CHARACTERIZATIONS OF COMPLETELY NONDETERMINISTIC STOCHASTIC PROCESSES

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A discrete weakly stationary Gaussian stochastic process $\{x(t)\}$, is completely nondeterministic if no non-trivial set from the σ-algebra generated by $\{x(t): t > 0\}$ lies in the σ-algebra generated by $\{x(t): t \leq 0\}$. In [8] Levinson and McKean essentially showed that a necessary and sufficient condition for complete nondeterminism is that the spectrum of the process is given by $|h|^2$ where h is an outer function in the Hardy space, H^2, of the unit circle in C with the property that $h/\bar{h}$ uniquely determines the outer function h up to an arbitrary constant. In this paper we consider several characterizations of complete nondeterminism in terms of the geometry of the unit ball of the Hardy space H^1 and in terms of Hankel operators.

1. **Introduction.** In [10] Sarason defines a property of a discrete weakly stationary Gaussian stochastic process, $\{x(t)\}$, which he calls complete nondeterminism. This condition is that no set from the future of the process (i.e. the σ-algebra generated by the random variables $x(t)$ for $t > 0$) lies in the past (i.e. the σ-algebra generated by $x(t)$ for $t \leq 0$), except for null sets and the complements of null sets. In the spectral representation this condition becomes the following. Let m be the spectral measure of the process and let $\mathcal{P}$ denote the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n \leq 0$ where functions are defined on **T**, the unit circle in C. Let $\mathcal{F}$ denote the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n > 0$. Then complete nondeterminism is equivalent to the condition that $\mathcal{P} \cap \mathcal{F} = \{0\}$. It is clear that this condition reflects a certain kind of independence (in a statistical sense) of the past, $\mathcal{P}$, and the future, $\mathcal{F}$.

It is of interest to characterize those measures m on **T** which lead to completely nondeterministic (cnd) processes. In [10] a necessary and sufficient condition for complete nondeterminism was stated as the measure m being absolutely continuous with respect to Lebesgue measure, $d\theta$, with log $(dm/d\theta)$ integrable. Unfortunately this characterization is incorrect. In [8, p. 105] Levinson and McKean essentially describe a partial characterization of cnd processes which we discuss in §3. This paper

continues an investigation into the problem of characterizing spectral measures of cnd processes.

In §2 we examine the relationship between complete nondeterminism and some other familiar kinds of independence of $\mathscr{P}$ and $\mathscr{F}$.

In §3 we restate the question in several ways which yield answers in terms of exposed points of the unit sphere of H^1 and certain Hankel operators.

In §4 we define k-step completely nondeterministic processes and provide the characterization of these processes in terms of cnd processes.

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2. Complete nondeterminism. A Gaussian process is called *deterministic* if its past determines the future, i.e., for each $t > 0$, $x(t)$ is measurable with respect to the past. This is translated in the spectral representation to the property that $\mathscr{P} = L^2(dm)$. A necessary and sufficient condition for this to occur is that $\log (dm/d\theta)$ be not integrable. Conversely the process is *indeterministic* if $\log (dm/d\theta)$ is integrable. A stronger restriction than indeterminism is that the process is *purely indeterministic* or *regular*. This an asymptotic independence condition which, in the spectral representation, is equivalent to $\bigcap_{k=1}^{\infty} \mathscr{F}_k = \{0\}$ where $\mathscr{F}_k$ is the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n \geq k$. This condition is often referred to by saying that the process has trivial remote future. Results of Szego [11], Kolmogorov [6] and Krein [7] show that $\{x(t)\}$ is regular if and only if m is absolutely continuous with respect to Lebesgue measure and $\log (dm/d\theta)$ is integrable. First we give an example of a process which is regular but not completely nondeterministic, thereby showing that the characterization in [10] is incorrect. We will use the following notation. L^1 (resp. L^2) is the space of integrable (resp. square integrable) functions on **T**. L^{∞} is the space of essentially bounded functions on **T**. We shall often regard functions in L^1 as extended harmonically into the open unit disc $D = \{z: |z| < 1\}$ by means of Poisson's formula. We let H^1 denote those functions in L^1 which have analytic extensions into the disc. We define H^2 and H^{∞} similarly. H^2 is a Hilbert space with orthonormal basis $\{z^n: n = 0, 1, 2, \ldots\}$. For standard results on the Hardy spaces we refer to [5].

