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GAUTHIER'S LOCALIZATION THEOREM ON MEROMORPHIC UNIFORM APPROXIMATION

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This paper provides a proof of the localization theorem of Gauthier, which states that a function on a closed subset which is essentially of finite genus in an open Riemann surface is uniformly approximable by global meromorphic functions if and only if it is uniformly approximable locally by local meromorphic functions. The proof relies upon previously published work of Gauthier and of the author and these two lemmas: a connected surface of infinite genus cannot be the union of a compact set and a collection of pair-wise disjoint open sets of finite genus; if a Laurent series for an isolated essential singularity is prescribed at a point of a compact Riemann surface, there can be found an an analytic function on the surface with a singularity only at the given point and with Laurent series at the point identical with the given series except possibly for the coefficients of powers greater than $-2g$, where g is the genus of the surface.

For purposes of this paper a Riemann surface will be a connected one-dimensional complex manifold without boundary, and all functions are single-valued. An open surface is one which is not compact. A subset of a surface is called *bounded* if its closure is compact, and it is said to be *essentially of finite genus* if it is contained in a open set each connected component of which is of finite genus.

In **[G1]** Gauthier states in an equivalent way the following important localization theorem.

THEOREM 1. *Let M be an open Riemann surface, E a closed subset essentially of finite genus, and f a function defined on E with values in the extended complex plane. Suppose each point of E is interior to a compact disc D in M with the property that on $D \cap E$ f is the uniform limit of functions which are meromorphic on $D \cap E$. Then f is the uniform limit on E of meromorphic functions on M.*

Well-known examples **[GH]**, **[S2]** show that Theorem 1 is not valid in full generality without the hypothesis that E be essentially of finite genus. Gauthier pointed out to me **[G2]** that implicit in the proof of Theorem 1 is an additional hypothesis:

$(*)$ $\qquad M - E$ is not bounded.

The purpose of this paper is the removal of the hypothesis (∗) from Gauthier's theorem. Let "Theorem 1(∗)" denote Theorem 1 with the addition of (∗). The proof of Theorem 1 will be based on Theorem 1(∗) and uses two lemmas: first, that a surface of infinite genus is not the union of a bounded set with a set which is essentially of finite genus, and second, that on a compact surface a prescribed isolated essential singularity can be realized, up to a meromorphic function, as the only singularity of a global function.

The proofs of the following statements will be given after the proof of Theorem 1.

LEMMA 2. *Suppose M is an open Riemann surface, $\{U_n\colon n \geq 1\}$ is a collection of pairwise-disjoint open sets, each one of finite genus, and K is compact. If $M = K \cup U_1 \cup U_2 \cup \cdots$, then M has finite genus.*

LEMMA 3. *Suppose R is a compact Riemann surface, p is a point of R, V is a neighborhood of p, and h is analytic on $V - \{p\}$. Then there is an analytic function H on $R - \{p\}$ such that $h - H$ has at worst a pole of order $2g - 1$ at p, where g is the genus of R.*

REMARK. For the proof of Theorem 1 it would be sufficient to have Lemma 3 without the bound $2g - 1$ on the order of the pole. This result can be deduced readily from the existence theorem for meromorphic sections of principal bundles, as stated in **[HC]** on page 649 and proved on the following pages. The proof below is both more elementary and easier, I believe, and provides a bound on the order of the pole.

COROLLARY 4. *If p is a point of a compact Riemann surface R, z is a local coordinate at p such that $z(p) = 0$, and a_n is a sequence of complex numbers with $|a_n|^{1/n} \to 0$ as $n \to \infty$, then there is an analytic function F on $R - \{p\}$ such that in the Laurent series for F at p in terms of z the coefficient of z^{-n} is a_n for all $n \geq 2g$, where g is the genus of R.*

LEMMA 5. *Let R be a compact Riemann surface, D an open disc in R with holomorphic coordinate z, p_n a sequence of distinct points of D converging to a point p of D, Q_n a principal part at p_n for each n, and $\delta > 0$. (That is, each Q_n is a finite linear combination of terms $(z - z(p_n))^{-k}$ for $k \geq 1$.) Then there is a meromorphic function h on $R - \{p\}$ whose only singularities are poles at $\{p_n\}$, whose principal part at p_n is Q_n for each n (that is, $h - Q_n$ has a removable singularity at p_n), and which satisfies the inequality $|h| < \delta$ on $R - D$.*

Proof of Theorem 1. In case $E = M$ the hypothesis implies that f is itself meromorphic on M and can therefore be used as the approximating function. So assume $E \neq M$.

