PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

# ENGEL'S THEOREM FOR A CLASS OF ALGEBRAS

ERNEST L. STITZINGER

**A condition of nilpotency is derived for a class of algebras which include the almost alternative algebras of A. A. Albert. This result is seen to be an extension of Engel's theorem. Some consequences are then considered.**

An algebra $A$ over a Noetherian ring $R$ is called almost alternative if it is power associative and satisfies the following identities

I.

$$u(vw) = \alpha_1(uv)w + \alpha_2 w(uv) + \alpha_3(vu)w + \alpha_4 w(vu) + \alpha_5(uw)v + \alpha_6 v(uw) + \alpha_7(wu)v + \alpha_8 v(wu)$$

and

II.

$$(vw)u = \beta_1 v(wu) + \beta_2(wu)v + \beta_3 v(uw) + \beta_4(uw)v + \beta_5 w(vu) + \beta_6(vu)w + \beta_7 w(uv) + \beta_8(uv)w$$

where $u$, $v$, $w \in A$ and $\alpha_i$, $\beta_i \in R$. In what follows the requirement of power associativity will not be needed. Lie, alternative and $(\gamma, \delta)$-algebras are contained in this class. It is the purpose of this note to give a criterion for nilpotency inspired by the Engel theorem in Lie algebras. However, it is not sufficient to assume that each multiplication is nilpotent. For let $A$ be a 3-dimensional algebra generated by $x$, $y$ and $z$ where $xy = z$ and $zx = y$ and all other multiplications between basis elements are 0. Then $A$ is a non-nilpotent algebra satisfying I and II with $\alpha_4 = \beta_4 = 1$ and all other $\alpha_i, \beta_i = 0$. Also each right and left multiplication by any element in $A$ is nilpotent. Note that $A$ is not power associative.

Let $R$ be a Noetherian ring. All algebras and modules over $R$ are assumed to be unital. Let $A$ be an algebra over $R$ satisfying I and II. For each $x \in A$, $R_x$ and $L_x$ will denote right and left multiplication of $A$ by $x$. Let $M$ be an $A$-bimodule (see [4, p. 25]) with induced representation $(S, T)$. Hence $S$ and $T$ satisfy identities derived from I and II and consequently $S_{xy}$, $T_{xy} \in \langle S_x, S_y, T_x, T_y \rangle$, the associative subalgebra of $\mathrm{End}(M)$ generated by $S_x$, $S_y$, $T_x$, $T_y$. In particular, if $x \in A$ and $y$ is in the subalgebra of $A$ which is generated by $x$, then $S_y$, $T_y \in \langle S_x, T_x \rangle$. This latter subalgebra will be denoted by $J(x)$. Also recall that a subset of $A$ which is closed under multiplication is called a Lie set.

THEOREM. *Let $A$ be an algebra over a Noetherian ring $R$ such that $A$ satisfies* I *and* II. *Let $M$ be an $A$-bimodule such that $M$ is a finitely generated $R$-module. Let $C$ be a Lie set in $A$ such that $C$ generates $A$. Suppose that for each $x \in C$, $J(x)$ is nilpotent. Then $A$ acts nilpotently on $M$.*

*Proof.* There exist Lie subsets $G$ of $C$ such that the algebra generated by $G$, $\langle G \rangle$, acts nilpotently on $M$ since the algebra generated by any $x \in C$ has this property. This follows from the remarks preceding the statement of the theorem. Also $\{x \in A;\ Mx = xM = 0\}$ is an ideal in $A$. Hence assume that the representation is faithful and that $A$ is finitely generated. Let $D$ be a Lie subset of $C$ such that $\langle D \rangle$ acts nilpotently on $M$ and $\langle D \rangle$ is maximal with this property. Since we may assume that $A$ contains $\langle D \rangle$ properly, $C \nsubseteq \langle D \rangle$. By assumption there exists $n$ such that $M\sigma_1 \cdots \sigma_n = 0$ for all possible $\sigma_1,\ldots,\sigma_n$ where $\sigma_i = S_{x_i}$ or $\sigma_i = T_{x_i}$ and each $x_i \in D$. Let $M_i = \{m \in M;\ m\sigma_1 \cdots \sigma_i = 0$ for all possible $\sigma_1,\ldots,\sigma_i\}$. Then $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$.

