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TOPOLOGICAL SPHERICAL SPACE FORM PROBLEM III: DIMENSIONAL BOUNDS AND SMOOTHING

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In the two earlier papers in this series we showed, that if the finite group π has period $2d$ in cohomology, and if for all primes p a subgroup of order $2p$ is cyclic, then there exists a free topological action by π on a sphere of dimension S^{2nd-1}, for some positive integer n. Two questions remained open, namely whether there also existed smooth actions, and whether n could be taken equal to one. In this short paper we prove that there exists a free smooth action of π on $S^{2e(\pi)-1}$, the sphere with the standard differentiable structure. Here $e(\pi)$ is the Artin-Lam induction exponent, that is, the least positive integer such that $e(\pi)1$ belongs to the ideal of the rational representation ring, generated by representations induced from cyclic subgroups. It turns out that $e(\pi) = d(\pi)$ or $2d(\pi)$, and that our result is geometrically the best possible, except for one class of groups.

Our theorem can actually be extracted from various papers already available, chiefly [**3**], [**4**], [**10**] and [**12**], but given its interest, we think it worthwhile to publish this guide to the material separately.

The outline of the argument is as follows. Starting with a periodic projective resolution for the cohomology of π, see [**8**], we choose the chain homotopy type so that, restricted to a hyperelementary subgroup ρ, the resolution is equivalent to a free resolution of period $2e(\pi)$. It now follows that the finiteness obstruction for the π-resolution vanishes, and that there is a finite geometric realisation, $Y(\pi)$. Following [**3**] we construct a smooth normal invariant, and show that the surgery obstruction to replacing $Y(\pi)$ by a homotopy equivalent smooth manifold vanishes. By variation inside the orbit of this manifold under the action of $L_{2e(\pi)}(\mathbf{Z}\pi)$ on $\mathcal{S}_0(Y(\pi))$ we show that the universal cover $Y(1)$ may be taken to be the standard sphere. As in the construction of the complex $Y(\pi)$, in proving the existence of a homotopy smoothing we reduce technical problems by the principle of induction to hyperelementary subgroups. These are necessarily either metacyclic or split extensions of a cyclic group of odd order by a binary dihedral group $D^*_{2^k}$. Once dihedral subgroups D_{2p} are excluded, such a 2-hyperelementary subgroup admits a fixed point free representation of real degree four or eight, see [**13**, p. 168 and 204].

We shall use the following notation. A group π with periodic cohomology will be called a $\mathcal{P}$-group; if p and q are prime, π satisfies the pq-condition if every subgroup of order pq is cyclic. The manifold $M(\pi)$ is a linear space form, if it is one of the manifolds of constant positive curvature, associated to a fixed point free representation, and classified in **[13]**. As in **[3]** the Poincaré complex $Y(\pi)$ is polarised by the choice of an orientation and identification of the fundamental group with the abstract group π.

1. Choice of the normalized Poincaré complex. We refer to **[10]** and **[13]** for the classification of $\mathcal{P}$-groups — in the solvable case the reader may find the table on page 179 of **[13]** useful. The main purpose of **[10]** was to refine the earlier classification to emphasise the hyperelementary subgroup structure. Thus a 2-hyperelementary subgroup of Type II has presentation:

$$\{u_1, u_2, u_3, u_4, x, y \mid u_i^{a_i} = 1, y^2 = x^{2^{k-2}}, y^4 = 1, x^y = x^{-1}, u_1^x = u_1,$$
$$u_1^y = u_1, u_2^x = u_2, u_2^y = u_2^{-1}, u_3^x = u_3^{-1}, u_3^y = u_3, u_4^x = u_4^{-1}, u_4^y = u_4^{-1}\},$$

where the a_i are pairwise coprime and odd. We denote this group by $Q(2^k a_2; a_3, a_4) \times \mathbf{Z}/a_1$, and define it to be

of Type IIK, if $a_3 = a_4 = 1$

of Type IIL, if $k \geq 4$ and $a_3 a_4 \neq 1$, and

of Type IIM, if $k = 3$ and two of a_2, a_3, a_4 are distinct from 1.

