PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

EXTENSIONS OF d/dx THAT GENERATE UNIFORMLY BOUNDED SEMIGROUPS

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$A \equiv -d/dx$ on $BC_0(R^+)$, $\mathfrak{D}(A) \equiv \{f \in BC_0(R^+) \mid f' \in BC_0(R^+)\}$, is an example of a maximal accretive operator that does not generate a contraction semigroup. It does, however, have extensions that generate uniformly bounded semigroups. A large class of such extensions are presented. The same is done with d/dx and $-d/dx$ on $C_0[0, 1]$.

Introduction. The theory of accretive operators generalizes the theory of symmetric operators on a Hilbert space. An operator, T, on a Hilbert space is accretive if $\mathrm{Re}\langle Tx, x\rangle \geq 0$ for all x in the domain of T. A maximal accretive operator is one that has no proper accretive extensions. An m-accretive operator is one that generates a strongly-continuous contraction semigroup. The "m" suggests "maximal", because in a Hilbert space, every maximal accretive operator is m-accretive. (See Theorem 0.) This need not be true in a general Banach space. Lumer and Phillips, in 1961, ([**1**], p. 688) first gave an example of a maximal accretive operator that is not m-accretive. The space is $C_0[0, 1]$, and the accretive operator is d/dx, with domain $\{f \mid f' \text{ exists}, f' \in C_0[0, 1]\}$. Lumer and Phillips show that any proper extension of this operator fails to be accretive. To verify that this operator is not m-accretive, one uses the well-known ([**2**], p. 240) fact, that, if T is a closed accretive operator, then $(1 + T)$ is one-to-one, and T is m-accretive if and only if the range of $(1 + T)$ is the entire space. The range of $(1 + d/dx)$, with the domain given above, can be explicitly calculated to be $\{g \in C_0[0, 1] \mid \int_0^1 e^r g(r)\,dr = 0\}$, which is not dense in $C_0[0, 1]$. Intuitively, this operator fails to be m-accretive because d/dx should generate the translation semigroup $\{T_s\}_{s\geq 0}$, defined by $(T_s f)(t) = f(t - s)$, but this semigroup does not take $C_0[0, 1]$ into itself.

However, it is interesting that the Lumer-Phillips operator d/dx has extensions that generate uniformly bounded semigroups. This is the same as saying that there exist equivalent norms on $C_0[0, 1]$ with respect to which d/dx has m-accretive extensions. One of the main aims of this paper is to present such extensions.

The operator $-d/dx$, on $BC_0(R^+)$, with domain $\{f \mid f' \text{ exists}, f' \in BC_0(R^+)\}$, is a related example of a maximal accretive operator that is not m-accretive. Obvious modifications of the Lumer-Phillips proof ([**1**], p. 688) show that this operator has no proper accretive extensions, while the range of $(1 - d/dx)$ can be explicitly calculated to be $\{f \in BC_0(R^+) \mid \int_0^\infty e^{-t}f(t)\,dt = 0\}$, so that the operator is not m-accretive. Again, the intuition is that $-d/dx$ should generate left-translation, however $BC_0(R^+)$ is not invariant under left-translation. Because this operator is easier to deal with, its extensions will be discussed first; we will then go on to treat the Lumer-Phillips operator.

The paper concludes with some open questions about a general theory of accretive-equivalent operators, that is, operators which become accretive under an equivalent norm.

We present the following facts for easy future reference. Most of the items need not be read until they are referred to.

Definitions and preliminaries.

1. If T is a linear operator, $\mathfrak{D}(T) \equiv$ domain of T.

2. A linear operator, A, on a Banach space, X, is *accretive* if, for all $x \in \mathfrak{D}(A)$, there exists $\Psi_x \in X^*$ such that $\|\Psi_x\| = 1$, $\Psi_x(x) = \|x\|$, and $\operatorname{Re}\Psi_x(Ax) \geq 0$.

