cong G_2(4) \times (Z_2 \times Z_2)$ and $R^\#$ consists of conjugates of j. As in (8.6) and (8.8) of **[2]** there exists a conjugate R^x of R such that $R^x \leq AR$, $R^x \cap A = 1$, and R^x projects to the center of a Sylow 2-subgroup of A.

Let V be the Lie algebra of $\bar{G}$ and write $V = [V, R] \oplus C_V(R)$. Then $[V, R] = V_1 \oplus V_2 \oplus V_3$, where $V_i = [V, R] \cap C(j_i)$ and $R^\# = \{j_1, j_2, j_3\}$. As in (11.11) we have $[A, V_i] \neq 1$ for $i = 1, 2, 3$, so by **[18]** $\dim(V) \geq \dim[V, R] \geq 3 \cdot \dim(V_1) \geq 3 \cdot 60 = 180$. The claim follows. Consequently, $|T_0| \geq (q - 1)^8 \geq 12^8$.

By (11.4)(iii), $|T_0|^2 \le X$, which by the above inequality is impossible for $q > 13$. Hence, $q = 13$. Now 13 has order 11 (modulo 23) and this forces $n \ge 10$. But then, $|T_0| \ge \frac{1}{11}(12)^{10}$, again contradicting $|T_0|^2 \le X$.

(11.21) $X \not\cong F_5$.

Proof. Suppose $X \cong F_5$ and let V be the Lie algebra of $\bar{G}$. If $p \ne 5$, then use (§4, II) of Harada **[16]** to conclude that X contains an extraspecial group of order 5^5 and an element inducing Z_4 on the center of the extraspecial group. Elementary arguments imply that $\dim(V) \ge 100$. Hence $n \ge 7$, equality possible only for $\bar{G} = E_7(K)$ or $C_7(K)$. We then have $|T_0| \ge \frac{1}{2}(12)^7$, contradicting $|T_0|^2 \le X$.

Suppose $p = 5$. Then $q = 5^a$ for $a \ge 2$ and from (11.1) we have $|T_0|$ dividing $3^6 \cdot 7 \cdot 11 \cdot 19$. Since $19 \mid |G_0|$, a primitive divisor argument shows that if $q = 25$, then $\Phi_{d_1}(25)$ divides $|G_0|$, for d_1 a multiple of 9. But then $\varphi(d_1) \ge \varphi(9) = 6$ and $|T_0| \ge \frac{1}{d}(24)^6$, a contradiction. For $q = 5^3$ use the prime 11 to obtain a contradiction, etc.

(11.22) $X \not\cong F_3$.

Proof. Suppose $X \not\cong F_3$. Then X contains a non-split extension of an elementary abelian group of order 2^5, by SL(5, 2) (Thompson **[27]**). This group has trivial multiplier, so (11.5) implies that $n \ge 7$, with equality possible only if $\bar{G} = E_7(K)$. Hence $|T_0| \ge \frac{1}{2}(q-1)^7$.

If $q \ge 23$, then this contradicts $|T_0|^2 \le |X|$. For the remaining cases argue as follows. Since 31 divides $|X|$, 31 divides $|G_0|$. So there is an integer d_1 such that $31 \mid \Phi_{d_1}(q) \mid |G_0|$. One checks that d_1 is a multiple of 30, 30 or 15, according to $q = 13$, 17, or 19. It follows that $n \ge 8$, equality possible only if $G_0 = E_8(q)$. Hence $|T_0| \ge (q-1)^8$ and this contradicts $|T_0|^2 \le |X|$.

We have now considered all possibilities for X, completing the proof of (10.1).

12. Some consequences of (10.1). In this section we derive some consequences of (10.1) and (10.2). Throughout this section assume that $p > 3$ and $q > 11$.

THEOREM (12.1). *The map $\bar{X} \to \bar{X}_\sigma$ is a bijection between the set of closed, connected, σ-invariant subgroups of $\bar{G}$ containing a maximal torus and the set of subgroups of G generated by maximal tori of G.*

The above theorem will be a consequence of the next result which gives additional information. In particular, the inverse of the map $\bar{X} \to \bar{X}_\sigma$ is described. We first need some notation.