The point of this subdivision is, that whereas all three types have cohomological period 4, types IIL and IIM only admit fixed free representations of real degree 8. Furthermore there is no free action by a group of type IIL on S^{4n-1}, if n is odd, see **[2]**. If $d(\pi)$ and $e(\pi)$ are as in the introduction, then $e(\pi) = d(\pi)$ unless $d(\pi) \equiv 2 \bmod 4$, and π contains subgroups of type IIL or IIM, in which case $e(\pi) = 2d(\pi)$. Write $d_p(\pi)$ for the p-period in cohomology.

For an arbitrary $\mathcal{P}$-group π and integer $n \equiv 0 \bmod d(\pi)$, the chain homotopy type of a projective resolution of period $2n$ is determined by a generator $g \in H^{2n}(\pi; \mathbf{Z})$, the k-invariant. The obstruction to replacing this projective resolution by a free one is an element $\theta(g) \in \tilde{K}_0(\mathbf{Z}\pi)$; from **[12]** we have the following result:

THEOREM 1. *With the notation above, suppose that $n \equiv 0 \bmod e(\pi)$ and that the restrictions $\{g_p\}$ of g to a representative family of Sylow subgroups $\{\pi_p\}$ are such that*

(i) *Let p be odd. If $d_p(\pi)$ is odd, or $n/d_p(\pi)$ even, g_p is an $e(\pi)$th power; if $d_p(\pi)$ is even and $n/d_p(\pi)$ odd, g_p is minus an $e(\pi)$th power.*

(ii) *If π_2 is cyclic, g_2 is $\pm e(\pi)$th power, depending on whether $n \equiv 0$ or* $2 \pmod 4$.

(iii) *Let* γ_1 *be the second Chern class of the standard embedding of* $D^*_{2^k}$ *in* $SU(2)$. *If* π_2 *is binary dihedral and* π *contains no subgroup of Type* IILM, *then* $g_2 = \lambda^n \gamma_1^{n/2}$.

(iv) *Otherwise,* $g_2 = ((1 + 2^{k-2})\lambda^2\gamma_1)^{n/2}$. (*Note that in this last case* $n \equiv 0 \pmod 4$.)

Then the finiteness obstruction $\theta(g)$ *equals zero.*

REMARK. The possibility that $n \equiv 2 \pmod 4$ is omitted from the treatment in [**12**]. It arises when π is metacyclic and of cohomological period 4. In the notation of [**13**] π is then isomorphic to $D^*_{2^k(2v+1)}$.

Proof. We refer the reader to [**12**, §§11–12] for the details. However the main idea is that $\theta(g)$ vanishes only if it does so on each hyperelementary subgroup ρ. The conditions on the restricted generators $g_p \in H^{2n}(\pi_p, \mathbf{Z})$ are chosen to ensure, that over ρ the projective resolution, truncated in dimension $2n$, is chain homotopy equivalent to a free resolution. For all hyperelementary subgroups of Type II, and for those of Type I, which are such that the automorphism group of the quotient p-group does not act faithfully on the normal cyclic subgroup, this free resolution is defined by a fixed point free representation of real degree $2n$. For the remaining groups of Type I, one uses the explicit free resolution constructed in [**12**, Thm. 12.1 and §14]. Since g can always be chosen to satisfy the numerical restrictions (i)–(iii), periodic free resolutions certainly exist in dimension $2n - 1$, with $n \equiv 0 \bmod e(\pi)$.

Let ρ be a 2-hyperelementary subgroup of the $\mathcal{P}$-group π, satisfying the $2p$-condition. A subgroup of ρ of order pq, where p and q are both odd primes, is necessarily contained in a cyclic normal subgroup of ρ. Hence ρ is the fundamental group of a linear space form. From the outline proof of Theorem 1 it follows that the covering complex $Y(\rho)$ corresponding to the subgroup ρ is homotopy equivalent to this linear space form. (If ρ is of Type I and ρ_2 acts faithfully on $\rho_{(\mathrm{odd})}$, then ρ contains a subgroup isomorphic to some dihedral group D_{2p}.) For groups of Types III–VI we have slightly more:

(i) The restriction of the generator g_2 to a binary dihedral group of order 8 (respectively 16) equals $\gamma_1^{n/2}$ (respectively $\gamma_1^{n/2}$ or $(9\gamma_1)^{n/2}$).