This is equivalent to: $\|(1 + tA)x\| \geq \|x\|$ for all $t \geq 0$.

3. A is *m-accretive* if it generates a one-parameter contraction semigroup, $\{e^{-tA}\}_{t\geq 0}$, that is, for all $x \in \mathfrak{D}(A)$, $-Ax = \lim_{t\to 0}(e^{-tA}x - x)/t$.

A closed operator, A, is m-accretive if and only if A is accretive, densely defined, and the range of $(1 + A)$ equals X.

The "m" is suppposed to suggest "maximal". (See Theorem 0.)

4. A is *m.e.-accretive* if it generates a uniformly-bounded semigroup, that is, there exists $M < \infty$ such that $\|e^{-tA}\| < M$, for all $t \geq 0$.

The "e" stands for "equivalent", because A is m.e.-accretive if and only if there exists an equivalent norm with respect to which A is m-accretive. (Let $|||x||| \equiv \sup_{t\geq 0}\|e^{-tA}x\|$.)

5. A is *accretive-equivalent* if there exists $\delta > 0$ such that for all $n \in N$, $t \geq 0$, $x \in \mathfrak{D}(A)$,

$$\|(1 + tA)^n x\| \geq \delta\|x\|.$$

A is accretive-equivalent if and only if there exists an equivalent norm with respect to which A is accretive.

6. $(\mathfrak{L}g)(s) \equiv \int_0^\infty e^{-st}g(t)\,dt$, the Laplace transform of g.

7. $(f * g)(t) \equiv \int_0^t f(t - r)g(r)\,dr$.

8. If $x, t \in R$, $g_x(t) \equiv g(x + t)$.

9. $BC(R^+) \equiv \{\text{bounded continuous } f\colon [0, \infty) \to \mathbf{C}\}$.

10. $BC_0(R^+) \equiv \{f \in BC(R^+) \mid f(0) = 0\}$.

11. $C_0[0, 1] \equiv \{\text{continuous } f\colon [0, 1] \to C \mid f(0) = f(1) = 0\}$.

12. δ_a is the measure defined by $\delta_a(f) \equiv f(a)$, for all $f \in BC(R^+)$. Note that $\mathfrak{L}(\delta_a)(s) = e^{-as}$, $(g * \delta_a)(t) = g(t - a)$.

13. $\delta \equiv \delta_0$, the Dirac delta function.

The following two Banach algebras will be used as technical tools during some proofs.

14. Let $\mathcal{Q} \equiv$ the Banach algebra spanned by $L^1(R^+)$ and δ, with convolution as multiplication.

The Gelfand transform for $\mathcal{Q}$ is the Laplace transform, that is, if Λ is in the maximal ideal space, then either there exists s, with $\operatorname{Re} s \geq 0$, such that $\Lambda(f + \alpha\delta) = (\mathcal{L}f)(s) + \alpha$, or $\Lambda(f + \alpha\delta) = \alpha$ (corresponds to $s = \infty$).

15. Let $\mathcal{B} \equiv$ the Banach algebra spanned by $\mathcal{Q}$ and δ_1.

$$\mathcal{B} = \left\{ f + \sum_{k=0}^{\infty} \alpha_k \delta_k \,|\, f \in L^1(R^+), (\alpha_k)_{k=0}^{\infty} \in l^1(N) \right\}.$$

If Λ is in the maximal ideal space of $\mathcal{B}$, then either

(i) there exists s, with $\operatorname{Re} s \geq 0$ such that

$$\Lambda\left(f + \sum_{k=0}^{\infty} \alpha_k \delta_k \right) = \mathcal{L}\left(f + \sum_{k=0}^{\infty} \alpha_k \delta_k \right)(s) = (\mathcal{L}f)(s) + \sum_{k=0}^{\infty} \alpha_k e^{-sk},$$

or

(ii) there exists s with $\operatorname{Re} s \geq 0$, such that $\Lambda = \Gamma_s$, where

$$\Gamma_s\left(f + \sum_{k=0}^{\infty} \alpha_k \delta_k \right) = \sum_{k=0}^{\infty} \alpha_k e^{-ks}.$$

To motivate the construction of the extensions (and open question #3 near the end of the paper), here is a proof of the following well-known theorem.