For a subset Δ of $\bar{\Sigma}$, let $\bar{G}(\Delta, \bar{T}) = \langle \bar{U}_\alpha \mid \alpha \in \Delta \rangle$. If $\Delta = \Delta^\sigma$, then $\bar{G}(\Delta, \bar{T})$ is σ-invariant and we set $G(\Delta, T) = O^{p'}(\bar{G}(\Delta, \bar{T})_\sigma)$. For $\bar{X} \leq \bar{G}$, let $\bar{\Sigma}(\bar{X}, \bar{T}) = \{\alpha \in \bar{\Sigma} \mid \bar{U}_\alpha \leq \bar{X}\}$, and for $X \leq G$, set $\bar{\Sigma}(X, T) = \bigcup_{X_i \leq X} \bar{\Sigma}_i$. If the maximal torus $\bar{T}$ is understood we abbreviate the above to $\bar{G}(\Delta)$, $G(\Delta)$, $\bar{\Sigma}(\bar{X})$, and $\bar{\Sigma}(X)$, respectively.

We say that a subset Δ of $\bar{\Sigma}$ is *$\bar{T}$-closed* if $\Delta = \Delta^\sigma = \bar{\Sigma}(\bar{G}(\Delta))$. This agrees with the concept introduced in §10. A final notation. For $Y \leq G$, let $\bar{G}(Y, T) = \bar{G}(\bar{\Sigma}(Y)) \cdot \bar{T}$, abbreviated to $\bar{G}(Y)$ when T is understood. We can now state

THEOREM (12.2). *Let $T_0 \leq Y \leq G$. Then*

(i) $\Delta = \bar{\Sigma}(Y)$ *is the unique $\bar{T}$-closed subset of $\bar{\Sigma}$ satisfying $G(\Delta)T_0 \trianglelefteq Y$.*

(ii) $\bar{G}(Y)$ *is independent of T_0. That is, if T_1 is a maximal torus of G with $T_1 \cap G_0 \leq Y$, then $\bar{G}(Y, T_1) = \bar{G}(Y, T)$.*

(iii) $Y \leq N(\bar{G}(Y))$, *T normalizes Y, and $G(\Delta)T \trianglelefteq YT$.*

(iv) *If $Y \geq T$ and if Y is generated by maximal tori of G, then $\bar{G}(Y, T)$ is the unique closed, connected, σ-invariant subgroup of $\bar{G}$ containing a maximal torus of $\bar{G}$ and having fixed point set Y.*

(v) *If $\bar{X} = \bar{X}^\sigma = \bar{X}^0 \geq \bar{T}$, then $X = \langle T^X \rangle$, where $X = \bar{X}_\sigma$.*

It is clear that (12.1) follows from (12.2) and that the inverse of the map $\bar{X} \to \bar{X}_\sigma$ is the map $Y \to \bar{G}(Y)$. The next several results aim at the proof of (12.2). First we characterize $\bar{T}$-closed subsets of $\bar{\Sigma}$ at the G-level.

(12.3) A σ-invariant subset Δ of $\bar{\Sigma}$ is $\bar{T}$-closed if and only if $\bar{\Sigma}(G(\Delta)) = \Delta$.

Proof. Suppose $\Delta = \Delta^\sigma$. First assume that $\bar{\Sigma}(G(\Delta)) = \Delta$. Clearly $\Delta \subseteq \bar{\Sigma}(\bar{G}(\Delta))$. If Δ is not closed, then there is some $\langle \sigma \rangle$-orbit $\bar{\Sigma}_i$ of $\bar{\Sigma}$ such that $\bar{X}_i \leq \bar{G}(\Delta)$ and $\bar{\Sigma}_i \nsubseteq \Delta$. However, $\bar{X}_i \leq \bar{G}(\Delta)$ implies $X_i \leq G(\Delta)$, so the assumption gives $\bar{\Sigma}_i \subseteq \Delta$, a contradiction. Therefore, Δ is closed.