Case I. $M - E$ is unbounded. Then Theorem 1(*) applies and the conclusion of Theorem 1 is valid.

Case II. $M - E$ is bounded. In this case Lemma 2 implies M is of finite genus. Therefore, M can be regarded as an open subset of a compact Riemann surface $\tilde{M}$ **[B]**, **[S1**, §3]. Select a point p_0 of $M - E$ and apply Theorem 1(*) to the surface $M - \{p_0\}$, the set E, and the function f (for $M - \{p_0\} - E$ is unbounded in $M - \{p_0\}$) to obtain a meromorphic function g on $M - \{p_0\}$ so that $|g - f| < \varepsilon/3$ on E. Now select an open disc D in $M - E$ which contains p_0. By Lemma 5 there is a meromorphic function h on $\tilde{M} - \{p_0\}$ so that $|h| < \varepsilon/3$ on $\tilde{M} - D$ and $h - g$ is analytic on $D - \{p_0\}$. By Lemma 3 there is an analytic function k on $\tilde{M} - \{p_0\}$ such that $k - (h - g)$ has at worst a pole at p_0. By Theorem 1 of **[KT]** or by Theorem 8 of **[S2]** (which does not depend on the present Theorem 1) there is a meromorphic function m on $\tilde{M}$ which satisfies $|m - k| < \varepsilon/3$ on $\tilde{M} - D$.

Now by construction $F = k - (h - g) - m$ is meromorphic on M, and we have

$$\begin{aligned} |F - f| &= |(k - m) + (g - f) - h| \\ &\leq |k - m| + |g - f| + |h| < 3\varepsilon/3 = \varepsilon \quad \text{on } E. \end{aligned}$$

Proof of Lemma 2. Suppose the hypotheses are valid, but M is of infinite genus. Richards **[R]**, **[S1**, §3] gives the following description for noncompact orientable connected surfaces of infinite genus. Let $\beta_0 \subseteq \beta$ be nonempty compact totally disconnected subsets of $\mathbf{R}$ and Δ_n be a sequence of open discs whose boundary circles $\partial\Delta_n$ lie in the open upper half-plane and whose cluster set (as $n \to \infty$) is exactly β_0. For $T \subseteq \mathbf{C}$ let $\tilde{T}$ be the reflection of T with respect to the real axis. Put $S = \mathbf{C} \cup \{\infty\} - \bigcup \tilde{\Delta}_n - \beta$ and define a mapping $\hat{}$ from S onto a surface $\hat{S}$ by identifying each point of each $\partial\Delta_n$ with its complex conjugate. For an appropriate pair β, β_0 the resulting $\hat{S}$ will be homeomorphic to M; for the remainder of the discussion let us identify M with $\hat{S}$.

Because K is compact in $M = \hat{S}$ it is the image of compact subset of S. Therefore, we may choose $x \in \beta_0$ and an open disc D centered at x so that $(D \cap S)\hat{}$ is disjoint from K. Since $D \cap S$ is connected and $\hat{}$ is continuous, $(D \cap S)\hat{}$ belongs entirely to a single U_j, which we may take to be U_1. Because $x \in \beta_0 \cap D$ there is a subsequence $\partial\Delta_{n(j)}$ converging to

x and lying entirely in D. Let $\gamma_j = (\partial\Delta_{n(j)})\hat{}$ be the corresponding circles in $\hat{S} = M$. Now $(D \cap S)\hat{} - \bigcup \gamma_j = (D \cap S - \bigcup \partial\Delta_{n(j)} - \bigcup \partial\tilde{\Delta}_{n(j)})\hat{}$ is connected, being the image in $\hat{S}$ of a connected subset of S. Therefore, $(D \cap S)\hat{}$ has infinite genus; so U_1, which contains $(D \cap S)\hat{}$, also has infinite genus, contrary to hypothesis.

