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FINITE-TO-ONE OPEN MAPPINGS ON CIRCULARLY CHAINABLE CONTINUA

PHILIP BARTICK AND EDWIN DUDA

The authors analyze the behavior of a finite-to-one open mapping on a hereditarily decomposable circularly chainable continuum. It is shown that such a mapping behaves similarly to an open mapping on a simple closed curve.

0. Introduction. In a previous paper (2), the authors completely described the behavior of a finite-to-one open mapping on a hereditarily decomposable chainable continuum. In this note we will completely describe the behavior of a finite-to-one open mapping on a hereditarily decomposable circularly chainable continuum. This result is a generalization of the following theorem of G. T. Whyburn (**8**): If X is a simple closed curve and $f(X) = Y$ is a non-constant open mapping onto a Hausdorff space, then Y is either a simple closed curve or an arc. If Y is a simple closed curve, then there is an integer n such that f is topologically equivalent to the mapping $w = z^n$ on the unit circle in the complex plane. If Y is a simple arc, then there exists an even integer k such that f is topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for $0 \leq x \leq 2\pi$ from the unit circle $r = 1$ in the plane to the interval $[-1, 1]$. We use the fundamental results of E. S. Thomas (**6**) implicitly.

1. Notation and definitions. The word mapping is used to denote a continuous function and a metric continuum X is decomposable if it can be expressed as the union of two proper subcontinua. A continuum X is hereditarily decomposable if every non-degenerate subcontinuum is decomposable. The interior of a subset D is denoted by $\mathrm{int}(D)$ and the closure of D is represented by $\mathrm{cl}(D)$. Two mappings $f(X) = Y$ and $g(Z) = W$ are topologically equivalent if there exist homeomorphisms $h(X) = Z$ and $k(W) = Y$ such that $kgh(x) = f(x)$ for all x in X. A mapping $f(X) = Y$ is irreducible if no proper subcontinuum of X maps onto Y under f. Suppose X is a continuum which admits a monotone mapping t onto the unit interval such that no point inverse has interior points. A point inverse $t^{-1}(r)$, $r \neq 0$, is called an element of subcontinuity from the left in the upper-semicontinuous decomposition of X induced by t if $t^{-1}(r) \subset \mathrm{cl}(t^{-1}[0, r))$. If $r \neq 1$ and $t^{-1}(r) \subset \mathrm{cl}(t^{-1}(r, 1])$, then $t^{-1}(r)$ is an element of subcontinuity from the right. The collection of elements of subcontinuity from the left are known to be a dense set in X (**6**) and the same is true of the elements of subcontinuity from the right.

2. Preliminary results.

LEMMA 1. *Let X be a hereditarily decomposable circularly chainable continuum. There is a monotone upper-semicontinuous decomposition $G(X)$ defined on X whose quotient space is a simple closed curve and each element of which has empty interior.*

Outline of proof. Let $X = A \cup B$ be a decomposition of X into proper subcontinua. We can assume that $A \cap B$ has no interior points. The continua A and B are chainable and hereditarily decomposable, so that by R. H. Bing's characterization (**1**) they admit monotone upper-semicontinuous decompositions $G(A)$ and $G(B)$ respectively whose quotient spaces are simple arcs, say $[a, b]$ and $[c, d]$ respectively. Furthermore, no element of $G(A)$ or $G(B)$ has interior points relative to X. We can assume that the elements corresponding to a and c meet, as do the elements corresponding to b and d. The monotone decomposition $G(X)$ is obtained by taking the union of the elements corresponding to a and c as one element, the union of the elements corresponding to b and d as another element, and the rest of the elements of $G(A)$ and $G(B)$ for the remainder of $G(X)$. This decomposition is upper-semicontinuous, its quotient space is a simple closed curve, and no element of it has interior points.

The following theorem established in (**2**) is stated in complete detail even though only the first portion is used in this paper.

THEOREM A. *Let $f(X) = Y$ be a finite-to-one open mapping, where X is a non-degenerate hereditarily decomposable chainable continuum and Y is a Hausdorff space. Then $X = \bigcup_{j=1}^{n} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, f is a homeomorphism on any continuum which is interior to K_j relative to X, and if $K_i \cap K_j$ is non-empty for $i \neq j$, then the intersection is contained in a single element K of $G(X)$. If K is an element of subcontinuity from one side, then it is an element of subcontinuity from both sides. If not, then K is the union of two homeomorphic subcontinua which meet in a single element of $G(K)$, f is one-to-one on each of them, and they have the same image. If f is irreducible (i.e., there is only one K_j), then f is a homeomorphism.*

LEMMA 2. *If $f(X) = Y$ is a finite-to-one open mapping, where X is hereditarily decomposable and circularly chainable, and Y is Hausdorff, then Y is hereditarily decomposable and either circularly or linearly chainable.*

Proof. By well-known theorems, Y is a metric continuum. The space Y is hereditarily decomposable, for if Y contained a non-degenerate indecomposable continuum L, then by an application of the Brouwer Reduction Theorem there would exist an indecomposable continuum in X which maps onto L. The space Y is circularly or linearly chainable by a theorem of E. Duda and J. Kell (**3**) or by applying a result of W. T. Ingram (**5**).