Now assume that Δ is closed. Hence, $\bar{\Sigma}(\bar{G}(\Delta)) = \Delta$ and we must prove $\bar{\Sigma}(G(\Delta)) = \Delta$. Clearly, $\Delta \subseteq \bar{\Sigma}(G(\Delta))$, so it will suffice to take $X_i \leq G(\Delta)$ and show $\bar{X}_i \leq G(\Delta)$. If X_i is a p-group, then by (3.9) of [4] there is a σ-invariant parabolic subgroup $\bar{P}$ of $\bar{G}(\Delta)\bar{T}$ such that $X_i \leq R_u(\bar{P})$ and $T_0 \leq \bar{P}$. By (6.4) we also have $\bar{T} \leq \bar{P}$. Therefore, $R_u(\bar{P})$ is a product of $\bar{T}$-root subgroups of $\bar{G}$ and using (6.9), we have $\bar{X}_i \leq R_u(\bar{P}) \leq \bar{G}(\Delta)$, as required.

Suppose X_i is a group of Lie type and let T_1 be a Cartan subgroup $G(\Delta)$ (a p-complement in a Sylow p-normalizer). Then $T_2 = C_{G(\Delta)T_0}(T_1)$ is a maximal torus of G_0 (by (9.1)), and we set $\overline{T}_2 = C_{\overline{G}}(T_2)^0$, a maximal torus of $\overline{G}$. Then $T_2 \leq G(\Delta)T_0 \leq \overline{G}(\Delta)\overline{T}$ and we claim that T_2 normalizes a σ-invariant maximal torus of $\overline{G}(\Delta)\overline{T}$. To see this first use (5.16) of **[25]** to get $T_2 \leq N(\overline{A}_1)$, where $\overline{A}_1$ is a σ-invariant maxial torus of $\overline{G}(\Delta)$. Then let $\overline{A} = \overline{A}_1 Z(\overline{G}(\Delta))^0$ and check that $\overline{A}$ is a maximal torus of $\overline{G}(\Delta)\overline{T}$. This proves the claim, so by (6.3) and (2.8) we must have $\overline{A} = \overline{T}_2$.

Set $Y = G(\Delta)$ and note that $\overline{G}(\Delta) \leq \overline{Y}(T_0)$. Since $G(\Delta) \leq \overline{G}(\Delta)$ and since each T_2-root subgroup contained in $G(\Delta)$ is a p-group, the argument in the second paragraph of the proof shows that $\overline{Y}(T_2) \leq \overline{G}(\Delta)$. On the other hand, (9.1) shows that $\overline{Y}(T_2) = \overline{Y}(T_0)$, so we now have $\overline{G}(\Delta) = \overline{Y}(T_0)$. By definition, $\overline{X}_i \leq \overline{Y}(T_0)$, so $\overline{X}_i \leq \overline{G}(\Delta)$ as required.

(12.4) Let $T_0 \leq Y \leq G$. Set $\Delta = \overline{\Sigma}(Y)$. Then Δ is the unique $\overline{T}$-closed subset of $\overline{\Sigma}$ satisfying $G(\Delta)T_0 \trianglelefteq Y$.

Proof. Let $\Delta = \overline{\Sigma}(Y)$. By (10.1), $Y(T_0)T_0 \trianglelefteq Y$ and $Y(T_0) = O^{p'}(\overline{Y}(T_0)_\sigma)$. From the definitions we have $\overline{Y}(T_0) = \overline{G}(\Delta)$, so $Y(T_0) = G(\Delta)$, which proves $G(\Delta)T_0 \trianglelefteq Y$. From (12.3), it follows that Δ is $\overline{T}$-closed. For if $X_i \leq G(\Delta)$, then $X_i \leq Y$ and $\overline{\Sigma}_i \subseteq \Delta$. Hence $\Delta = \overline{\Sigma}(G(\Delta))$.