(ii) If $v > 1$, and π contains a subgroup T_v^* (respectively O_v^*), $d_3(\pi) = 1$ (respectively $n/d_3(\pi)$ is even). In both cases g_3 is an nth. power.

By inspection of the top dimensional Chern class of the fixed point free representations of T_v^* and O_v^* [**13**, pages 199–202] and [**9**], we see that the k-invariant g can be chosen so that any covering complex with fundamental group isomorphic to one of these groups is also homotopy equivalent to a linear space form.

If $v = 1$, more care is needed, since for the faithful representations τ of T_1^* and $o_\pm$ of O_1^* in SU(2), the mod 3 second Chern class is *minus* a square. If $d_3(\pi) = 2$ (Types IV, V and VI), then, without loss of generality taking $n = e(\pi)$, g_3 is an $e(\pi)$th (respectively $-e(\pi)$th) power if $e(\pi)$ is divisible by 4 (respectively otherwise). By acting on $S^{2e(\pi)-1}$ via the direct sum of copies of τ or $o_{\pm 1}$, we can again ensure that $Y(T_1^*)$ or $Y(O_1^*)$ is homotopy equivalent to a linear space form. However for groups of Type III, $d_3(\pi) = 1$, and when $n \equiv 2 \bmod 4$, in constructing a smooth normal invariant, we need the full force of the theorem below.

2. The smooth normal invariant and surgery. In our discussion of normal invariants we start by assuming that $Y(\pi)$ is an arbitrary finite Poincaré complex of dimension $2n - 1$ wtih $n \equiv 0 \bmod d(\pi)$, and then apply our result to the very special complex constructed in the previous section. Let $O \to G$ be the natural map of the stable orthogonal group into the space of stable self homotopy equivalences of the sphere. The set of smooth normal invariants over Y (stable O-bundles with spherical Thom class) is in $(1 - 1)$ correspondence with the homotopy set $[Y, G/O]$. Denote such a bundle by ν, and its lift to a p-Sylow covering space of Y by ν_p.

THEOREM 2. *Let $Y(\pi)$ be a finite $(2n - 1, \pi)$-polarized complex, such that for each 2-hyperelementary subgroup ρ contained in π, $Y(\rho)$ is homotopy equivalent to a manifold. There is a smooth normal invariant ν the restriction of which to a 2-Sylow covering complex $Y(\pi_2)$ coincides with the normal invariant of a linear space form.*

Proof. Since the map $BO \to BG$ is a map of infinite loop spaces, $k(\ \ ) = [\ \ , B(G/O)]$ is a respectable cohomology theory with a transfer, from which it follows that

$$k(Y(\pi)) \to \coprod_{p \mid [\pi:1]} k\big(Y(\pi_p)\big)$$

is a monomorphism. But π_p is either cyclic or binary dihedral; in both cases $Y(\pi_p)$ is homotopy equivalent to a smooth manifold, and the existence of normal invariants follows. Next we pick a normal invariant with a good restriction ν_2, by checking the $\mathcal{P}$-groups type by type. If π is solvable (I–IV), π is a split extension of a metacyclic group by a group π_2, T_v^* or O_v^* of coprime order. Since π retracts onto the quotient, either

$H^*(\pi, A)_2 \cong H^*(\pi_2, A)$ (Types I and II), or

$H^*(\pi, A)_{2,3} \cong H^*(T_v^*, A)$ or $H^*(O_v^*, A)$ (Types III and IV).

Comparing the spectral sequences $H^*(\ \ , G/O^*(\mathrm{pt})) \Rightarrow G/O^*(\ \ )$ for π and its quotient group, we see that the isomorphism holds with H^* replaced by the cohomology theory defined by maps into the iterated loop spaces of G/O. The result now follows at once for groups of Type I or II. For a group of Type III we note that the argument just given also shows that