Proof of Lemma 3. Given R, p, and h we shall produce H in a uniform manner. We shall obtain a disc D in R containing p, a local coordinate z on D for which $z(p) = 0$, an integer $N \leq 2g$, a constant A, and for each $n \geq N$ a meromorphic function f_{-n} on R having p as its only pole such that $f_{-n} - z^{-n}$ has at worst a pole of order $(N - 1)$ at p and such that $|f_{-n}| \leq A^n |z|^n$ on D. Expressing h as $\Sigma_{-\infty}^{\infty} a_n z^n$, we can choose $\Sigma_{-\infty}^{-N} a_n f_n$ for H. If $g = 0$, R is the sphere and Lemma 3 is straightforward and well known. So assume $g \geq 1$.

The Weierstrass gap theorem implies the existence of an integer $N \leq 2g$ such that for every $n \geq N$ there is a meromorphic function having a pole of order n at p and no other poles. (Since $g > 0$, N must be greater than 1, for a meromorphic function having a simple pole would define a conformal equivalence of R with the sphere.) Fix such an integer N and select for each k, $N \leq k \leq 2N - 1$, a meromorphic function F_{-k} having a pole of order k at p and no other poles. Define a local coordinate z at p by $z = \sqrt[N]{1/F_{-N}}$, which makes sense because $1/F_{-N}$ has an N-fold zero at p. Let D be a disc in R defined by $\{q\colon |z(q)| \leq r_0\}$, where r_0 is so small that z is an analytic homeomorphism on D. For simplicity let us write $D = \{|z| \leq r_0\}$.

By an induction define meromorphic functions f_{-k} as follows: for $k = N$ put $f_{-N} = F_{-N} = z^{-N}$. Define $f_{-N-1} = \alpha F_{-N-1} + \beta f_{-N}$, where α is chosen so as to make the coefficient of z^{-N-1} equal to 1 and β is chosen to make the coefficient of z^{-N} equal to 0 in the Laurent series for f_{-N-1} in powers of z. Define $f_{-N-2} = \alpha' F_{-N-2} + \beta' f_{-N-1} + \gamma' f_{-N}$, where the constants are chosen so as to make $f_{-N-2} = z^{-N-2}$ + powers of z greater than $-N$. And so forth, until f_{-2N+1} is obtained. Each f_{-k} is a linear combination of F_{-k} and the previously obtained f's.

Now $z^k f_{-k}$ is analytic on D; let K_k be the maximum of $|z^k f_{-k}|$ on D. $1 \leq K_k < \infty$ for $N \leq k \leq 2N - 1$. Put $A = (1 + \Sigma_N^{2N-1} K_k)^{1/N}$ and let r denote $|z|$. By induction we shall obtain for every $n \geq N$ a function f_{-n} with these properties:

(a) f_{-n} is meromorphic on R and the only singularity of f_{-n} is a pole of order n at p;

(b) $f_{-n} = z^{-n}$ + powers of z greater than $-N$;

(c) $|f_{-n}| \leq A^n r^{-n}$ on D.

Note that (a), (b), and (c) hold for $N \le n \le 2N - 1$, by construction of $f_{-N}, \ldots, f_{-2N+1}$, by the estimate $|f_{-n}| \le K_n r^{-n}$, and by the evident inequality $A^n \ge A^N \ge K_n$.