THEOREM 0. *If T is an acccretive operator on a Hilbert space, H, then it has an m-accretive extension.*

Proof. Let $\bar{T} \equiv$ the closure of T. $\bar{T}$ is also accretive ((**2**), p. 240). Let $Z \equiv$ the orthogonal complement of the range of $(1 + \bar{T})$, and let $\mathcal{D}(S) \equiv \mathcal{D}(\bar{T}) \oplus Z$, $Sx \equiv \bar{T}x$, for all $x \in \mathcal{D}(\bar{T})$, $Sz \equiv z$, for all $z \in Z$.

First, note that S is accretive. Indeed, for all $x \in \mathcal{D}(\bar{T})$, and $z \in Z$, we have

$$\begin{aligned} \langle x + z, S(x + z) \rangle &= \langle x, \bar{T}x \rangle + \langle z, \bar{T}x \rangle + \langle x, z \rangle + \|z\|^2 \\ &= \langle x, \bar{T}x \rangle + \|z\|^2 - \langle z, x \rangle + \langle x, z \rangle \\ &= \langle x, \bar{T}x \rangle + \|z\|^2 + 2i \operatorname{Im}\langle x, z \rangle, \end{aligned}$$

so that

$$\mathrm{Re}\langle x + z, S(x + z)\rangle = \mathrm{Re}\langle x, \bar{T}x\rangle + \|z\|^2 \geq 0;$$

that is, S is accretive.

Since the range of $(1 + S)$ equals H, S is m-accretive (see Definition 13). □

Let $A \equiv -d/dx$ on $BC_0(R^+)$, $\mathfrak{D}(A) \equiv \{f \in BC_0(R^+) \mid f' \in BC_0(R^+)\}$.

Suppose B is an m.e.-accretive extension of A. Since the range of $(1 + A)$ equals $\{f \in BC_0(R^+) \mid (\mathfrak{L}f)(1) = 0\}$, a set of co-dimension one in $BC_0(R^+)$ and $(1 + B)$ is one-to-one, $\mathfrak{D}(B)$ must equal $\mathfrak{D}(A) + \mathrm{span}\{g\}$, for some $g \in BC_0(R^+)$. I will assume g is differentiable, with $g'(0) = 1$. Then

$$\begin{aligned} Bf &= B(f - f'(0)g) + B(f'(0)g) \\ &= -d/dx(f - f'(0)g) + f'(0)Bg = f'(0)(g' + Bg) - f'. \end{aligned}$$

Thus, extensions of $-d/dx$ with the following form are considered:

DEFINITION. $(B_\phi f)(x) \equiv f'(0)\phi(x) - f'(x)$, where $\phi \in BC(R^+)$, $\phi(0) = 1$. $\mathfrak{D}(B_\phi) = \{f \in BC_0(R^+) \mid f' \in BC(R^+)\}$.

For which ϕ is B_ϕ m.e.-accretive? Here is a sufficient condition.

THEOREM 1. *Suppose* (1) *ϕ' exists a.e., with $(\phi + \phi') \in L^1(R^+)$, and* (2) *for all s such that* $\mathrm{Re}\, s \geq 0$, *$(\mathfrak{L}\phi)(s)$ exists, and is nonzero. Then B_ϕ is m.e.-accretive. For all $f \in \mathfrak{D}(B_\phi)$, there exists $\Lambda(f)$ such that $\phi * \Lambda(f) = f$; the semigroup generated by B_ϕ is given by the following formula*:

$$F_\phi(t)f(x) = f(x + t) - (\phi_x * \Lambda(f))(t).$$

Proof. Because $\phi(0) = 1$, we have

$$\mathfrak{L}(\delta + \phi + \phi')(s) = (1 + s)\mathfrak{L}\phi(s),$$

so $(\delta + \phi + \phi')$ is an element of $\mathfrak{A}$ (Definition 14) whose Gelfand transform never vanishes. It follows that it is invertible in $\mathfrak{A}$, that is, there exists $k \in \mathfrak{A}$ such that

$$(\delta + \phi + \phi') * k = \delta.$$

For all $f \in BC_0(R^+)$, let

$$F_\phi(t)f(x) \equiv f(x + t) - \phi(x)(k * f)(t) - ((\phi_x + \phi_x') * k * f)(t).$$

For all $t \geq 0$,

$$\|F_\phi(t)f\|_\infty \leq \|f\|_\infty + \|\phi\|_\infty \|k * f\|_\infty + \|\phi + \phi'\|_1 \|k * f\|_\infty$$
$$\leq (1 + \|\phi\|_\infty \|k\|_1 + \|\phi + \phi'\|_1 \|k\|_1)\|f\|_\infty.$$

So $F_\phi(t)$ is a uniformly bounded family of operators.

For all $f \in \mathfrak{D}(B_\phi)$, let $\Lambda(f) \equiv k * (f + f')$.

Since

$$\phi(x)f(t) + ((\phi_x + \phi_x') * f)(t) = (\phi_x * (f + f'))(t),$$

we have the desired form

$$F_\phi(t)f(x) = f(x+t) - (\phi_x * \Lambda(f))(t);$$
$$(\phi * \Lambda(f))(t) = (k * (\phi(0)f + (\phi + \phi') * f))(t)$$
$$= (k * (\delta + \phi + \phi') * f)(t) = f(t),$$

by the definition of k. This shows that $F_\phi(t)f(0) = 0$, so that $F_\phi(t)f \in BC_0(R^+)$. Note also that $F_\phi(0)f(x) = f(x)$, for all $x \geq 0, f \in BC_0(R^+)$.

To show that $F_\phi(t)$ is the semigroup generated by B_ϕ, one must show that $-B_\phi F_\phi(t)f = d/dtF_\phi(t)f$, for all $f \in \mathfrak{D}(B_\phi)$. The computation follows:

$$\begin{aligned}
&(B_\phi F_\phi(t)f)(x) + \frac{d}{dt}(F_\phi(t)f)(x) \\
&= (-f'(x+t) + (\phi_x' * \Lambda(f))(t)) + \phi(x)(f'(t) - (\phi' * \Lambda(f))(t)) \\
&\quad + (f'(x+t) - (\phi_x' * \Lambda(f))(t) - \phi(x)\Lambda(f)(t)) \\
&= \phi(x)(f'(t) - (\phi' * \Lambda(f))(t)) - \phi(x)\Lambda f(t) \\
&= \phi(x)(f'(t) - (\Lambda(f)(t) + \phi' * \Lambda(f))(t)) \\
&= 0;
\end{aligned}$$

this can be seen by differentiating both sides of $\phi * \Lambda(f) = f$, to get

$$\phi(0)\Lambda(f)(t) + (\phi' * \Lambda(f))(t) = f'(t). \quad \square$$

REMARK 1. When $\phi(x) = e^{\alpha x}$, with $\operatorname{Re} \alpha \leq 0$, then $F_\phi(t)f(x) = f(x+t) - e^{\alpha x}f(t)$. This class of semigroups is due to Chernoff (unpublished), and was the starting point for this paper.

REMARK 2. Condition (1) of Theorem 1 is not a necessary condition on ϕ; take $\phi(x) = \cos x + \sin x$, then

$$(F_\phi(t)f)(x) = f(x+t) - (\cos x + \sin x)f(t) - 2\sin x(e^{-t} * f(t)),$$

which is a uniformly bounded semigroup.

However, condition (2) is almost necessary.