For a regular process we can write $dm = wd\theta = |H| d\theta = |h|^2 d\theta$ where H is an outer function in H^1 and h is an outer function in H^2. w is known as the spectral density of the process.

PROPOSITION 1. *There is a regular process which is not completely nondeterministic.*

Proof. Let $w(e^{i\theta}) = |1 + e^{i\theta}|^2 = |1 + z|^2$ and put $dm = wd\theta$. Since $\log|1 + z|^2 \in L^1$ this process is regular. However $(1 + \bar{z})^{-1} \in \mathscr{P} \cap \mathscr{F}$. This follows since $1 + z$ is outer. For we have

$$\lim_{n \to \infty} \int_{\mathbf{T}} |1 - p_n(1 + z)|^2 \, dz = 0$$

for some sequence p_n of polynomials in z;

hence

$$\int_{\mathbf{T}} |(1 + \bar{z})^{-1} - zp_n|^2 |1 + \bar{z}|^2 \, dz \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \int_{\mathbf{T}} |(1 + \bar{z})^{-1} - zp_n|^2 |1 + z|^2 \, dz \to 0 \quad \text{as } n \to \infty;$$

i.e. $(1 + \bar{z})^{-1} \in \mathscr{F}$. Similarly

$$\int_{\mathbf{T}} |(1 + \bar{z})^{-1} - \bar{p}_n|^2 |1 + z|^2 \, dz$$

$$= \int_{\mathbf{T}} |1 - p_n(1 + z)|^2 \, dz \to 0 \quad \text{as } n \to \infty;$$

i.e. $(1 + \bar{z})^{-1} \in \mathscr{P}$.

We next obtain a simple necessary and sufficient condition for complete nondeterminism. It is straightforward to see that if m is singular with respect to Lebesgue measure then $\mathscr{P} \cap \mathscr{F} \neq \{0\}$. This, together with earlier comments means that in considering cnd processes we can restrict our attention to regular processes.

We wish to rephrase our question in terms of L^2 rather than $L^2(m)$. We have $dm = |h|^2 \, d\theta$. Consider the mapping $T: L^2(m) \to L^2$ given by $Tf = hf$. It is easily verified that T is an isometry of $L^2(m)$ onto L^2. Also T maps $\mathscr{F}$ onto $H_0^2 = \{f \in H^2 : f(0) = 0\}$, and T maps $\mathscr{P}$ onto $(h/\bar{h})\overline{H^2}$ where $\overline{H^2} = \{\bar{f} : f \in H^2\}$.

PROPOSITION 2. *A process is not cnd if and only if $h/\bar{h} = \alpha(F/\bar{F})$ where $F \in H^2$ is outer and α is inner with $\alpha(0) = 0$.*

Proof. Using the isometry T we see that $\mathscr{P} \cap \mathscr{F} \neq \{0\}$ if and only if there are non-zero functions g_1, g_2 in H^2 such that

$$zg_1 = (h/\bar{h})\bar{g}_2$$

$$\Leftrightarrow z(g_1/h) = (\bar{g}_2/\bar{h}) \text{ and } z(g_2/h) = (\bar{g}_1/\bar{h})$$

$$\Rightarrow z(g_1 + g_2)/h = \overline{(g_1 + g_2)}/\bar{h}.$$

Hence $\mathscr{P} \cap \mathscr{F} \neq \{0\}$ if and only if there exists a function $G \in H^2$ such that $zG/h = \bar{G}/\bar{h}$. If we use the inner-outer factorization of G then this equality becomes

$$z\phi F/h = \overline{\phi F}/\bar{h} \quad \text{where } \phi \text{ is inner and } F \in H^2 \text{ is outer,}$$

$$\Rightarrow h/\bar{h} = \alpha(F/\bar{F}) \quad \text{and} \quad \alpha(0) = 0.$$