3. Main result.

THEOREM 1. *Let $f(X) = Y$ be a finite-to-one open mapping, where X is hereditarily decomposable and circularly chainable and Y is a Hausdorff space. Then Y is hereditarily decomposable and either linearly or circularly chainable. If Y is circularly chainable, then f is exactly n-to-one for some n and a local homeomorphism, and $X = \bigcup_{j=1}^{n} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, and if $K \subset \operatorname{int}(K_j)$ is a continuum, then f is one-to-one on K. If Y is linearly chainable, then there is an even integer k such that $X = \bigcup_{j=1}^{k} K_j$, where each K_j is a continuum, $f(K_j) = Y$ for each j, and if $K \subset \operatorname{int}(K_j)$ is a continuum, then f is one-to-one on K.*

Proof. By Lemma 2, Y is a hereditarily decomposable linearly or circularly chainable continuum. By Lemma 1, there is an upper-semicontinuous decomposition $G(X)$ generating a monotone mapping g_1 onto a unit circle S_1, such that each point-inverse of g_1 is an element of $G(X)$. If Y is circularly chainable, there is a similar upper-semicontinuous decomposition $G(Y)$ generating a monotone map g_2 onto a unit circle S_2. If Y is linearly chainable, then by Bing's theorem (1), there is a monotone decomposition $G(Y)$ generating a monotone map g_3 of Y onto the interval $I = [-1, 1]$. Each point-inverse of g_1, g_2, or g_3, or equivalently, each element of $G(X)$ or $G(Y)$, has empty interior.

It was proved in (2) that if $T \in G(X)$, then $f(T) \in G(Y)$, and if $L \in G(Y)$, then $f^{-1}(L) = \bigcup_{i=1}^{m} L_i$, where $L_i \cap L_j = \varnothing$ if $i \neq j$ and each $L_i \in G(X)$.

If Y is circularly chainable, define a mapping f_1 of S_1 onto S_2 by $f_1(z) = g_2 f g_1^{-1}(z)$. The mapping f_1 is well-defined, since f preserves decomposition elements, and f_1 is continuous, finite-to-one and open, since f has these properties. By the quoted theorem of Whyburn, f_1 is a local homeomorphism which is topologically equivalent to z^n for some integer n. If I_1 is an interval in S_1 of length less than $2\pi/n$, then $f_1 \mid I_1$ is a homeomorphism of I_1 onto $f_1(I_1)$. The set $g_1^{-1}(I_1)$ is a continuum, and it is a component of $f^{-1}f(g_1^{-1}(I_1))$, so $f \mid g_1^{-1}(I_1)$ is a finite-to-one open mapping. Furthermore, $f \mid g_1^{-1}(I_1)$ is irreducible, and hence by Theorem A, it is a homeomorphism. We now know that f is a local homeomorphism and exactly n-to-one.

Let S_1 be divided into n non-overlapping arcs $I_j, j = 1,\ldots,n$, each of length $2\pi/n$, and let $K_j = g_1^{-1}(I_j)$ for each j. Then X is the union of the $K_j, f(K_j) = Y$ for each j, and if $K \subset \text{int}(K_j)$ then $f|K$ is one-to-one.

Suppose Y is linearly chainable. Define the map f_2 from S_1 to I by $f_2(z) = g_3 f g_1^{-1}(z)$. As before, f_2 is a continuous finite-to-one open mapping of S_1 onto I, and by Whyburn's theorem, f_2 is topologically equivalent to the mapping $f(1, x) = \sin(kx/2)$ for x between 0 and 2π and k an even integer. In this case, $S_1 = \bigcup_{j=1}^{k} I_j$ and $f_2 | I_j$ is a homeomorphism of I_j onto I. Let $K_j = g_1^{-1}(I_j)$ for each j. Then $f(K_j) = Y$ for each j, and if L is any continuum in the interior of K_j, then $f| L$ is a finite-to-one open mapping of L onto $f(L)$. Furthermore, it is irreducible, so that $f| L$ is a homeomorphism by Theorem A.

The following example illustrates that when Y is linearly chainable, $f| K_j$ is not necessarily a homeomorphism. Let X be the closure of the graph of $y = \sin(1/x)$, $x \neq 0$, $x \in [-1, 1]$. Let the endpoints of X be a and b. In the unique minimal monotone decomposition $G(X)$ given by Bing's theorem, the only non-degenerate element is the interval from $(0, -1)$ to $(0, 1)$ on the y-axis. Let q be the quotient map obtained by identifying each point (x, y) in X with $(-x, -y)$. The mapping q is open and exactly 2-to-one except at the origin. Let Y be a disjoint copy of X with endpoints c and d corresponding to a and b, respectively. Let Z be the circularly chainable continuum obtained by identifying c with a and b with d. Let p be the mapping of Z onto X obtained by folding Z. The mapping $f = q \circ p$ is a finite-to-one open mapping of Z onto $q(X)$. The continuum $q(X)$ is chainable. There are four continua K_1, K_2, K_3, K_4 given by Theorem 1 such that $Z = \bigcup_{j=1}^{4} K_j$ and $f(K_j) = q(X)$ but $f| K_j$ is not one-to-one, even though $f| \text{int}(K_j)$ is one-to-one.

We thank the referee for pointing out that Lemma 2 of this paper appears in a more general form in (**4**) and that Lemma 1 is also true in a more general form in (**7**).

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University of Miami
Coral Gables, FL 33124