Now consider any element of the form $\sigma_1 \cdots \sigma_{i-1}\tau\sigma_i \cdots \sigma_{2n}$ where each $\sigma_i$ is as defined above, $\tau = S_a$ or $\tau = T_a$ and $i = 1,\ldots,2n-1$. This product must be 0. Next consider any $y \in A$ with $2n+1$ factors of which $2n$ are from $D$. Both $S_y$ and $T_y$ are the sum of terms of the above type, hence $S_y = T_y = 0$. Hence $y = 0$. Now there exists a least positive integer $m$ such that $C\tau_1 \cdots \tau_m \subseteq \langle D \rangle$ for all $\tau_1,\ldots,\tau_m$ where each $\tau_i = R_{x_i}$ or $\tau_i = L_{x_i}$ and $x_i \in D$. Then there exists $z = x\tau_1 \cdots \tau_{m-1} \notin \langle D \rangle$ and $x \in C$. Now $z \in C$ since $C$ is a Lie set and if $y$ is the product of $z$'s, no matter how associated, then $y \in C$, and using I and II, $yD$, $Dy \subseteq D$. If $F$ is the union of $D$ and all such $y$, then $F$ is a Lie subset of $C$ and $\langle F \rangle$ contains $\langle D \rangle$ properly.

It remains only to show that $\langle F \rangle$ acts nilpotently on $M$. Under the conditions on the representation, $S_z$ and $T_z$ leave each $M_i$ invariant and $J(z)$ acts nilpotently on each factor. Furthermore $S_y$, $T_y \in J(z)$ where $y$ is any product of $z$'s. Then the chain may be refined to one such that $F$, and hence $\langle F \rangle$, annihilates each factor. This contradiction establishes the result.

The following are extensions of results on Lie algebras to the present setting.

COROLLARY 1. *Let $A$ and $R$ be as in the theorem and let $N$ be a nilpotent ideal of $A$. Let $U(N, A)$ be the associative subalgebra of* End$(A)$ *generated by $R_x$, $L_x$ for all $x \in N$. Then $U(N, A)$ is contained in the radical of $U(A, A)$.*

*Proof.* Since $N$ is a nilpotent ideal of $A$, $J(n)$ is nilpotent for each $n \in N$. Hence $U(N, A)$ is nilpotent by the theorem. Hence these exists a chain $0 = A_0 \subseteq \cdots \subseteq A_k = A$ where $A_j = \{x \in A; Nx + xN \subseteq A_{j-1}\}$. Clearly $AA_j + A_jA \subseteq A_j$. Let $\sigma = \sigma_1 \cdots \sigma_n$ where $\sigma_i = S_{x_i}$ or $\sigma_i = T_{x_i}$, $x_i \in A$, and at least one $x_i \in N$. Then $T$ annihilates each factor in the chain. Hence the ideal of $U(A, A)$ generated by $U(N, A)$ has nilpotent length $k$.

Using the regular representation, the theorem becomes

COROLLARY 2. *Let $A$ be finitely generated over the Noetherian ring $R$. Suppose that $A$ satisfies* I *and* II. *Let $C$ be a Lie set in $A$ such that $C$ generates $A$. If $J(x)$ is nilpotent for all $x \in C$, then $A$ is nilpotent.*

The next application deals with the nilpotency of algebras which admit regular automorphisms. This is an extension of a Lie algebra result of Jacobson [3]. Note that algebras satisfying I and II remain in this class under extension of the base.

COROLLARY 3. *Let $A$ be a finite dimensional algebra over a field. Suppose that $A$ satisfies* I *and* II. *Let $\Phi$ be an automorphism of $A$ such that $\Phi^p = I$ where $p$ is a prime. Suppose that $\Phi$ has no non-zero fixed points. Then $A$ is nilpotent.*

*Proof.* We may assume that the base is algebraically closed. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $\Phi$, all of which are $p$-roots of unity other than 1, and let $A = A_{\alpha_1} \oplus \cdots \oplus A_{\alpha_n}$ be the decomposition of $A$ into characteristic subspaces. For roots $\alpha$, $\beta$, $A_\alpha A_\beta \subseteq A_{\alpha\beta}$. Hence $C = UA_{\alpha_i}$ is a Lie set in $A$. Let $x \in A_\alpha$ and $\beta$ be any $p$-root of unity. Then there exists $i$, $1 \leq i \leq p$, such that $\beta\alpha^i = 1$. If $\sigma$ is the product of $i$ terms, each of which is $R_x$ or $L_x$, then $A_\beta \sigma \subseteq A_1 = 0$. Hence if $\sigma$ is the product of $p$ or more such terms, then $A_\gamma \sigma = 0$ for all roots $\gamma$. Hence $A\sigma = 0$. Now the product of $p$ elements from $J(x)$ can be expressed as the sum of terms each with at least $p$ factors all of the form $R_x$ or $L_x$. Hence $J(x)^p = 0$. Therefore $A$ is nilpotent.