Conversely

$$h/\bar{h} = \alpha(F/\bar{F}), \qquad \alpha(0) = 0$$

$$\Rightarrow (h/\bar{h})\bar{F} = z(\beta F) \quad \text{where } \alpha = z\beta$$

$$\Rightarrow \mathscr{P} \cap \mathscr{F} \neq \{0\} \quad \text{by the above.}$$

The same reasoning yields the following result for $k \geq 1$:

$$\mathscr{P} \cap \mathscr{F}_k \neq \{0\} \Leftrightarrow h/\bar{h} = \alpha(F/\bar{F}) \quad \text{where } F \in H^2 \text{ is outer and } \alpha \text{ is inner}$$

with α having a zero at the origin of order at least k.

Another strictly stronger property than regularity is that of minimality. Introduced by Kolmogorov [6] this property says that a process is minimal if the value of the random variable $x(0)$ cannot be predicted without error from the values of the random variables $\{x(t): t \neq 0\}$. In other words a process is not minimal if it is possible to perfectly interpolate any value of the process from knowledge of the remaining values of the process. Kolmogorov [6] proved that a process is minimal if and only if w^{-1} is in L^1.

It is immediately of interest to examine the relationship between minimal processes and completely nondeterministic processes.

PROPOSITION 3. *If the process* $\{x(t)\}$ *is minimal then it is completely nondeterministic. On the other hand there exist completely nondeterministic processes which are not minimal.*

Proof. Suppose $\{x(t)\}$ is minimal. Then by Kolmogorov's theorem $h^{-1} \in H^2$. Using Proposition 2 we argue by contradiction. For suppose

$\{x(t)\}$ is not completely nondeterministic. Then $(h/\bar{h}) = \alpha(f/\bar{f})$ where f is outer and α is inner with $\alpha(0) = 0$. This equality implies $\bar{f}/\bar{h} = \alpha(f/h)$. The LHS is in $\overline{H^1}$ and the RHS is in H_0^1 which forces both sides to be zero and thus $f = 0$ which is a contradiction. This proves the first statement of the proposition. An example of a process which yields the second statement is given by $w = |1 + z|$. In this case $h = (1 + z)^{1/2}$ and $h/\bar{h} = z^{1/2}$. By Kolmogorov's criterion this process is not minimal. On the other hand suppose $h/\bar{h} = \alpha(f/\bar{f})$ for f outer, α inner with $\alpha(0) = 0$. Then

$$z^{1/2} = \alpha(f/\bar{f}) = z\phi(f/\bar{f}) \quad \text{with } \phi \text{ inner}$$

$$\Rightarrow z^{1/2}\phi f = \bar{f} \Rightarrow z(\phi f)^2 = (\bar{f})^2.$$

The LHS is in H_0^1 and the RHS is in $\overline{H^1}$. Again this forces both sides to be zero and hence $f = 0$ which gives a contradiction. Thus the process with $w = |1 + z|$ is completely nondeterministic.

Let $\mathcal{P}_k$ be the span in $L^2(m)$ of the exponentials $e^{in\theta}$ with $n \leq k$. A minimal process is one for which the function 1 does not belong to the closed linear span of $\mathcal{P}_1$ and $\mathcal{F}_1$ i.e. $1 \notin \mathcal{P}_1 \vee \mathcal{F}_1$. There is a similar restatement of the condition of completely nondeterminacy. Let $\mathcal{P}_1 + \mathcal{F}_1 = \{f \in L^2(m): f = g + h \text{ with } g \in \mathcal{P}_1, h \in \mathcal{F}_1\}$.

PROPOSITION 4. *A regular process is completely nondeterministic if and only if* $1 \notin \mathcal{P}_1 + \mathcal{F}_1$.

