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A GENERALIZATION OF THE GLEASON-KAHANE-ZELAZKO THEOREM

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In this paper, we consider two classes of commutative Banach algebras, which include $C^n(T)$, $\mathrm{Lip}_\alpha(T)$, $BV(T)$, $L^1 \cap L^p(G)$, $A^p(G)$, $L^1 \cap C_0(G)$, l^p, c_0, and $C_0(S)$. We characterize ideals of finite codimension in these two classes of algebras and thereby partially answer a question suggested by C. R. Warner and R. Whitley.

In [5] and [9], A. M. Gleason, J. P. Kahane and W. Zelazko gave independently the following characterization of maximal ideals: Let A be a commutative Banach algebra with unit element. Then a linear subspace M of codimension 1 in A is a maximal ideal in A if and only if it consists of noninvertible elements, or equivalently, each element of M belongs to some maximal ideal. This interesting result as first proved depended on the Hadamard Factorization Theorem.

This characterization of maximal ideals was extended in [15] and [16] to algebras without identity. In [16], C. R. Warner and R. Whitley also gave a characterization of ideals of finite codimension in $L^1(R)$ and $C[0,1]$. They showed: Let A be any one of $L^1(R)$ and $C(S)$, where S is a compact subset of R. If M is a closed subspace of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal. In fact, M is the intersection of n regular maximal ideals. Also in [16], C. R. Warner and R. Whitley suggested the following question: For what locally compact abelian group G does $L^1(G)$ have the property of $L^1(R)$ described above?

In this paper, we partially answer this question and generalize the work of C. R. Warner and R. Whitley. In this paper, two methods are introduced; One uses the Baire category theorem and the other generalizes the ideas of Theorems 2 and 4 in [16].

THEOREM 1. *Let A be a commutative Banach algebra with a countable maximal ideal space $\mathfrak{M}$. If M is a closed subspace of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal, which is the intersection of n regular maximal ideals.*

Proof. From the hypothesis, we know that $M \subset \bigcup I_{s_1 s_2 \cdots s_n}$, where $I_{s_1 s_2 \cdots s_n}$ denotes the space $\{x \in A : \hat{x}$ vanishes at $s_1, s_2, \ldots, s_n\}$ and the union is taken over all sets of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. Since $\mathfrak{M}$ is countable, the union is a countable union. By the Baire category theorem, $M \subset I_{s_1 s_2 \cdots s_n}$ for some set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. If not, for any set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$, we have $M \cap I_{s_1 s_2 \cdots s_n} \subsetneqq M$. By the open mapping theorem, we find that $M \cap I_{s_1 s_2 \cdots s_n}$ is of first category in M and so the union $\bigcup (M \cap I_{s_1 s_2 \cdots s_n})$ is of first category in M. This implies that M is of first category in itself and contradicts the fact that M is a Banach space. Therefore $M \subset I_{s_1 s_2 \cdots s_n}$ for some set of distinct elements $s_1, s_2, \ldots, s_n$ in $\mathfrak{M}$. Since M and $I_{s_1 s_2 \cdots s_n}$ are of codimension n in A, $M = I_{s_1 s_2 \cdots s_n}$. We have completed the proof.

EXAMPLE 2. Any of the following spaces has the property described in Theorem 1: $C^n(T)$; $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \leq 1$; $BV(T)$; $L^p(G)$, $1 \leq p \leq \infty$, or $A^p(G)$ or $C(G)$, or any normed ideal in $L^1(G)$, where G is a metrizable compact abelian group; l^p, $1 \leq p < \infty$, and c_0 (cf. [**1, 2, 4, 7, 8, 10, 11, 12, 14**]).

REMARK 3. The structure of a metrizable compact abelian group can be found in [**12**, Theorem 2.2.6]. It is well-known that the maximal ideal space of l^∞ coincides with the Stone-Čech compactification βZ^+, whose cardinal number is uncountable. (See [**2**, pp. 58] and [**3**, pp. 244].) Therefore Theorem 1 cannot be applied to this case. Theorem 1 answers the question suggested by C. R. Warner and R. Whitley for $L^1(G)$ in the case G is compact and metrizable.

The following theorem extends the results presented in Theorem 1 to another kind of algebra while not hypothesizing that M be closed. (Compare this with Theorem 1 and [**16**, Theorems 2 and 4].) This theorem generalizes Theorems 2 and 4 in [**16**].