For the uniqueness of Δ argue as follows. Let $\Omega \subseteq \overline{\Sigma}$ with $\Omega = \Omega^\sigma$ and $G(\Omega)T_0 \trianglelefteq Y$. If X_i is any T-root subgroup of G, then $\langle T_0^{X_i}\rangle = X_i T_0$ (see (6.7) and (7.1)). So if $X_i \leq Y$, we have $X_i \leq G(\Omega)$. This implies that $\overline{\Sigma}(G(\Omega)) = \overline{\Sigma}(Y)$. If, in addition, Ω is $\overline{T}$-closed, then (12.3) yields $\Omega = \overline{\Sigma}(G(\Omega)) = \overline{\Sigma}(Y) = \Delta$, and Δ is unique.

(12.5) Let $T_0 \leq Y \leq G$. Then
(i) $Y \leq N(\overline{G}(Y))$;
(ii) $T \leq N_G(Y)$; and
(iii) $G(\overline{\Sigma}(Y))T \trianglelefteq YT$.

Proof. Let $\Delta = \overline{\Sigma}(Y) = \overline{\Sigma}(Y, T)$. By (12.4), $G(\Delta)T_0 \trianglelefteq Y$ and Δ is $\overline{T}$-closed. As in the proof of (12.4) we have $G(\Delta) = Y(T_0)$. Let T_2 be a p-complement of the normalizer of a Sylow p-subgroup of $Y(T_0)T_0$. By (9.1), T_2 is a maximal torus of G_0, so $\overline{T}_2 = C_{\overline{G}}(T_2)^0$ is a maximal torus of $\overline{G}$.

By (10.2) and (9.1) we have $Y(T_0) = Y(T_0)(T_2)$. If D is any T_2-root subgroup of G, then $\langle T_2^D\rangle = DT_2$. As $\langle T_2^Y\rangle \leq Y(T_0)T_0$, we conclude that $Y(T_2) \leq Y(T_0)$, and hence $Y(T_0) = Y(T_2)$. From (9.1) we also have $\overline{Y}(T_2) = \overline{Y}(T_0)$ and $\overline{Y}(T_0)\overline{T} = \overline{Y}(T_2)\overline{T}_2$. The Frattini argument yields $Y = Y(T_2)N_Y(T_2)$.

Since $\bar{G}(\Delta) = \bar{Y}(T_0)$, we have $\bar{G}(Y) = \bar{Y}(T_0)\bar{T} = \bar{Y}(T_2)\bar{T}_2$. By (2.5) $\bar{G}(Y)_\sigma = (\bar{G}(\Delta)\bar{T})_\sigma = G(\Delta)T$ and also $\bar{G}(Y)_\sigma = Y(T_0)T = Y(T_2)T_3$, where $T_3 = (\bar{T}_2)_\sigma$.

As $Y(T_2) \le \bar{Y}(T_2)$ and $N_Y(T_2) \le N(\bar{Y}(T_2)) \cap N(\bar{T}_2)$, we necessarily have $Y = Y(T_2)N_Y(T_2) \le N(\bar{G}(Y))$, proving (i). By the above, $Y \le N(\bar{G}(Y)_\sigma) = N(G(\Delta)T)$, so this will prove (iii), once (ii) is proved.

Now, $Y \le N(\bar{G}(Y))$, so Y normalizes $\bar{G}(Y)_\sigma = Y(T_0)T = Y(T_2)T_3 = Y(T_0)T_3$. The group $N_Y(T_2)$ also normalizes T_3, so $[N_Y(T_2), T_3] \le T_3 \cap G_0 = T_2$, and $[N_Y(T_2), Y(T_0)T_3] \le Y(T_0)T_2 = Y(T_0)T_0$. Letting $\sim$ denote images modulo $Y(T_0)$ we use the above to conclude $[Y^\sim, T^\sim] = [N_Y(T_2)^\sim, T^\sim] = [N_Y(T_2)^\sim, T_3^\sim] \le T_0^\sim \le Y^\sim$. This proves (ii) and completes the proof of (12.5).