Proof. Assume f is a non-zero element of $\mathcal{P} \cap \mathcal{F}$. Then, for some $k \geq 1$, $f \in \mathcal{F}_k$ but $f \notin \mathcal{F}_{k+1}$ (since $\bigcap_{k=1}^{\infty} \mathcal{F}_k = \{0\}$). Hence $f = ae^{ik\theta} + f_1$ where $a \neq 0$ and $f_1 \in \mathcal{F}_{k+1}$. This implies

$$e^{ik\theta} = (f - f_1)/a \in \mathcal{P} + \mathcal{F}_{k+1}$$

$$\Rightarrow 1 \in (e^{-ik\theta}\mathcal{P}) + \mathcal{F}_1 \subseteq \mathcal{P}_1 + \mathcal{F}_1.$$

Conversely assume that $1 \in \mathcal{P}_1 + \mathcal{F}_1$. Then $1 = f_1 + f_2$ with $f_1 \in \mathcal{P}_1, f_2 \in \mathcal{F}$. Hence $e^{i\theta}f_1 = e^{i\theta} - e^{i\theta}f_2 \in \mathcal{F}$. But $e^{i\theta}f_1 \in \mathcal{P}$. Hence $e^{i\theta}f_1 \in \mathcal{P} \cap \mathcal{F}$.

3. Exposed points of the ball in H^1 and Hankel operators. It is well known (see [2]) that the extreme points of the unit ball of H^1 are given by the outer functions F in H^1 with $\|F\|_1 = 1$. It is also well known that an H^1 function F of unit norm is not determined by its argument.

In [8, p. 205] Levinson and McKean showed that for continuous processes the dimension of $\mathcal{P} \cap \mathcal{F}_0 = 1$ if and only if $h/\bar{h}$ determines the outer function h up to a constant. In this section we consider this

follows that $z^k \phi F^2 \in S_{h/\bar{h}}$. Conversely if $f \in S_{h/\bar{h}}$ and $f(z_1) = f(z_2) = \cdots = f(z_k) = 0$ where $z_j \in D$ ($1 \le j \le k$) then it is easy to verify that a positive multiple of

$$g(z) = z^k f(z) \prod_{j=1}^{k} (z - z_j)^{-1} (1 - \bar{z}_j z)^{-1}$$

is in $S_{h/\bar{h}}$. Factorize g as $g = z^k bF$ where b is inner and $F \in H^2$ is outer. Since $ag \in S_{h/\bar{h}}$ for some $a > 0$ it follows that $h/\bar{h} = z^k bF/\bar{F}$ showing that $\mathscr{P} \cap \mathscr{F}_k \ne \{0\}$.

Note that Proposition 5 yields the version of the Levinson and McKean result as applied to cnd processes: namely, a process is cnd if and only if $\arg(h/\bar{h})$ is the argument of a unique H^1 function.

Since we have expressed the characterization of completely nondeterministic processes in terms of an extremum problem it is not suprising that there is a version of the problem in terms of the norms of Hankel operators, which are closely related to extremum problems on H^1.

Let P be the orthogonal projection of L^2 onto H^2. Recall that the Hankel operator with symbol $\phi \in L^\infty$ is the bounded operator from H^2 to $L^2 \ominus H^2$ defined by

$$H_\phi(f) = (I - P)(\phi f) \qquad (f \in H^2).$$

The norm of H_ϕ is given by $\| H_\phi \| = d(\phi, H^\infty) = \inf_{f \in H^\infty} \| \phi - f \|_\infty$. It is straightforward to show from first principles that the process $\{x(t)\}$ is not cnd if and only if $H_{\bar{h}/h}$ attains the norm of 1 on the unit sphere of H^2. In fact more is true.

In [1] it is essentially shown that H_ϕ attains its norm on the unit sphere on H^2 if and only if $\phi = f + \lambda \psi$ where $f \in H^\infty$, $\lambda > 0$ and $|\psi| = 1$ a.e. on $\mathbf{T}$ with S_ψ^- containing more than one function. Also if $\| \phi \|_\infty = 1$ then H_ϕ attains the norm 1 if and only if $|\phi| = 1$ a.e. on $\mathbf{T}$ and S_ψ^- contains more than one function [1]. There is another result of this type which does not seem to have appeared in the literature.

PROPOSITION 7. $\| H_\phi \| < \| \phi \|_\infty \Rightarrow \phi = f + \lambda \psi$ where $f \in H^\infty$, $\lambda > 0$ and $|\psi| = 1$ a.e. on $\mathbf{T}$ with S_ψ^- containing exactly one function.