$$G/O(Y(T_v^*)) \cong G/O(Y(Z/3^v)) + G/O(Y(D_8^*))^{\mathrm{inv}}.$$

Now the unique free linear action of D_8^* on S^{2n-1} extends to T_v^*, hence its normal invariant is invariant with respect to the induced $\mathbf{Z}/3^v$-action. It follows that we can first lift this choice of ν_2 back to $Y(T_v^*)$, and then using the first decomposition, back to $Y(\pi)$. Note that this conclusion is independent of the homotopy type of $Y(T_v^*)$. For Type IV, as in the topological case we note that the obstructions to factoring a map $Y(\pi_2) \to G/O$ through $Y(\pi)$ are detected by the obstructions to factoring $Y(\pi_2 \cap T_v^*) \to G/O$ through $Y(T_v^*)$, and we have just seen that these vanish. For the remaining groups (Types V and VI) we need the same small diagram of subgroups as in [**3**, 3.3], and [**4**, 1.10].

LEMMA 3. *If π is of Type* V, *a smooth normal invariant ν_2 extends to $Y(\pi)$ if and only if the restriction to each of the subgroups isomorphic to D_8^* in π_2 extends to a normal invariant for $Y(T_1^*)$.*

Proof. Clearly this condition is necessary. To prove sufficiency, let us take the sequence of obstructions in $H^i(p_*, \pi_{i-1}(G/O))$ to extending ν_2 to ν, defined over all of $Y(\pi)$. Here $p\colon Y(\pi_2) \to Y(\pi)$ denotes the covering map. We first note, that given the splitting of the cohomology theory $k(\ \ )$, odd torsion does not affect the extension problem, and hence the obstruction can only be non-zero, when

$$H^i(\pi, \pi_{i-1}(G/O))_2 \to H^i(\pi_2, \pi_{i-1}(G/O))$$

fails to be an isomorphism. This only occurs in dimensions congruent to 1 or 2 modulo 4, and by periodicity one sees that it is enough to consider $H^2(\pi_2, A) \simeq A/2A + A/2A$. The binary dihedral group $D_{2^{t+2}}^* = \{x, y\colon x^{2^{t+1}} = 1, x^{2^t} = y^2, x^y = x^{-1}\}$ contains two subgroups isomorphic to D_8^*, generated respectively by $\{y, x^{2^{t-1}}\}$ and $\{xy, x^{2^{t-1}}\}$, and an element in $H^2(\pi_2, A)$ is detected by restriction to one or another of them. By hypothesis the restriction of ν_2 extends to $Y(T_1^*)$, which implies that the restricted obstruction as an element of $H^2(D_8^*, A)$ is invariant under the induced action of $\mathbf{Z}/3$. But the only such element is trivial, from which it follows that ν_2 extends to ν.

For a group of Type VI we need to combine this lemma with an argument analogous to that used to reduce Type IV to Type III.

In order to complete the proof of the theorem, choose ν_2 in $[Y(\pi_2), G/O]$ to be the normal invariant of a linear space form. (This can certainly be done, as in the first paragraph of the proof, since any finite polarised complex is homotopy equivalent to a linear space form, see for example [9, Theorem 6].) Over the subgroups $\langle y, x^{2^{t-1}}\rangle$ and $\langle xy, x^{2^{t-1}}\rangle$ ν_2 reduces to the normal invariant of the unique linear space form with fundamental group D_8^*. The action defining this extends to T_1^* and we are done.

REMARK. For solvable groups of Types III and IV, which map onto T_v^* and O_v^*, for $v \geq 2$, the simple argument for Types I and II actually applies. This is because conditions on the finiteness obstruction imply that a (2, 3)-Sylow covering complex is already homotopy equivalent to a linear space form. The argument fails for T_1^* and O_1^*, because of the existence of non-linear homotopy types, which are finitely realisable, see [8], [9] and [12].

Before proving the existence of space forms corresponding to the complex $Y(\pi)$, we need one technical lemma on surgery obstruction groups. For convenience we work in the category of weak simple homotopy types, that is we calculate torsion in the image $Wh'(\mathbf{Z}\pi)$ of the homomorphism $Wh(\mathbf{Z}\pi) \to Wh(\mathbf{Q}\pi)$. The associated L-groups are denoted by $L'_*(\mathbf{Z}\pi)$.