(12.6) Let $T_0 \le Y \le G$. If T_1 is any maximal torus of G with $T_2 = T_1 \cap G_0 \le Y$, then $\bar{G}(Y, T_1) = \bar{G}(Y, T)$.

Proof. Let $\Delta = \bar{\Sigma}(Y, T)$, so $G(\Delta)T_0 \trianglelefteq Y$ by (12.4). Also, $G(\Delta) = Y(T_0)$, by (10.1). By (10.2), $Y(T_0)/O_p(Y) = E(Y/O_p(Y))$, so $Y(T_0) = Y(T_2)$. Also, $Y(T_0) = G(\Delta, T)$ and $Y(T_2) = G(\Delta_1, T_1)$, where $\Delta_1 = \bar{\Sigma}(Y, T_1)$. By (12.5)(iii) Y normalizes $\bar{G}(Y, T) = \bar{G}(\Delta)\bar{T}$. In particular, $T_2 \le N(\bar{G}(\Delta)\bar{T})$.

We claim that $\bar{T}_1 \le \bar{G}(\Delta)\bar{T}$, where $\bar{T}_1 = C_{\bar{G}}(T_1)^0$. Let $\bar{Q} = R_u(\bar{G}(\Delta)\bar{T})$ and $\bar{Z}/\bar{Q}$ the connected center of $\bar{G}(\Delta)\bar{T}/\bar{Q}$. So $\bar{G}(\Delta)\bar{T}/\bar{Z}$ is semisimple and (5.16) of **[25]** shows that T_2 normalizes a σ-invariant maximal torus of $\bar{G}(\Delta)\bar{T}/\bar{Z}$. So there is a σ-invariant maximal torus $\bar{A}$ of $\bar{G}$, with $\bar{A} \le \bar{G}(\Delta)\bar{T}$ and $T_2 \le N(\bar{A}\bar{Q})$. Let $A = \bar{A}_\sigma$ and $Q = \bar{Q}_\sigma$. Then T_2 normalizes $(\bar{A}\bar{Q})_\sigma = AQ$ and T_2AQ/Q is a solvable p'-group. By Hall's theorem T_2 is contained in a Hall p'-group of T_2AQ, so T_2 normalizes A^x, for some $x \in Q$. But A^x is a maximal torus of G, whence $T_2 = A^x \cap G_0$ by (6.3). Then (2.6) gives $\bar{A}^x = \bar{T}_1$, and the claim is proved. Therefore, $\bar{G}(\Delta)\bar{T} = \bar{G}(\Delta)\bar{T}_1$.

Let $\Omega = \bar{\Sigma}(\bar{G}(\Delta), \bar{T}_1)$ so that $\bar{G}(\Delta) = \bar{G}(\Omega, \bar{T}_1)$. Clearly Ω is $\bar{T}_1$-closed and $G(\Omega, T_1) = G(\Delta)$. By (2.5)(iv), $G(\Omega, T_1)T_2 = G(\Delta)T_0 \trianglelefteq Y$, so (12.4) forces $\Omega = \bar{\Sigma}(Y, T_1)$. At this stage we have $\bar{G}(Y, T_1) = \bar{G}(Y, T)$, as desired.

(12.7) Assume that Y is generated by G conjugates of maximal tori of G. Then $Y = \bar{X}_\sigma$ for a unique closed connected, σ-invariant subgroup $\bar{X}$ of $\bar{G}$ such that $\bar{X}$ contains a maximal torus of $\bar{G}$.

Proof. Let T_1, T_2 be maximal tori of G with $T_1 \le Y \ge T_2$. Let $\Delta_i = \bar{\Sigma}(Y, T_i)$ for $i = 1, 2$. Then $G(\Delta_i, T_i)T_i \trianglelefteq Y$ for $i = 1, 2$ (see

(12.5)(iii)). By (12.6), $\bar{G}(Y, T_1) = \bar{G}(Y, T_2)$, so taking fixed points (and applying (2.5)) we have $G(\Delta_1, T_1)T_1 = G(\Delta_2, T_2)T_2$. Fixing T_1 and letting T_2 vary over all maximal tori of G contained in Y, we conclude that $Y = G(\Delta_1, T_1)T_1$ and this gives the existence of $\bar{X}$.