Proof. Without loss of generality we assume that $\| \phi \|_\infty = 1$. Suppose $\| H_\phi \| < 1$. Then by [3] there exists $\psi \in L^\infty$ such that (i) $\phi - \psi \in H^\infty$ and (ii) $\psi = \bar{F}/|F|$ for some $F \in H^1$, $F \ne 0$. Now (i) $\Rightarrow H_\psi = H_\phi$ and so $\| H_\psi \| < 1$. So there exists $g \in H^\infty$ such that $\|(\bar{F}/|F|) - g\|_\infty = a < 1$ which gives that $|\arg(gF)| < b < \pi/2$. Hence $(gF)^{-1} \in H^1$ (since $gF \ne 0$

on D and if G is analytic on D and $|\arg G| < b < \pi/2$ then $G \in H^p$ for all $p < \pi/2b)$. Thus $g(gF)^{-1} \in H^1 \Rightarrow F^{-1} \in H^1$. Now $F/|F| = \bar{\psi}$ so that a positive multiple of F is in S_ψ^-. Then $F^{-1} \in H^1$ implies that S_ψ^- contains one and only one function (if $G \in S_\beta$ and $G^{-1} \in H^1$ then $S_\beta = \{G\}$–see [2, Theorem 8] and use the fact that positive $H^{1/2}$ functions are constant).

Note however that S_ψ^- containing exactly one function does not necessarily imply that $\|H_\psi\| < \|\psi\|_\infty$. For example if $h = (1 + z)^{1/2}$, and we take $\psi = \bar{h}/h$ it can be shown that $\|H_\psi\| = 1$ but, as we saw in the proof of Proposition 3, $|h|^2$ corresponds to a cnd process so that $S_{h/\bar{h}} = \{h^2\}$.

4. k-Step completely nondeterministic processes.

The Helson-Szego theorem gives an elegant characterization of the spectral densities of processes where $\mathcal{P}$ and $\mathcal{F}_1$ are at positive angle i.e. $\sup\{|\langle f, g \rangle| : f \in \text{unit ball of } \mathcal{P}, g \in \text{unit ball of } \mathcal{F}\} < 1$. See [10] for a discussion of this result. Note that such processes are necessarily cnd. In the course of characterizing processes with $\mathcal{P}$ and $\mathcal{F}_k$ at positive angle Helson and Sarason [4] showed that the spectral density of such a process, w, is given by $w = |p|^2 w_1$ where w_1 is the spectral density of a process with $\mathcal{P}$ and $\mathcal{F}_1$ at positive angle and p is a polynomial of degree less than k with zeros all on **T**.

In this section we show that there is a similar theory relating processes with $\mathcal{P} \cap \mathcal{F}_k = \{0\}$ and cnd processes. We call a process k-step completely nondeterministic (k-cnd) if $\mathcal{P} \cap \mathcal{F}_k = \{0\}$ but $\mathcal{P} \cap \mathcal{F}_{k-1} \neq \{0\}$ ($k \geq 1$). For simplicity we let M_k denote the image of $P \cap \mathcal{F}_k$ in L^2 under the isometry T. Thus $M_k = \{0\}$ if and only if $\mathcal{P} \cap \mathcal{F}_k = \{0\}$.

PROPOSITION 8. *Suppose that* $w = |p|^2 w_0$, *where* p *is a polynomial of degree* k *with all* k *zeros on* **T**, *and* $w_0 \in L^1$. *Then* $\mathcal{P} \cap \mathcal{F}_k \neq \{0\}$.

Proof. We show that $1/\bar{p} \in \mathcal{P} \cap \mathcal{F}_k$.

