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ON A RADON-NIKODYM PROBLEM FOR VECTOR-VALUED MEASURES

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The purpose of this paper is to show that if m is a Banach space-valued measure with finite variation on a σ-algebra, then the variation $|m|$ of m has a Radon-Nikodym derivative with respect to m.

This Radon-Nikodym derivative takes its values in the dual of the Banach space, is integrable in Dinculeanu's sense and may be chosen of norm as close to one as we want.

From this theorem we deduce that if m and m' are Banach space-valued measures on the same σ-algebra, such that $m \ll m'$, then m has a Radon-Nikodym derivative with respect to m' and this derivative is m'-integrable in Dinculeanu's sense if we assume that the image space of m has Radon-Nikodym property.

1. Introduction. The general setting is the following. T will denote a set, $\mathfrak{A}$ a σ-algebra of subsets of T, E a Banach space with dual E' and m a measure from $\mathfrak{A}$ to E, with finite variation.

We will show that for every $\varepsilon > 0$, there exists a function f from T to E' which is strongly measurable, integrable with respect to m in Dinculeanu's sense, such that $|m|(A) = \int_A f\,dm$ for every A in $\mathfrak{A}$ and $1 \leq |f| < 1 + \varepsilon$.

Let us now recall the different ways to define integrable functions with respect to an operator-valued measure. Let E and F be Banach spaces, m a measure from $\mathfrak{A}$ to $\mathfrak{B}(E, F)$ and f a measurable function from T to E.

(a) f is integrable in Dobrakov's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that for every A in $\mathfrak{A}$, the sequence $(\int_A f_n\,dm)_{n\geq 1}$ is convergent in F. The limit of this sequence is then denoted by $\int_A f\,dm$.

(b) f is integrable in the author's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that

$$\lim_{n,p\to\infty} \int |f_n - f_p|\, d|x' \circ m| = 0$$

uniformly in x' in E', $\|x'\| \leq 1$.

(c) f is integrable in Dinculeanu's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that

$$\lim_{n,p\to\infty} \int |f_n - f_p|\, d|m| = 0$$

i.e. the function $|f|$ is $|m|$-integrable.

It is known that the strongest requirement is in Dinculeanu's definition and that the weakest is in Dobrakov's. We also see that Dinculeanu's definition is very strong because the measure has to be of finite variation for the definition to make sense and even in that case, the different spaces of integrable functions may be different (see **[1]** for an example of functions which are integrable in the author's sense and not in Dinculeanu's sense). The best answer we could have to our problem is thus to find a Dinculeanu-integrable function.

It is very easy to show that it is possible to have a Dobrakov-integrable function (using Rybakov's theorem).

Rybakov's theorem (**[3]** and **[5]**) states that if m is a vector measure, there exists an x' in E' such that $m \ll |x' \circ m|$ (and it may be shown that the set of such x' is dense in E'). As $|m| \ll x' \circ m$, there exists a function g which is $(x' \circ m)$-integrable and such that $|m|(A) = \int_A g\, d(x' \circ m) = \int_A g \cdot x'\, dm$ for A in $\mathfrak{A}$. The function $g \cdot x'$ is easily seen to be integrable in Dobrakov's sense.

As the following example will show, it is not possible to prove that each Radon-Nikodym derivative of $|m|$ with respect to m coming from Rybakov's theorem is integrable in Dinculeanu's sense. On the other hand, this example will also show that by choosing more carefully the x' satisfying Rybakov's theorem, it is possible, in that particular case, to have a function which is Dinculeanu-integrable.

But as we don't have any way to insure the selection of a "good" x', Rybakov's theorem may not be used to prove our theorem.

2. Example. Let $\mathfrak{A}$ be the Borel tribe of $[0, 1]$ and $E = L^1([0, 1])$ the space of μ-integrable functions on $[0, 1]$ where μ denotes the Lebesgue measure.