LEMMA 4. *If i denotes the inclusion i: $\{1\} \to \pi$, then the induced restriction homomorphism i^*: $L'_0(\mathbf{Z}\pi) \to L'_0(\mathbf{Z})$ is an epimorphism for all finite groups π.*

Proof. As usual write Σ for the sum of the group elements in π, and consider the Milnor square

$$\begin{array}{ccc} \mathbf{Z}\pi & \xrightarrow{k} & \mathbf{Z}\pi/(\Sigma) \\ \downarrow j & & \downarrow \\ \mathbf{Z} & \to & \mathbf{Z}/[\pi; 1] \end{array}$$

If $X = SK_1(\mathbf{Z}\pi) + \{\pm 1\} + \pi/[\pi; \pi]$, $\bar{X}$ is its image in $K_1(\mathbf{Z}\pi/\langle\Sigma\rangle)$, and L_i^X is defined by those quadratic forms with determinant in X, then there is a "mixed" Mayer-Vietoris exact sequence, see [7]

$$\cdots \to L_0^X(\mathbf{Z}\pi) \xrightarrow{j_* + k_*} L_0^{\{\pm 1\}}\mathbf{Z} + L_0^{\bar{X}}(\mathbf{Z}\pi/\langle\Sigma\rangle) \to L_0^{\{\pm 1\}}(\mathbf{Z}/[\pi:1]) \to \cdots$$

From [11, 1.4] we have that $L_0^s(\hat{\mathbf{Z}}) = 0$, hence that $L_0^{\{\pm 1\}}(\mathbf{Z}) = L_0^s(\mathbf{Z})$ maps onto $L_0^s(\mathbf{Z}/[\pi:1]) = L_0^{\{\pm 1\}}(\mathbf{Z}/[\pi:1])$. By exactness there exists $x \in L_0^X(\mathbf{Z}\pi)$, mapping to the pair $(x_0, 0)$ in $L_0^{\{\pm 1\}}(\mathbf{Z}) + L_0^{\bar{X}}(\mathbf{Z}\pi/\langle\Sigma\rangle)$. We claim that $i^*(x) = x_0$. Since $L_0^{\{\pm 1\}}(\mathbf{Z})$ maps momomorphically into

$L_0^{\{\pm 1\}}(\mathbf{R})$, it is enough to check this over $\mathbf{R}$, when the long exact sequence above decomposes into trivial short exact sequences. The class $x_{\mathbf{R}}$ is represented by a form on $(\mathbf{R})^{\oplus n} = \mathbf{R}^{\oplus n} + (\mathbf{R}\pi/\langle \Sigma \rangle)$, and the projection onto the first factor equals $x_{0\mathbf{R}}$. Clearly $i^* x_{\mathbf{R}} = x_{0\mathbf{R}}$ — forget the π-action.

We are now ready for

THEOREM 5. *Let π be a $\mathcal{P}$-group, satisfying the $2p$-condition for all primes p. Then there exists a free smooth action of π on S^{2n-1} for all $n \equiv 0$ mod $e(\pi)$.*

Proof. From Theorems 1 and 2 it follows, that in all dimensions $2n - 1$ with $n \equiv 0 \bmod e(\pi)$, there exists

(i) a finite Poincaré complex $Y^{2n-1}(\pi)$ such that for each 2-hyperelementary subgroup $\rho \subset \pi$, $Y(\rho)$ is homotopy equivalent to a smooth manifold, and

(ii) a normal invariant $\nu(Y(\pi))$ such that $\nu(Y(\pi_2))$ equals the normal invariant of a linear space form.

As in [3] we now show that the surgery obstruction $\sigma(Y)$ to replacing $Y(\pi)$ by a smooth manifold vanishes. Note that condition (ii) implies that $\sigma(Y(\rho)) = 0$ for every 2-subgroup of π. The principle of Brauer induction for automorphisms of quadratic forms, [1] and [**11**], implies that σ vanishes if its restrictions to the class of p-elementary subgroups (p = odd) and 2-hyperelementary subgroups all vanish. In the first case $\rho = \rho_2 + \rho'$, where ρ' is cyclic of odd order. However, by [**11**, Theorem 2.4.2]

$$L'_*(\mathbf{Z}(\rho_2 + \rho')) \cong L'_*(\mathbf{Z}\rho_2) + \tilde{L}_*(\mathbf{R}(\rho_2 + \rho')),$$

and hence, since $Y(\pi)$ has odd dimension, $\sigma(Y(\rho)) = \sigma(Y(\rho_2)) = 0$.