(i) $1/\bar{p} \in \mathcal{P}$: without loss of generality we can assume that

$$1/\bar{p} = \prod_{j=1}^n \left(1 - \bar{z}/\bar{\zeta}_j\right)^{-n_j}, \qquad |\zeta_j| = 1,$$

$$= \sum_{j=1}^n q_{n_j-1}(\bar{z})\left(1 - \bar{z}/\bar{\zeta}_j\right)^{-n_j} \quad \text{where } q_{n_j-1} \text{ is a polynomial.}$$

Now $(1 - \bar{z}/\bar{\zeta}_j)^{-n_j}$ can be approximated by polynomials in $\bar{z}$ in $L^2(m)$. In fact

$$\int_{\mathbf{T}} \left| (1 - \bar{z}/\bar{\zeta}_j)^{-n_j} - \left\{ 1 + \frac{n-1}{n}(\bar{z}/\bar{\zeta}_j) + \frac{n-2}{n}(\bar{z}/\bar{\zeta}_j)^2 \right. \right.$$
$$\left. \left. + \cdots + (1/n)(\bar{z}/\bar{\zeta}_j)^{n-1} \right\}^{n_j} \right|^2 w(\theta) d\theta$$

$$= \int_{\mathbf{T}} \left| 1 - \left[(1 - \bar{z}/\bar{\zeta}_j) \left\{ 1 + \frac{n-1}{n}(\bar{z}/\bar{\zeta}_j) \right. \right. \right.$$
$$\left. \left. \left. + \cdots + (1/n)(\bar{z}/\bar{\zeta}_j)^{n-1} \right\} \right]^{n_j} \right|^2 w_2(\theta) d\theta$$

$$\text{where } w_2 = w/|1 - z/\zeta_j|^{n_j}$$

$$= \int_{\mathbf{T}} \left| 1 - \left[1 - (1/n)\left\{ \bar{z}/\bar{\zeta}_j + \cdots + (\bar{z}/\bar{\zeta}_j)^n \right\} \right]^{n_j} \right|^2 w_2(\theta) d\theta$$

$\to 0 \quad$ as $n \to \infty$ by Lebesgue's dominated convergence theorem.

Hence $1/\bar{p} \in \mathcal{P}$.

(ii) $1/\bar{p} \in \mathcal{F}_k$: $1/\bar{p} = z^k/z^k\bar{p} = z^k/q_k$ where $q_k = z^k\bar{p}$ is also a polynomial of degree k in z. The same construction as in (i) shows that $1/q_k$ can be approximated by polynomials in z in $L^2(m)$. Hence $1/q_k \in \mathcal{F}_0$ and $1/\bar{p} \in \mathcal{F}_k$.

REMARK. This proposition implies that if we restrict our attention to cnd processes then the strong mixing condition implies the property that $\mathcal{P}$ and $\mathcal{F}_1$ be at positive angle; (see [4], [10, p. 77] for definitions). For if the angle between $\mathcal{P}$ and $\mathcal{F}_n$ is converging to $\pi/2$ as $n \to \infty$ then, for some k, $\mathcal{P}$ and $\mathcal{F}_k$ are at a positive angle which implies by [4] that $w = |p|^2 w_1$ for some trigonometric polynomial p where w_1 is the spectrum of a process for which $\mathcal{P}$ and $\mathcal{F}_1$ are at a positive angle. If the process is cnd then Proposition 5 implies that p must have zero degree. In general the strong mixing condition does not imply that $\mathcal{P}$ and $\mathcal{F}_1$ are at positive angle (e.g. take $h = 1 + z$).

PROPOSITION 9. *Let $k \geq 1$. Suppose $M_k = \{0\}$ but $M_{k-1} \neq \{0\}$. Then $h/\bar{h} = z^{k-1}F/\bar{F}$ where F is outer and $|F|^2$ corresponds to a cnd process.*

Proof. $\mathcal{P} \cap \mathcal{F}_{k-1} \neq \{0\}$ implies that $h/\bar{h} = z^{k-1}\alpha(F/\bar{F})$ where $F \in H^2$ is outer, α is inner. Suppose that α is non-constant. Then we can find constants a, b such that $0 \neq aF + b(\alpha F) \in H_0^2$. Then

$$z^{k-1}(aF + b\alpha F)/h = \overline{(\bar{a}\alpha F + \bar{b}F)}/\bar{h}.$$

Then LHS $\in \mathcal{F}_k$ and the RHS $\in \mathcal{P}$. Hence $\mathcal{P} \cap \mathcal{F}_k \neq \{0\}$. This contradiction implies that α is a constant λ, which can then be absorbed into F. $M_k = \{0\}$ implies that F must correspond to a cnd process.