PROPOSITION 2. *Suppose B_ϕ is* m.e-*accretive. Then, for all s such that* $\operatorname{Re} s > 0$, $(\mathfrak{L}\phi)(s) \neq 0$.

Proof. Fix $f \in \mathfrak{D}(B_\phi)$, and write $u(t, x) \equiv F_\phi(t)f(x)$, $v(s, x) \equiv (s + B_\phi)^{-1}f(x)$, where $\operatorname{Re} s > 0$.

Then $v(s, x) = \int_0^\infty e^{-st}u(t, x)\,dt$, so

$$\begin{aligned} \partial v/\partial x(s, x) - \phi(x)\partial v/\partial x(s,0) &= \int_0^\infty e^{-st}\big(-B_\phi u(t, x)\big)\,dt \\ &= \int_0^\infty e^{-st}\partial u/\partial t(t, x)\,dt = -u(0, x) + s\int_0^\infty e^{-st}u(t, x)\,dt \\ &= -f(x) + sv(s, x), \end{aligned}$$

which yields the differential equation

$$\partial v/\partial x(s, x) - sv(s, x) = -f(x) + \phi(x)w(s),\ v(s,0) = 0, \quad \text{where } w(s) \equiv \partial v/\partial x(s,0).$$

This has the solution

$$v(s, x) = -e^{sx}\int_0^x e^{-sr}(-f(r) + \phi(r)w(s))\,dr.$$

Since $v(s, x)$ is bounded as $x \to \infty$, one must have $\mathfrak{L}(-f + \phi w(s))(s) = 0$, or

$$\mathfrak{L}(f)(s) = w(s)\mathfrak{L}(\phi)(s)$$

This holds for all $f \in \mathfrak{D}(B_\phi)$; by choosing f such that $\mathfrak{L}f(s) \neq 0$, one concludes that $\mathfrak{L}\phi(s)$ cannot be zero. □

If B is the most general one-dimensional extension of A, with $\mathfrak{D}(B) = \mathfrak{D}(A) \oplus \operatorname{span}\{g\}$, one can show directly that

$$\begin{aligned} &(s + B)^{-1}f(x) \\ &\quad = \mathfrak{L}(f_x)(s) - \mathfrak{L}f(s)\left(\frac{-g(x) + s\mathfrak{L}(g_x)(s) - \mathfrak{L}((Bg)_x)(s)}{s\mathfrak{L}g(s) - \mathfrak{L}(Bg)(s)}\right) \end{aligned}$$

provided $s\mathfrak{L}g(s) - \mathfrak{L}(Bg)(s) \neq 0$. Using the resolvents to show that B is m.e.-accretive requires that one show that there exists $M < \infty$ such that

$$\|(s + B)^{-n}\| < M/s^n, \quad \text{for all } s > 0,\ n \in N.$$

(Hille-Yosida, Phillips Theorem, (**2**), p. 247). But this looks difficult.

A result similar to Theorem 1 holds for $-d/dx$ on $C_0[0,1]$; the major difference here is that $\mathfrak{D}(B_\phi)$ may be different, for different choices of ϕ.

THEOREM 3. *Suppose ϕ' exists, and is in $L^1[0,1]$, $\mathfrak{L}\phi(s) \neq 0$, for all s such that* $\operatorname{Re} s \geq 0$, $\phi(x) = 0$, *for all* $x \notin [0,1]$, $\phi(0) = 1$, *and* $|\phi(1)| < 1$. *Then* $(B_\phi f)(x) \equiv \phi(x)f'(0) - f'(x)$, *with*

$$\mathfrak{D}(B_\phi) \equiv \{f \in C_0[0,1] \mid f' \in C[0,1], f'(1) = \phi(1)f'(0)\},$$

is m.e.-accretive. The semigroup generated by B_ϕ has the same form as in Theorem 1.