In the second case, condition (i) implies that the surgery problem for ρ is one of maps between manifolds, which is again solved by lifting to ρ_2. Hence $Y(\pi)$ is homotopy equivalent to a smooth manifold $M(\pi)$, such that the universal cover $M(1)$ is a homotopy sphere Σ^{2n-1}.

So far we know that there is a homotopy smoothing $f: M(\pi) \to Y(\pi)$, such that the covering pair $f_2: M(\pi_2) \to Y(\pi_2)$ is normally cobordant to $g_2: N(\pi_2) \to Y(\pi_2)$, where $N(\pi_2)$ is a linear space form. Thus $N(1) = S^{2n-1}$, and f_2 and g_2 belong to the same orbit in $g_0(Y(\pi))$ under the action of $L'_{2n}(\mathbf{Z}\pi_2)$. Suppose that $[f_2]x = [g_2]$. If $2n \equiv 0 \bmod 4$, by Lemma 4 there exists $y \in L'_{2n}(\mathbf{Z}\pi)$ such that

$$i^*_{\pi \to \{1\}}(-y) = i^*_{\pi_2 \to \{1\}}(x).$$

Now act on the smoothing (M, f) by y; the image has universal cover diffeomorphic to the standard sphere. If $2n \equiv 2 \bmod 4$, π is necessarily of Type I, that is, a semidirect product of two cyclic groups of coprime orders. If π_2 is normal, it breaks off as a direct summand, and we can again use [**11**, 2.4.2] to lift x from an element $y \in L'_{2n}(\mathbf{Z}\pi)$. Otherwise π retracts onto π_2, and the same holds for the groups L'_{2n} by the splitting theorem (4.1.2) in [**11**]. Finally, for the sake of completeness, we consider the case when π_2 is trivial, π once more of Type I, and $2n \equiv 2 \bmod 4$. Choose a prime q such that π retracts onto π_q, then there is no problem in choosing $\nu(Y(\pi))$ to satisfy condition (ii) above with 2 replaced by q. By [**11**, 2.4.3] $L'_2(\mathbf{Z}\pi) = L'_2(\mathbf{Z}\pi_q) = \mathbf{Z}/2$ (detected by the classical Arf invariant). Therefore, by choice of the normal invariant, the universal covering manifold $\tilde{M}(\pi)$ is the standard rather than the Kervaire sphere.

We summarize the implications of our theorem type by type.

For I, IIK, III, IVK, V, VIK, and VIL ($p \equiv 1 \bmod 4$), $d(\pi) = e(\pi)$ and free actions exist in the lowest possible dimension. Note that no group of Type III or V contains a 2-hyperelementary subgroup of Type IIL or IIM, [**10**, 5.2 and 5.4]. For Type VI, when $p \equiv 1 \bmod 4$, $d(\pi)$ is divisible by four, and the presence of subgroups of Type IIL is irrelevant.

For IIL, IVL and VIL ($p \equiv -1 \bmod 4$), $e(\pi) = 2d(\pi)$, but there is a topological obstruction to the existence of free actions in dimension $2d(\pi) - 1$, [**2**]. In these cases Theorem 5 is the best possible geometric result.

The situation for IIM is complicated. With the notation of section one, take $a_1 = a_4 = 1$ and a_2 and a_3 to be distinct odd primes. Abbreviate $Q(8a_2 : a_3, 1)$ by Q. Both the finiteness obstruction $\theta_4(Q) \in \tilde{K}_0(\mathbf{Z}Q)$ in dimensions congruent to 3 (modulo 8), and the surgery obstruction $\sigma_{8k+3}(Y) \in L^h_3(\mathbf{Z}Q)$ of the associated surgery problem come into play. In [**6**] R. J. Milgram proves that $\theta_4(Q)$ lies in the subgroup $D(\mathbf{Z}Q)$ of the projective class group, and can evaluate it in many cases.

There is a discussion of the surgery obstruction in [**5**]. However at the time of writing this part of the theory is in a somewhat fluid state.

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