THEOREM 4. *Let A be a commutative Banach algebra with involution $x \to x^*$ satisfying $\hat{x}^* = \hat{x}^-$. Suppose that there is an element x_0 in A, with $\hat{x}_0$ never zero, and that there is a one-to-one real-valued function ϕ on the maximal ideal space $\mathfrak{M}$ of A such that $\hat{x}_0 \phi^j = \hat{x}_j$ for some x_j in A $(1 \leq j \leq n)$. If M is a subspace (not a priori closed) of codimension n in A with the property that each element in M belongs to at least n regular maximal ideals, then M is an ideal which is the intersection of n regular maximal ideals.*

Proof. Without loss of generality, we may assume that $\hat{x}_0$ is real-valued. Let $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1}$ denote the cosets in the quotient space A/M corresponding to $x_0, x_1, \dots, x_{n-1}$. If $\lambda_0 \bar{x}_0 + \lambda_1 \bar{x}_1 + \cdots + \lambda_{n-1}\bar{x}_{n-1} = \bar{0}$, then $\lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{n-1} x_{n-1} \in M$ and so the equation $\lambda_0 + \lambda_1 \phi(s) + \cdots + \lambda_{n-1}\phi(s)^{n-1} = 0$ has n distinct solutions in s. This implies that the polynomial $\lambda_0 + \lambda_1 t + \cdots + \lambda_{n-1} t^{n-1}$ has n distinct zeros, which occurs only if all λ_j's are zero. Hence $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{n-1}$ form a basis for A/M.

There exist scalars $\lambda_0, \dots, \lambda_{n-1}$ such that $x_n - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1}$ is in M. Denote this element of M by m_0. We claim that $\hat{m}_0$ is real-valued. By hypothesis and since $m_0 \in M$, we find that the equation $\lambda_0 + \lambda_1 \phi(s) + \cdots + \lambda_{n-1}\phi(s)^{n-1} = \phi(s)^n$ has n distinct solutions, say $s_1, s_2, \dots, s_n$. We write down these relations as follows:

$$\lambda_0 + \lambda_1 \phi(s_1) + \cdots + \lambda_{n-1}\phi(s_1)^{n-1} = \phi(s_1)^n,$$
$$\vdots$$
$$\lambda_0 + \lambda_1 \phi(s_n) + \cdots + \lambda_{n-1}\phi(s_n)^{n-1} = \phi(s_n)^n.$$

By hypothesis, we know that $\phi(s_1), \phi(s_2), \dots, \phi(s_n)$ are n distinct real numbers. By Cramer's rule, we find that $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are all real and so $\hat{m}_0$ is real-valued. As we saw above, $\hat{m}_0$ vanishes exactly at $s_1, s_2, \dots, s_n$.

Let m be an element in M with $\hat{m}$ real-valued. We have $m + im_0 \in M$ and so the equation $\hat{m}(s) + i\hat{m}_0(s) = 0$ has n distinct solutions in s. This implies that $\hat{m}(s_1) = \cdots = \hat{m}(s_n) = 0$, because $\hat{m}_0$ vanishes exactly at $s_1, s_2, \dots, s_n$.

Fix m in M. There exist scalars $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ such that $m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1}$ is in M. We have $m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \in M$ and so the equation $2\mathrm{Re}\, \hat{m}(s) - \lambda_0 \hat{x}_0(s) - \cdots - \lambda_{n-1}\hat{x}_0(s)\phi(s)^{n-1} = 0$ has n distinct solutions in s. By Cramer's rule, we find that $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are all real. On the other hand, we have $-m + m^* - \lambda_0 x_0 - \cdots - \lambda_{n-1} x_{n-1} \in M$ and so the equation $-2i\,\mathrm{Im}\, \hat{m}(s) - \lambda_0 \hat{x}_0(s) - \cdots - \lambda_{n-1}\hat{x}_0(s)\phi(s)^{n-1} = 0$ has n distinct solutions in s. By Cramer's rule, we find that $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are all pure imaginary. Combining these two results we find that all λ_j's are zero. This shows that m^* is in M.

We know that

$$m = 2^{-1}(m + m^*) + i\big[(2i)^{-1}(m - m^*)\big],$$

where the Fourier-Gelfand transforms of $m + m^*$ and $(2i)^{-1}(m - m^*)$ are real-valued. From the results presented in the preceding two paragraphs, we find that $\hat{m}$ vanishes at $s_1, s_2, \ldots, s_n$ for every m in M. This says that $M \subset I_{s_1 s_2 \cdots s_n}$, where $I_{s_1 s_2 \cdots s_n}$ denotes the space $\{x \in A : \hat{x}$ vanishes at $s_1, s_2, \ldots, s_n\}$. Since M and $I_{s_1 s_2 \cdots s_n}$ are of codimension n in A, $M = I_{s_1 s_2 \cdots s_n}$. We have completed the proof.

EXAMPLE 5. Any of the following spaces has the property described in Theorem 4: $C^n(T)$; $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \leq 1$; $BV(T)$; $L^1 \cap L^p(G)$, $1 \leq p \leq \infty$, or $A^p(G)$ or $L^1 \cap C_0(G)$, or any normed ideal in $L^1(G)$ which is invariant under involution, where G is either a metrizable compact abelian group or the direct product of the real line R and a metrizable compact abelian group; l^p, $1 \leq p < \infty$, and $C_0(S)$, where S is any closed subset of $R \times Z^\infty$.

Example 5 follows immediately from the following lemma:

LEMMA 6. *The following two types of algebras have the property described in Theorem* 4:

(i) *Any normed ideal in* $L^1(G)$ *which is invariant under involution, where G is a metrizable compact abelian group or the direct product of R and such a G.*

(ii) $C_0(S)$, *where S is any closed subset of* $R \times Z^\infty$.