PROPOSITION 10. *Let $k \geq 0$. If $M_k = M_{k+1}$ then $M_k = \{0\}$.*

Proof. Observe that $M_k = [(h/\bar{h})H_0^2 + z^{k-1}\overline{H^2}]^{\perp}$. Thus

$$(*) \qquad M_k = M_{k+1} \Rightarrow z^k \in (h/\bar{h})H_0^2 + z^{k-1}\overline{H^2}.$$

This gives

$$z^{k+1} \in (h/\bar{h})zH_0^2 + z^k\overline{H^2} \subseteq (h/\bar{h})H_0^2 + z^{k-1}\overline{H^2} \quad \text{by } (*).$$

Similarly $z^{k+2}, z^{k+3}, \ldots$ belong to $(h/\bar{h})H_0^2 + z^{k-1}\overline{H^2}$. But this implies that $L^2 = \bigvee_{j=1}^{\infty} z^{k+j}\overline{H^2} \subseteq M_k^{\perp}$. Hence $M_k = \{0\}$.

THEOREM 11. *The spectral density w is the spectral density of a k-cnd process if and only if $w = |p|^2 w_0$ where p is a polynomial of degree $k-1$ with all of its zeros on $\mathbf{T}$ and w_0 is the spectral density of a cnd process.*

Proof. Suppose $w = |h|^2$ corresponds to a k-cnd process. Since $\dim(M_{j+1}^{\perp}/M_j^{\perp}) \leq 1$, it follows from Proposition 10 that $\dim(M_{j-1}/M_j) = 1$ for $j = 1, \ldots, k$. Thus $\dim M_0 = k$. Now $M_0 = (h/\bar{h})H^2 \cap H^2 = [(h/\bar{h})H_0^2 + \bar{z}\overline{H^2}]^{\perp}$. Also Proposition 9 shows that $h/\bar{h} = z^{k-1}F/\bar{F}$ where $w_0 = |F|^2$ is the spectral density of a cnd process. It is easy to check that $z^jF \in M_0$ for $j = 0, \ldots, k-1$. Hence $M_0 = \bigvee\{F, zF, \ldots, z^{k-1}F\}$. But $h \in M_0$ and so $h = pF$ where p is a polynomial of degree $k-1$. It is clear that since $|h|^2$ is k-cnd p must have exactly $k-1$ roots on $\mathbf{T}$ counting multiplicities. Conversely, if $w = |p|^2 w_0$ where p, w_0 are as in the theorem, then Proposition 8 shows that $\mathcal{P} \cap \mathcal{F}_{k-1} \neq \{0\}$. The fact that w_0 corresponds to a cnd process gives $\mathcal{P} \cap \mathcal{F}_k = \{0\}$.

REMARK. There are some intriguing questions concerning $S_{h/\bar{h}}$ which are related to the work of this Section. In [2] it was shown that $S_{h/\bar{h}} = \{h^2\}$ implies that h is strong outer, i.e. $h/(z - \lambda) \notin H^2$ for all $\lambda \in \mathbf{T}$. It would be of interest to construct a strong outer function h with $S_{h/\bar{h}} \neq \{h^2\}$. In such a case $\mathcal{P} \cap \mathcal{F}_k$ is infinite dimensional for all $k \geq 1$. This follows since if $0 \neq \dim M_1 = n < \infty$ then, by the reasoning in the proof of Theorem

11, $|h|^2 = |p|^2 w_0$ where p has n zeros on $\mathbf{T}$ which contradicts the strong outer property of h. Thus $\dim M_1 = \infty$, and $\dim M_{k+1} \geq \dim(M_1) - k = \infty$ for all $k \geq 1$. This, together with [2, Lemma 4.6], provides evidence for the following conjecture: if S_ϕ contains more than one function one of which is strong outer, then S_ϕ contains a function with an inner factor which is not a finite Blaschke product. Note that it is easy to construct examples of processes for which $\mathscr{P} \cap \mathscr{F}_k \neq \{0\}$ for all $k > 1$. The spectral density $w = |1 + B|^2$ is such an example if B is an inner function which is not a finite Blaschke product.

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