Proof. Let $\alpha \equiv \phi(1)$. The result follows from the calculations below:
(i)

$$\mathfrak{L}(\phi')(s) = \int_0^1 e^{-st}\phi'(t)\,dt = \alpha e^{-s} - 1 + s\mathfrak{L}\phi(s)$$
$$= \mathfrak{L}(\alpha\delta_1 - \delta)(s) + s\mathfrak{L}\phi(s).$$

(ii) If $x \leq 1, f \in \mathfrak{D}(B_\phi)$, then

$$(\phi_x * f')(t) = \int_{x+t-1}^{t} \phi(x+t-r)f'(r)\,dr$$
$$= \phi(x)f(t) - \phi(1)f(x+t-1) + \int_{x+t-1}^{t} \phi'(x+t-r)\,dr$$
$$= \phi(x)f(t) - \alpha f(x+t-1) - (\phi_x' * f)(t).$$

(iii) Suppose $w(t,x) \equiv (\phi_x * h)(t)$, with h continuous. Then

$$\frac{\partial w}{\partial t} = \frac{d}{dt}\int_{x+t-1}^{t} \phi(x+t-r)h(r)\,dr$$
$$= \phi(x)h(t) - \alpha h(x+t-1) + (\phi_x' * h)(t).$$

(iv) With the same w as in (iii),

$$\frac{\partial w}{\partial x} = -\alpha h(x+t-1) + (\phi_x' * h)(t).$$

By (i), $\mathfrak{L}(\phi + \phi' + \delta - \alpha\delta_1)(s) = (1+s)\mathfrak{L}\phi(s)$. Also (see Definition 15), $\Gamma_s(\phi + \phi' + \delta - \alpha\delta_1) = 1 - \alpha e^{-s} \neq 0$, for all s such that $\operatorname{Re} s \geq 0$, because $|\alpha| < 1$.

Thus there exists $k \in \mathfrak{B}$ (Definition 15) such that

$$(\phi + \phi' + \delta - \alpha\delta_1) * k = \delta.$$

For all $f \in \mathfrak{D}(B_\phi)$, define

$$\big(F_\phi(t)f\big)(x) \equiv f(x+t) - (\phi_x * (f+f') * k)(t).$$

Since $\phi(1 + t) = 0$, for all $t > 0$, $(F_\phi(t)f)(1) = 0$, for all $t \geq 0$. To see that $(F_\phi(t)f)(0) = 0$, note that by (ii)

$$\begin{aligned}(\phi * (f + f') * k)(t) &= (f(t) - \alpha f(t-1) + (\phi + \phi') * f(t)) * k(t) \\ &= (f * (\delta - \alpha\delta_1 + \phi + \phi') * k)(t) = f(t),\end{aligned}$$

by the definition of k.

So $(F_\phi(t)f)(0) = 0 = (F_\phi(t)f)(1)$, that is,

$$F_\phi(t)f \in C_0[0,1], \quad \text{for all } f \in \mathfrak{D}(B_\phi).$$

Also by (ii),

$$\begin{aligned}(F_\phi(t)f)(x) &= f(x+t) \\ &\quad + (\phi(x)f(t) - \alpha f(x+t-1) + (\phi_x + \phi_x') * f(t)) * k(t),\end{aligned}$$

so

$$\begin{aligned}&\|F_\phi(t)f\|_\infty \\ &\quad \leq \|f\|_\infty + (\|\phi\|_\infty \|f\|_\infty + |\alpha|\,\|f\|_\infty + \|\phi + \phi'\|_1 \|f\|_\infty)\|k\|_1,\end{aligned}$$

so that $F_\phi(t)$ extends to a uniformly bounded family of operators on $C_0[0,1]$.