If m is defined by $m(A) = \varphi_A$, it is clear that μ is the variation of m.

For g in $L^\infty([0, 1])$, $g \circ m = g \cdot \mu$ (i.e. $g(m(A)) = \int_A g\, d\mu$ for A in $\mathfrak{A}$), so that an x' in E' satisfying Rybakov's theorem is a g in $L^\infty([0, 1])$ such that the set $\{t \mid g(t) = 0\}$ is μ-null.

For such a g, $\mu(A) = \int_A f\, d(g \cdot \mu) = \int_A f\, d(g \circ m)$ for A in $\mathfrak{A}$, where f is defined by $f(t) = (g(t))^{-1}$ if $g(t) \neq 0$ and $f(t) = 0$ if $g(t) = 0$.

From that equality, it easily follows that $\mu(A) = \int_A \hat{g}\, dm$ for A in $\mathfrak{A}$, where $\hat{g}$: $[0, 1] \to L^\infty([0, 1])$ is given by $\hat{g}(t) = g/g(t)$ if $g(t) \neq 0$ and $\hat{g}(t) = 0$ if $g(t) = 0$. $\hat{g}$ is clearly integrable in Dobrakov's sense and for $\hat{g}$ to be integrable in Dinculeanu's sense, the function $t \to \|\hat{g}(t)\|$ has to be μ-integrable, which means that the function

$$t \to \begin{cases} (|g(t)|)^{-1} & \text{if } g(t) \neq 0, \\ 0 & \text{if } g(t) = 0, \end{cases}$$

has to be μ-integrable.

This is of course true when g is chosen such that 0 is not in the closure of the range of g, which gives a wide choice of "good" g's.

3. Results. Before proving the two results, we now state an exhaustion lemma which will be used in the proof of the first theorem.

EXHAUSTION LEMMA. *Let m: $\mathfrak{A} \to E$ be a vector measure and μ a positive measure on $\mathfrak{A}$. Suppose P is a property of m such that:*

(a) *if m has property P on $A \in \mathfrak{A}$, then m has property P on every $B \in \mathfrak{A}$, contained in A.*

(b) *every set $A \in \mathfrak{A}$ of positive μ-measure contains a set $B \in \mathfrak{A}$ of positive measure such that m has property P on B.*

Then there exists a sequence $(A_n)_{n\geq 0}$ of disjoint members of $\mathfrak{A}$ such that $T = \bigcup_{n=0}^{\infty} A_n$ where $\mu(A_0) = 0$ and m has property P on each A_n for $n \geq 1$.

This lemma is a simpler version and an immediate consequence of the exhaustion lemma stated in [3].

THEOREM 1. *If m is an E-valued measure on $\mathfrak{A}$, with finite variation $|m|$ and if $\varepsilon > 0$, there exists f: $T \to E'$, m-integrable in Dinculeanu's sense, such that $1 \leq |f| \leq 1 + \varepsilon$ and $|m|(A) = \int_A f\,dm$ for every A in $\mathfrak{A}$.*

Proof. Let $x' \in S(E') = \{x' \mid x' \in E', \|x'\| = 1\}$. From the classical Radon-Nikodym theorem, there exists an $|m|$-integrable function $f_{x'}$ such that $|f_{x'}| \leq 1$ and $x'(m(A)) = \int_A f_{x'}\,d|m|$ for every A in $\mathfrak{A}$.

Let $A_{x'} = \{t \mid |f_{x'}(t)| \geq (1 + \varepsilon)^{-1}\}$.

It is easy to see that $\bigcup_{x'\in S(E')} A_{x'}$ has an $|m|$-negligible complement but we will show that a countable union of $A_{x'}$'s already has that property by using the exhaustion lemma.

We say that a set A in $\mathfrak{A}$ has property P if there exists $x' \in S(E')$ such that A is contained in $A_{x'}$.