So for the induction let us assume that we have $f_{-N}, \ldots, f_{-(n-1)}$ satisfying (a), (b), and (c) for an integer $n \ge 2N$. To obtain f_{-n} write the Laurent series for f_{-n+N} as $z^{-n+N} + \sum_{j>-N} \alpha_j z^j$, by (b), and define the auxiliary function

$$\tilde{f}_{-n} = f_{-n+N} f_{-N} = \left(z^{-n+N} + \sum_{j>-N} \alpha_j z^j \right) z^{-N}$$

$$= z^{-n} + \sum_{-N+1}^{0} \alpha_j z^{j-N} + \sum_{1}^{\infty} \alpha_j z^{j-N}.$$

Put $f_{-n} = \tilde{f}_{-n} - \sum_{-N+1}^{0} \alpha_j f_{j-N}$. From (a) and (b) for $f_{-N}, \ldots, f_{-2N+1}$, from the explicit Laurent series for $\tilde{f}_{-n}$, and from the equation $\tilde{f}_{-n} = f_{-n+N} f_{-N}$ we have (a) and (b) for f_{-n}. To prove (c) we make some estimates. Cauchy's estimate and (c) for f_{-n+N} yield

$$|\alpha_j| = \left| \int_{|z|=r} f_{-n+N} z^{-j-1} \, dz \right| (2\pi)^{-1} \le A^{n-N} r^{-n+N-j}.$$

Now for $-N + 1 \le j \le 0$ we have $N \le -j + N \le 2N - 1$ and therefore $|f_{j-N}| \le K_{N-j} r^{j-N}$ on D, by the original estimate for f_{j-N}. Therefore

$$|\alpha_j f_{j-N}| \le A^{n-N} r^{-n+N-j} K_{N-j} r^{j-N} = A^{n-N} r^{-n} K_{N-j}.$$

Combining these inequalities for the various j with the estimate $|\tilde{f}_{-n}| = |f_{-n+N}| |f_{-N}| \le A^{n-N} r^{-n+N} r^{-N} = A^{n-N} r^{-n}$ we obtain

$$|f_{-n}| \le A^n r^{-n} A^{-N} \left(1 + \sum_{-N+1}^{0} K_{N-j} \right)$$

$$= A^n r^{-n} A^{-N} \left(1 + \sum_{0}^{N-1} K_{N+j} \right) = A^n r^{-n},$$

and the inductive step is complete.

Now we write the Laurent series for h at p as $h = \sum_{-\infty}^{\infty} a_n z^n$, and we have $|a_n|^{1/|n|} \to 0$ as $n \to -\infty$, because h is analytic in a deleted neighborhood of p. Therefore $H = \sum_{-\infty}^{-N} a_n f_{-n}$ converges uniformly by (c) on every small circle $\{|z| = r\}$, hence uniformly on every set $R - \{|z| < r\}$ by the maximum principle, for the f's are singular only at p. Thus, H is analytic on $R - \{p\}$, and if we integrate the series for H term by term against $z^{-k-1} dz$ around any small circle $\{|z| = r\}$, we see that the Laurent series for H is identical with that for h in the exponent range $-\infty$ to $-N$. Hence, $h - H$ has at worst a pole at p of order $N - 1 \le 2g - 1$.

Corollary 4 follows immediately from Lemma 3.

Proof of Lemma 5. This is actually a special case of Lemma 13 of **[S2]**, in which M is $R - \{p\}$, $P = \varnothing$, and $P' = \{p_n\colon n \geq 1\}$. However, the following proof is more accessible. Choose a sequence of open discs D_n in D which shrink to p, label $D = D_0$, put $P_n = \{p_k \mid p_k \in D_{n-1} - D_n\}$ for $n \geq 1$, and apply Corollary 11 of **[S2]** to obtain for each n a meromorphic function m_n on R whose pole set is $P_n \cup \{p\}$, whose principal part at $p_j \in P_n$ is Q_j for each $p_j \in P_n$, and which satisfies the inequality $|m_n| < \delta 2^{-n}$ on $R - D_{n-1}$. Then Σm_n converges normally (i.e., the tail of the series converges uniformly on compact sets) on $R - \{p\}$ and defines a meromorphic function on $R - \{p\}$ whose pole set is $\{p_k\colon k \geq 1\}$, whose principal part at p_k is Q_k for all k, and whose modulus is less than δ on $R - D_0 = R - D$.

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