Proof. Let A be a normed ideal in $L^1(G)$ which is invariant under involution, where G is either a metrizable compact abelian group or the direct product of the real line R and a metrizable compact abelian group. From Theorems 2.2.2 and 2.2.6 in **[12]** we find that Γ is of the form $\Gamma_1 \times \Gamma_2$, where Γ_1 is $\{0\}$ or R and Γ_2 is countable. Write Γ_2 as $\{\gamma_1, \gamma_2, \ldots\}$. Define a function ϕ on Γ as follows:

$$\phi(\gamma_m) = m \quad \text{if } \Gamma_1 = \{0\},$$

$$\phi(x, \gamma_m) = \frac{x}{(1 + 4\pi^2 x^2)^{1/2}} + m \quad \text{if } \Gamma_1 = R,$$

then ϕ is a one-to-one real-valued function on Γ.

Choose an integrable function h_0 on G with the following property:

$$\hat{h}_0(\gamma_m) = e^{-m^2} \quad \text{if } \Gamma_1 = \{0\},$$

$$\hat{h}_0(x, \gamma_m) = e^{-(x^2 + m^2)} \quad \text{if } \Gamma_1 = R.$$

It is well-known that Γ is sigma-compact, say $\Gamma = \bigcup_{j=1}^{\infty} K_j$, where K_j are compact subsets of Γ. There exists functions g_j in A such that $\hat{g}_j \geq 0$ on Γ and $\hat{g}_j = 1$ on K_j. Define

$$g_0 = \sum_{j=1}^{\infty} \frac{g_j}{j^2 \|g_j\|_A} \quad \text{and} \quad f_0 = g_0 * h_0,$$

then f_0 is in A and $\hat{f}_0$ is never zero.

For the case $\Gamma_1 = R$ we have

$$\begin{aligned}
\hat{f}_0(x, \gamma_m)\phi(x, \gamma_m)^j &= \hat{g}_0(x, \gamma_m) e^{-(x^2+m^2)} \left[\frac{x}{(1 + 4\pi^2 x^2)^{1/2}} + m \right]^j \\
&= \hat{g}_0(x, \gamma_m) e^{-(x^2+m^2)} \sum_{k=0}^{j} \binom{j}{k} x^k \hat{G}_1(x)^k m^{j-k} \\
&= \hat{g}_0(x, \gamma_m) \sum_{k=0}^{j} \binom{j}{k} e^{-x^2} x^k \hat{G}_1(x)^k e^{-m^2} m^{j-k} \\
&= \hat{g}_0(x, \gamma_m) \sum_{k=0}^{j} \binom{j}{k} \hat{H}_k(x) \hat{G}_1(x)^k e^{-m^2} m^{j-k} \\
&= \hat{g}_0(x, \gamma_m) \hat{F}_j(x, \gamma_m) \\
&= \hat{f}_j(x, \gamma_m)
\end{aligned}$$

where

$$\binom{j}{k} = \frac{j(j-1)(j-2)\cdots(j-k+1)}{k!}, \qquad \binom{j}{0} = 1,$$

$$G_1(x) = \frac{1}{(4\pi)^{1/2}} \frac{1}{\Gamma(1/2)} \int_0^{\infty} e^{-\pi x^2/\delta} e^{-\delta/4\pi} \frac{d\delta}{\delta},$$

$$\hat{H}_k(x) = e^{-x^2} x^k,$$

$$F_j = \sum_{k=0}^{j} \binom{j}{k} \Big(H_k * \underbrace{G_1 * \cdots * G_1}_{k \text{ terms}} \Big) \left(\sum_{m=1}^{\infty} e^{-m^2} m^{j-k} \gamma_m \right),$$

$$f_j = g_0 * F_j.$$

The definition of G_1 can be found in [**13**, pp. 132]. The existence of integrable functions H_k on R is based on the fact that the function e^{-x^2} is rapidly decreasing. We have $G_1 \in L^1(R)$, $H_k \in L^1(R)$ and the functions

$$\sum_{m=1}^{\infty} e^{-m^2} m^{j-k} \gamma_m$$

are integrable. This implies that $F_j \in L^1(G)$ and so f_j is in A. This result is also true for the case $\Gamma_1 = \{0\}$; with minor modifications the preceding proof applies.

It remains to show (ii). Let S be any closed subset of the space $R \times Z^\infty$. From Theorem XI.6.5 in [**3**] we find that S is locally compact. It is well-known that $R \times Z^\infty$ is the dual group of $R \times T^\omega$. (See [**12**, §2.2].) Take $G = R \times T^\omega$ and define ϕ and h_0 as above. Denote the restriction of $\hat{h}_0$ on S by f_0 and the restriction of ϕ on S by itself, then $f_0 \in C_0(S)$, f_0 is never zero, ϕ is one-to-one and real-valued and $f_0\phi^j \in C_0(S)$ for all j. (Here we use the assumption that S is closed.) We have completed the proof.

The problem of characterizing the ideals of finite codimension for $L^1(R^2)$ and $C(D)$, D the closed unit disk, raised in [**16**] remains open.

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