To show that $-B_\phi F_\phi(t)f = dF_\phi(t)f/dt$, for all $f \in \mathfrak{D}(B_\phi)$, let $h \equiv (f + f') * k$, so that

$$F_\phi(t)f(x) = f(x+t) - (\phi_x * h)(t), \qquad \phi * h = f.$$

By (iii),

$$\frac{d}{dt}F_\phi(t)f(x) = f'(x+t) - (\phi(x)h(t) - \alpha h(x+t-1) + (\phi_x' * h)(t))$$

By (iv),

$$\begin{aligned}B_\phi F_\phi(t)f(x) &= \phi(x)(f'(t) - (\phi' * h)(t) + \alpha h(t-1)) \\ &\quad - (f'(x+t) - (\phi_x' * h)(t) + \alpha h(x+t-1)).\end{aligned}$$

Thus,

$$\begin{aligned}&B_\phi F_\phi(t)f(x) + \frac{d}{dt}F_\phi(t)f(x) \\ &\quad = \phi(x)(f'(t) + \alpha h(t-1) - h(t) - (\phi' * h)(t)) = 0,\end{aligned}$$

by (iii), since $f(t) = (\phi * h)(T)$. □

A slight modification of the extensions of $-d/dx$ gives m.e.-accretive extensions of d/dx on $C_0[0, 1]$, the original Lumer-Phillips example.

Note that if $\phi(1) \neq 0$, then for all $f \in \mathfrak{D}(B_\phi)$,

$$(B_\phi f)(x) = \frac{\phi(x)}{\phi(1)} f'(1) - f'(x),$$

since

$$f'(1) = \phi(1) f'(0).$$

Because $e^{t(d/dx)}$ is left-translation, when considering extensions of $-d/dx$ on $C_0[0, 1]$, it seems more natural to modify the behavior of the operator at $x = 0$. When considering extensions of d/dx, where right-translation is being perturbed, one prefers the following form:

DEFINITION. $(C_\psi f)(x) \equiv f'(x) - \psi(x) f'(1)$, where $\psi(1) = 1$, $\psi(x) = 0$, for all $x \notin [0, 1]$, $\psi \in C[0, 1]$. $\mathfrak{D}(C_\psi) \equiv \{f \in C_0[0, 1] \mid f' \in C[0, 1], f'(0) = \psi(0) f'(1)\}$.

COROLLARY 4. *Suppose* Ψ' *exists and is in* $L^1[0, 1]$, $\int_0^\infty e^{-st}\psi(1-t)\,dt \neq 0$, *for all* s *such that* $\operatorname{Re} s \geq 0$, *and* $|\psi(0)| < 1$. *Then* C_ψ *is m.e.-accretive on* $C_0[0, 1]$. *The semigroup generated by* C_ψ *has the form*

$$G_\psi(t) f(x) = f(x - t) - \int_0^t \psi(x + r - t) h(r)\, dr,$$

where

$$f(1 - t) = \int_0^t \psi(1 + r - t) h(r)\, dr.$$

Proof. Define the isometry $U: C_0[0, 1] \to C_0[0, 1]$ by: $(Uf)(t) \equiv f(1 - t)$. Then $d/dx = U^{-1}(-d/dx)U$, $C_\psi = U^{-1} B_{U\psi} U$. Since $U\psi$ satisfies the hypotheses of Theorem 3, there exists $F_{U\psi}(t)$, a uniformly bounded semigroup generated by $B_{U\psi}$. Let $G_\psi(t) \equiv U^{-1} F_{U\psi}(t) U$. $G_\psi(t)$ is a uniformly bounded semigroup generated by C_ψ.

To see that G_ψ has the desired form, let $\phi \equiv U\psi$, and note that

$$F_\phi(t) g(y) = g(y + t) - (\phi_y * h)(t),$$

where

$$(\phi * h)(t) = g(t).$$

Thus,

$$G_\psi(t)f(x) \equiv \left(U^{-1}F_\phi(t)Uf\right)(x) = \left(F_\phi(t)Uf\right)(1-x)$$
$$= Uf(1-x+t) - \int_0^t \phi(1-x+t-r)h(r)\,dr$$
$$= f(x-t) - \int_0^t \psi(x+r-t)h(r)\,dr;$$
$$f(1-t) = Uf(t) = \int_0^t \phi(t-r)h(r)\,dr = \int_0^t \psi(1+r-t)h(r)\,dr. \quad \square$$