Now suppose $Y = \bar{X}_\sigma$, with $\bar{X}$ closed, connected, σ-invariant, and containing a maximal torus $\bar{T}_1$ of $\bar{G}$. We may take $\bar{T}_1$ to be σ-invariant. Set $T_1 = (\bar{T}_1)_\sigma$. Let $\Delta_1 = \bar{\Sigma}(\bar{X}, \bar{T}_1)$, so $\bar{X} = \bar{G}(\Delta_1)\bar{T}_1$. Hence $G(\Delta_1) \trianglelefteq Y$ and $Y/G(\Delta_1)$ is a p'-group. This implies that each T_1-root subgroup of G contained in Y is actually in $G(\Delta_1)$. That is, $\bar{\Sigma}(G(\Delta_1)) = \bar{\Sigma}(Y, T_1)$. But Δ_1 is $\bar{T}_1$-closed. Thus (12.3) shows that $\Delta_1 = \bar{\Sigma}(Y, T_1)$ and then $\bar{X} = \bar{G}(Y, T_1)$, proving uniqueness.

At this point all parts of Theorem (12.2) have been established, with the exception of (v). Suppose $\bar{X}$ is as in (12.2)(v) and let $X = \bar{X}_\sigma$. By (10.1), $X(T_0)T_0 = \langle T_0^X \rangle \trianglelefteq X$. Also, $X(T_0) = O^{p'}(\bar{X}(T_0)_\sigma) \geq O^{p'}(\bar{X}_\sigma)$. Since $X = O^{p'}(\bar{X}_\sigma)T$, we have the result.

We close this section with some results on generation. From now on, $\bar{T}$ is fixed. We thus delete mention of $\bar{T}$ and T from the earlier notation and say $\Delta \subseteq \bar{\Sigma}$ is closed if it is $\bar{T}$-closed.

(12.8) Let $\Omega_1, \ldots, \Omega_k$ be closed subsets of $\bar{\Sigma}$. Then

$$\langle G(\Omega_1), \ldots, G(\Omega_k) \rangle = O^{p'}(\langle \bar{G}(\Omega_1), \ldots, \bar{G}(\Omega_k) \rangle_\sigma).$$

Proof. Let $\Omega = \bar{\Sigma}(\langle \bar{G}(\Omega_1), \ldots, \bar{G}(\Omega_k) \rangle)$. Then $\bar{G}(\Omega) \geq \bar{G}(\Omega_i)$ for $i = 1, \ldots, k$ and Ω is closed. Let $Y = \langle G(\Omega_1), \ldots, G(\Omega_k) \rangle$ and set $\Delta = \bar{\Sigma}(Y)$. By (12.4) we have $G(\Delta)T_0 \trianglelefteq YT_0$ and it follows that $Y = G(\Delta)$.

Suppose $\bar{\Sigma}_j \subseteq \Omega_i$ for some i and j. Then $\bar{X}_j \leq \bar{G}(\Omega_i)$ and $X_j \leq G(\Omega_i) \leq Y$, proving $\bar{\Sigma}_j \subseteq \bar{\Sigma}(Y) = \Delta$. We conclude $\Omega_i \subseteq \Delta$ for $i = 1, \ldots, k$. Therefore, $\bar{G}(\Delta) \geq \langle \bar{G}(\Omega_1), \ldots, \bar{G}(\Omega_k) \rangle$. Since Δ is closed (by (12.4)), we have $\Omega \subseteq \Delta$. Hence, $G(\Omega) \leq G(\Delta)$. On the other hand, $\bar{G}(\Omega) \geq \bar{G}(\Omega_i)$ for $i = 1, \ldots, k$, so $G(\Omega) \geq \langle G(\Omega_1), \ldots, G(\Omega_k) \rangle = Y = G(\Delta)$. This proves $G(\Omega) = G(\Delta) = Y$, which proves the result.

The following results extend (6.10) to arbitrary collections of T-root subgroups.