Again let $A$ satisfy I and II and for $x \in A$, let $E_A(x) = \{y \in A;\ yJ(x)^k = 0$ for some $k = 1, 2, \ldots\}$. Then

LEMMA. *$E_A(x)$ is a subalgebra of $A$.*

The Frattini subalgebra $\phi(A)$ of an arbitrary algebra $A$ has been investigated by Towers [5]. In this direction we obtain a generalization of a theorem of D. W. Barnes.

COROLLARY 4. *Let $A$ be a finite dimensional algebra over a field. Suppose that $A$ satisfies* I, II *and is flexible. Let $N$ be an ideal of $A$ such that $N \subseteq \phi(A)$ and $A/N$ is nilpotent. Then $A$ is nilpotent.*

*Proof.* Suppose not. Then there exists $x \in A$ such that $J(x)$ is not nilpotent. For $\sigma \in J(x)$ let $A_{0\sigma}$ and $A_{1\sigma}$ be the Fitting null and one spaces of $A$ with respect to $\sigma$. Since $A$ is flexible, $J(x)$ is abelian and $A_{0\sigma}$ and $A_{1\sigma}$ are $\tau$-invariant for each $\tau \in J(x)$. Since $A/N$ is nilpotent, $A_{1\sigma} \subseteq N$ for each $\sigma$, hence $B = \bigcap A_{0\sigma}$ supplements $N$. Also $E_A(x) \subseteq B$. Now $J(x)$ restricted to $B$ is a nilalgebra, hence it is nilpotent. Hence $E_A(x) = B$. Then $A = E_A(x) + N = E_A(x) + \phi(A)$, a contradiction since $E_A(x)$ is a subalgebra of $A$.

The final result concerns the existence of Cartan subalgebras, nilpotent self-normalizing subalgebras. Clearly these can not exist in general. However, we have the following result.

COROLLARY 5. *Let $A$ be a finite dimensional algebra over a field. Suppose that $A$ is solvable, flexible and satisfies* I *and* II. *Then $A$ contains a Cartan subalgebra.*

*Proof.* Induct on the dimension of $A$. Let $M$ be minimal ideal of $A$. There exists $B \subseteq A$ such that $B \supseteq M$ and $B/M$ is a Cartan subalgebra of $A/M$. If $B \neq A$, then there exists a Cartan subalgebra $C$ of $B$. $C$ is nilpotent and we claim that $C$ is self-normalizing in $A$. Let $D = N_A(C)$, the normalizer of $C$ in $A$. Since $C + M/M$ is a Cartan subalgebra of $B/M$, $C + M = B$. If $x \in D$, then $x \in N_A(B) = B$. Hence $D \subseteq B$. Therefore $D = N_B(C) = C$ and $C$ is a Cartan subalgebra of $A$.

Suppose $B = A$. Then $A/M$ is nilpotent and we may assume that $A$ is not nilpotent. By the above corollary, $M \not\subseteq \phi(A)$. Hence $A$ contains a maximal subalgebra $T$ which complements $M$ in $A$ since $T \cap M$ is an ideal in $T + M = A$. $T$ is nilpotent and $N_A(T) \cap M$ is an ideal in $N_A(T) + M = A$. Then either $N_A(T) \cap M = 0$, $N_A(T) = T$ and $T$ is a Cartan subalgebra of $A$ or $N_A(T) \cap M = M$. Then $T$ is an ideal in $A$ and $A = A/T \cap M$ is nilpotent, a contradiction.

## References


[1] A. A. Albert, *Almost alternative algebras*, Portugals Math., **8** (1949), 23–26.
[2] D. W. Barnes, *On the cohomology of soluble Lie algebras*, Math. Z., **101** (1967), 343–349.
[3] N. Jacobson, *A note on automorphism and derivations of Lie algebras*, Proc. Amer. Math. Soc., **6** (1955), 281–283.
[4] R. D. Schafer, *An Introduction to Non-associative Algebras*, Academic Press, New York, 1966.
[5] D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc., (3) **2** (1973), 440–462.


Received August 4, 1981.

North Carolina State University
Raleigh, NC 27650