The calculations in the proof of Theorem 3 can be carried out explicitly when $\phi(x) \equiv e^{-x}$. Then $(\phi + \phi' + \delta - \alpha\delta_1) = \delta - e^{-1}\delta_1$; we want $k \in \mathfrak{B}$ such that

$$\left(\delta - e^{-1}\delta_1\right) \dot{*}\, k = \delta.$$

This means that

$$(\mathfrak{L}k)(s) = \frac{1}{1-e^{-(1+s)}} = \sum_{j=0}^{\infty} (e^{-(1+s)})^j$$
$$= \mathfrak{L}\left(\sum_{j=0}^{\infty} e^{-j}\delta_j\right)(s), \quad \text{when } \operatorname{Re} s \geq 0.$$

So $k = \sum_{j=0}^{\infty} e^{-j}\delta_j$.

$$\left(F_\phi(t)f\right)(x) = f(x+t)$$
$$+ \left(\phi(x)f(t) - \alpha f(x+t-1) + (\phi_x + \phi_x') * f(t)\right) * k(t)$$
$$= f(x+t) + \left(e^{-x}f(t) - e^{-1}f(x+t-1)\right) * \sum_{j=0}^{\infty} e^{-j}\delta_j$$
$$= \sum_{j=0}^{\infty} e^{-j}\left(f(x+t-j) - e^{-x}f(t-j)\right).$$

Thus Chernoff's original example has to be modified to get a uniformly bounded semigroup on $C_0[0,1]$: if $\phi(x) \equiv e^{\alpha x}$, then B_ϕ, on $C_0[0,1]$, generates the semigroup defined by

$$F_\phi(t)f(x) = \sum_{k=0}^{\infty} e^{\alpha k}\left(f(x+t-k) - e^{\alpha x}f(t-k)\right),$$

where f is zero outside $(0,1)$; note that, for a fixed x and t, all but two terms in the above sum are zero. $F_\phi(t)$ is uniformly bounded if and only if $\operatorname{Re}\alpha \leq 0$.

If $\psi(x) \equiv e^{-\lambda}e^{\lambda x}$, then C_ψ generates the semigroup defined by

$$G_\psi(t)f(x) = \sum_{k=0}^{\infty} e^{-\lambda k}\big(f(x+k-t) - e^{-\lambda}e^{\lambda x}f(1+k-t)\big).$$

(This can be obtained from the formula for F_ϕ, using the definition of G_ψ given in the proof of Corollary 4.) $G_\psi(t)$ is uniformly bounded if and only if $\operatorname{Re}\lambda \geq 0$.

Open Questions. A general theory of accretive-equivalent operators (definition 5) may be more desirable than the usual restriction to a particular norm. In all known cases, a maximal accretive, but not *m*-accretive, operator appears to occur merely because of an unlucky choice of the norm, generating the topology of the space. The disadvantage is that there seems to be no analogue of the "$\operatorname{Re}\Psi_x(\Lambda x) \geq 0$" definition of accretive.

I would like to raise the following questions:

1. Does every accretive-equivalent operator have an m.e.-accretive extension (on the same space)?

It is unknown whether every accretive operator has an *m*-accretive extension (possibly on a larger space), so a sub-question of (1) is:

2. If A has an *m*-accretive extension, possibly on a larger space, does it have an m.e.-accretive extension on the original space?

Another related question is:

3. If A is accretive, and the range of $(1 + A)$ is complemented, does A have an m.e.-accretive extension?

A positive answer to (2) would be helpful in getting an example of an accretive operator which fails to have an *m*-accretive extension on a larger space, since it would then be sufficient to find an accretive operator with no m.e.-accretive extensions on the original space.

This work is a part of the author's Ph.D thesis, at the University of California, Berkeley.

References

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Received September 24, 1981.

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