THEOREM (12.9). *Let $X_{i_1}, \ldots, X_{i_k}$ be T-root subgroups of G. Then*

$$\langle X_{i_1}, \ldots, X_{i_k} \rangle = O^{p'}(\langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k} \rangle_\sigma).$$

Proof. Set $Y = \langle X_{i_1}, \ldots, X_{i_k} \rangle$ and $\Delta = \bar{\Sigma}(Y)$. Apply (12.4) to YT_0 and conclude that $Y = G(\Delta)$ and Δ is closed. For $j = 1, \ldots, k$, $\bar{\Sigma}_{i_j} \subseteq \Delta$, so $\bar{X}_{i_j} \leq \bar{G}(\Delta)$. Therefore, $O^{p'}(\langle \bar{X}_{i_1}, \ldots, \bar{X}_{i_k} \rangle_\sigma) \leq O^{p'}(\bar{G}(\Delta)) = G(\Delta) = Y = \langle X_{i_1}, \ldots, X_{i_k} \rangle$. Since the other containment is obvious, the proof is complete.

THEOREM (12.10). *Let S be an arbitrary set of p'-elements of G. Then*

(i) *If $\bar{G}$ is simply connected, then $\langle C_{G_1}(a) : s \in S \rangle = G_1 \cap \langle C_{\bar{G}}(s) : s \in S \rangle$.*

(ii) *If $\bar{G}$ is simply connected, then $G_1 = \langle C_{G_1}(s) : s \in S \rangle$ if and only if $\bar{G} = \langle C_{\bar{G}}(s) : s \in S \rangle$.*

(iii) *If $\bar{G} = \langle C_{\bar{G}}(s)^0 : s \in S \rangle$, then $G_1 = \langle C_{G_1}(s) : s \in S \rangle$.*

(iv) *If $S \subseteq T$ with T a maximal torus of G, then $G_1 = \langle E(C_{G_1}(s)) : s \in S \rangle$ if and only if $\bar{G} = \langle E(C_{\bar{G}}(s)) : s \in S \rangle$.*

Proof. Set $X = \langle C_{G_1}(s) : s \in S \rangle$ and $\tilde{X} = \langle C_{\bar{G}}(s)^0 : s \in S \rangle$. Fix $s \in S$ and T a maximal torus of G with $s \in T$. Then $T_0 = T \cap G_0$ and $T_1 = T \cap G_1$ are maximal tori of G_0 and G_1, respectively. Let $\bar{T}$ be the unique maximal torus of $\bar{G}$ containing T. Then $T_0 \leq T_1 \leq X$ and $\bar{T} \leq \tilde{X}$.

Let $\bar{C} = C_{\bar{G}}(s)^0$ and $C = \bar{C}_\sigma$. By (10.1) we have $X(T_0) \trianglelefteq X$. Also T normalizes $X(T_0)$. Let $\bar{X}$ be the unique σ-invariant, connected subgroup of $\bar{G}$ satisfying $\bar{T} \leq \bar{X}$ and $\bar{X}_\sigma = X(T_0)T$ (use (12.1)). Then (12.2)(iv) implies $\bar{X} = \bar{G}(X, T)$. Since $C_G(s) = C_{G_1}(s)T$, we have $\bar{G}(C, T) \leq \bar{G}(X, T) = \bar{X}$. On the other hand, $\bar{G}(C, T) = \bar{C}$ (this follows from (12.1) as both have fixed point set under σ equal to C). So $\bar{C} \leq \bar{X}$ and letting s vary we obtain $\tilde{X} \leq \bar{X}$.