We only have to show that if B in $\mathfrak{A}$ is such that $|m|(B) \neq 0$, there exists A in $\mathfrak{A}$, $A \subseteq B$ such that $|m|(A) \neq 0$ and A has property P. As $|m|(B) \neq 0$, there exists $B_1, \ldots, B_n$ in $\mathfrak{A}$, mutually disjoint, such that

$$B = \bigcup_{i=1}^{n} B_i \quad \text{and} \quad \sum_{i=1}^{n} \|m(B_i)\| > \frac{1}{1+\varepsilon}|m|(B) = \frac{1}{1+\varepsilon}\sum_{i=1}^{n} |m|(B_i).$$

It follows that $\|m(B_i)\| > 1/(1 + \varepsilon)\,|m|(B_i)$ for some i and we may choose x' in $S(E')$ such that $|x' \circ m|(B_i) > 1/(1 + \varepsilon)\,|m|(B_i)$. From this inequality and the definition of $A_{x'}$, we easily see that $|m|(B_i \cap A_{x'}) \neq 0$ and $B_i \cap A_{x'}$ is the A we are looking for. So by the exhaustion lemma, there exists a sequence $(A_n)_{n\geq 0}$ in $\mathfrak{A}$, of disjoint members such that T is the union of the sequence, $|m|(A_0) = 0$ and for $n \geq 1$, there exists x'_n in $S(E')$ such that $A_n \subseteq A_{x'_n}$.

The function $f = \sum_{n=1}^{\infty} 1/f_{x'_n} \cdot x'_n \cdot \varphi_{A_n}$ is strongly measurable, integrable in Dinculeanu's sense, $1 \leq |f| \leq 1 + \varepsilon$ and it is clear that for A in $\mathfrak{A}$, $|m|(A) = \int_A f\, dm$.

REMARK. We easily see from the proof of the above theorem that if E is the dual of a space F, the values of f may be chosen in F instead of in F'' for we choose x' such that $x'(m(B_i)) > (1/(1 + \varepsilon))\,|m|(B_i)$ from the inequality $\|m(B_i)\| > (1/(1 + \varepsilon))\,|m|(B_i)$ and in this particular case, such an x' may be chosen in F.

From Theorem 1, we are now able to deduce a rather general and interesting Radon-Nikodym theorem for two Banach space-valued measures, one being absolutely continuous with respect to the other.

THEOREM 2. *Let $\mathfrak{A}$ be a σ-algebra of subsets of a set T and E a Banach space. Then, E has the Radon-Nikodym property if and only if for every Banach space F, m: $\mathfrak{A} \to E$ and m': $\mathfrak{A} \to F$ with finite variation, such that $m \ll m'$, there exists a function f: $T \to \mathfrak{B}(F, E)$, integrable in Dinculeanu's sense, such that $m(A) = \int_A f\, dm'$ for A in $\mathfrak{A}$.*

Proof. The necessity of the Radon-Nikodym property for E is obvious.

Let m, m' have finite variation such that $m \ll m'$.

It follows that $m \ll |m'|$ and that there exists an $|m'|$-Bochner integrable function f_1: $T \to E$ such that $m(A) = \int_A f_1\, d|m'|$ for every A in $\mathfrak{A}$.

On the other hand, by Theorem 1, there exists a Dinculeanu-integrable function f_2: $T \to F'$ such that $1 \leq |f_2| \leq 2$ and $|m'|(A) = \int_A f_2\, dm'$ for A in $\mathfrak{A}$.

Let f: $T \to \mathfrak{B}(F, E)$ defined by $f(t)(y) = f_1(t) \cdot f_2(t)(y)$. The function $t \to \|f(t)\|$ is clearly integrable with respect to $|m'|$ which means that f is Dinculeanu-integrable with respect to m'.

It is now very easy to see that $m(A) = \int_A f\, dm'$ for A in $\mathfrak{A}$ which completes the proof.

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