From $\tilde{X} \leq \bar{X}$ and $\bar{X}_\sigma = X(T_0)T$, we immediately have (iii). Suppose $\bar{G}$ is simply connected. Then $C_{\bar{G}}(s) = C_{\bar{G}}(s)^0$ for each $s \in S$. In particular $\tilde{X}_\sigma \geq \langle C_G(s) : s \in S \rangle \geq X$. Letting s, T be as before we have $\tilde{X}_\sigma \geq \langle X, T \rangle \geq X(T_0)T = \bar{X}_\sigma \geq \tilde{X}_\sigma$. Consequently, $\bar{X}_\sigma = \tilde{X}_\sigma$ and (12.1) yields $\tilde{X} = \bar{X}$. Also $\langle X, T \rangle = X(T_0)T = XT$. Therefore, $\tilde{X} \cap G_1 = \bar{X} \cap G_1 = \bar{X}_\sigma \cap G_1 = XT \cap G_1 = X(T \cap G_1) = XT_1 = X$, proving (i). If $X = G_1$, then $XT = G$, forcing $\bar{X} = \bar{G}$. Hence $\tilde{X} = \bar{G}$. Combining this with (iii) we have (ii).

Finally, suppose $S \subseteq T$ for T a maximal torus of G. For purposes of proving (iv) we may assume $\bar{G}$ is simply connected. For $s \in S$, $C_{\bar{G}}(s) = E(C_{\bar{G}}(s))\bar{T}$ and $C_G(s) = E(C_G(s))T$ (by (2.9) and its proof). Hence $\tilde{X} = \langle E(C_G(s)) : s \in S \rangle \bar{T}_1$. From the previous paragraph $\tilde{X} = \bar{X}$, so $\tilde{X}_\sigma = \bar{X}_\sigma = XT$. (iv) now follows from (12.1).

(12.11) Let $S_1,\ldots,S_k$ be subsets of T. Suppose that for each $\alpha \in \bar{\Sigma}^+$, there is some $i \in \{1,\ldots,k\}$ with $S_i \subseteq C(\bar{U}_\alpha)$. Then $G_0 = \langle E(C_{G_0}(S_i)) \mid i = 1,\ldots,k \rangle$.

Proof. For $1 \leq i \leq k$, $C_{\bar{G}}(S_i)^0$ is a reductive group and $(C_{\bar{G}}(S_i)^0)'$ a semi-simple group. Hence $E(C_{G_0}(S_i)) \geq O^{p'}((C_{\bar{G}}(S_i)^0_\sigma))$. By (12.4), $E(C_{G_0}(S_i)) = G(\Omega_i)$ for a unique closed subset Ω_i of $\bar{\Sigma}$ (if $E(C_{G_0}(S_i)) = 1$, set $\Omega_i = \varnothing$ and $G(\Omega_i) = 1$).

By (12.8), $\langle G(\Omega_1),\ldots,G(\Omega_k)\rangle = O^{p'}(\langle \bar{G}(\Omega_1),\ldots,\bar{G}(\Omega_k)\rangle_\sigma)$. If $\alpha \in \bar{\Sigma}^+$ with $S_i \leq C(\bar{U}_\alpha)$, then S_i centralizes $\langle \bar{U}_{\pm\alpha}\rangle \leq (C_{\bar{G}}(S_i)^0)'$. So if $\alpha \in \bar{\Sigma}_j$, we have $\bar{X}_j$, $\bar{X}_j^* \leq (C_{\bar{G}}(S_i)^0)'$ and then X_j, $X_j^* \leq G(\Omega_i)$. Since Ω_i is closed, we conclude $\bar{\Sigma}_j, \bar{\Sigma}_j^* \subseteq \Omega_i$. These remarks and our assumption show that $\langle \bar{G}(\Omega_1),\ldots,\bar{G}(\Omega_k)\rangle = \bar{G}$, and the result follows.

(12.12) Let T be a maximal torus of G and $R \leq T$. Then $G_0 = \langle E(C_{G_0}(R_1)) \mid R_1 \leq R \text{ and } R/R_1 \text{ cyclic}\rangle$.

Proof. Let $R \leq T$ and $\alpha \in \bar{\Sigma}^+$. Then R induces a cyclic group on $\bar{U}_\alpha$, hence $R_1 = C_R(\bar{U}_\alpha)$ has cyclic quotient group. So (12.12) follows from (12.11).

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Received July 21, 1980. Research supported in part by NSF Grant MCS 78-01944.

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