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NATURAL TRANSFORMATIONS OF TENSOR-PRODUCTS OF REPRESENTATION-FUNCTORS I, COMBINATORIAL PRELIMINARIES

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The present paper furnishes some combinatorial preliminaries towards a study of natural transformations between tensor products of shape functors $\wedge^{\alpha}$ and co-shape functors $\vee_{\alpha}$. The main result is the construction of an explicit basis for the module defined by (1) below; an apparently new result used for this purpose, which may be of some independent interest, is a 'column-free' expression for the Young idempotent NPN (in Young's terminology) associated with a partition, given by 1.2 below.

Introduction. In the following, the reader will be assumed to be familiar with the concepts and results of [**1**] and [**2**].

Let $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n$ be partitions, and let A be a commutative ring. The present paper is the first of a series concerned with the A-module, denoted by

$$(1) \qquad \operatorname{Nat}\operatorname{Tsf}_A(\alpha_1 \times \cdots \times \alpha_m, \beta_1 \times ,\ldots, \times \beta_n),$$

which consists of all natural transformations from the functor

$$\wedge_A^{\alpha_1} \otimes \cdots \otimes \wedge_A^{\alpha_m} \colon \mathbf{Mod}_A \to \mathbf{Mod}_A,\ E \mapsto \wedge_A^{\alpha_1} E \otimes \cdots \otimes \wedge_A^{\alpha_m} E$$

into the similar functor $\wedge_A^{\beta_1} \otimes \cdots \otimes \wedge_A^{\beta_n}$ (If A is a field this is equivalent to studying the space of interwining operators between the two representations of $\mathrm{GL}(E)$ with representation-modules $\wedge_A^{\alpha_1} E \otimes \cdots \otimes \wedge_A^{\alpha_m} E$ and $\wedge_A^{\beta_1} E \otimes \cdots \otimes \wedge_A^{\beta_n} E$ respectively (provided $\dim E$ is sufficiently great).

When A is a $\mathbf{Q}$-algebra, a generating set for the A-module (1) is furnished by the "exchange-transformations" given by Def. 3–6 below and a free basis by the subset of these given by Def. 3–8 (In the case $m = 2$, $n = 1$ this furnishes a more precise version of the Littlewood-Richardson rule (which only specifies the cardinality of such a basis).) The general case does not seem to be an immediate consequence of this special case; the attempt to reduce to the special case in the obvious way, by using the associativity of the tensor product, leads to the problem next to be discussed (and yields a second, different free basis for 1), related to that first mentioned by a generalization of the Robinson-Schensted correspondance, to which it reduces when all the α's and β's equal the partition $\langle 1 \rangle$.

(2) In computing with these 'exchange-transformations', it is first of all necessary to describe the 'recombination-laws' which express a composite of two such, as a linear combination of exchange-transformations. This problem in representation-theory seems hitherto to have been studied in detail only by the physicists in certain special cases (cf. for instance the discussion of 'Racah coefficients' (='$6-j$ symbols' = 'recoupling coefficients') in [**3**], p. 299 et seq.).

This problem of 'recombinations' is in fact not difficult on the level of representation-theory; its main difficulty is that of presenting a certain combinatorial complexity. The purpose of the present paper is to sketch some combinatorial concepts, which the author has found useful in studying these questions, as a preliminary to further work shortly to appear.

The main idea is, roughly, to treat $\bigwedge^{\alpha_1} E \otimes \cdots \otimes \bigwedge^{\alpha_m} E$ in a fashion independent, not only of an arbitrary choice of basis for E, but also (as far as possible) of an arbitrary choice of ordering of the set $\{\alpha_1, \ldots, \alpha_n\}$ of partitions, and in a manner which uses only the row structure (but not the column structure) within each 'tableau' α_i. For certain questions (e.g. when a 'standard basis' is desired) specific choices of such orderings, or even of a specific ordered basis for E, become in fact necessary; in questions so intimately related to the representation-theory of symmetric groups as these, however, an arbitrary choice of ordering can be a step as significant as an arbitrary choice of basis for E. Thus, we define in Section One below a category **Fin-2-Sets** of partitions α, and a category **Fin-3-Sets** of unordered sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$ of partitions, and in §2 treat $\bigwedge^{\alpha} E$, $\bigwedge^{\alpha_1} E \otimes \cdots \otimes \bigwedge^{\alpha_n} E$ functorially over these categories as well as over the category of R-modules E; the study of (2) involves a further category **Fin-4-Sets**. In this context, note especially Def. 1–2 below, which gives a construction (which the author believes to be new) for a suitable Young quasi-idempotent, in terms which depend only on the row-structure of the associated tableau (but *not* involving its column-structure, i.e. independent of the particular choice of ordering of the elements within each row).

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1. Some set-theoretic concepts. For any set E, we denote by $\mathfrak{S}(E)$ the group of bijections of E (written to the *left* of the elements of E on which they act, so the group operation is read from right to left: $(\sigma \cdot \sigma')(e) = \sigma(\sigma'(e))$.

If E, E' are finite sets with the same cardinal:

$$\#E = \#E'$$

then two bijections

$$\iota, \iota' \colon E \to E'$$

will be called *equally oriented* if the element $\iota^{-1} \circ \iota'$ in $\mathfrak{S}(E)$ is even; this is an equivalence relation, whose equivalence classes will be called *orientations from E into E'*. If σ_1 is an orientation from E_1 to E_2, σ_2 an orientation from E_2 to E_3, then all bijections

$$\iota_2 \circ \iota_1 \colon E_1 \to E_3$$

(where ι_1, ι_2 are bijections belonging to σ_1, σ_2 respectively) are in the same orientation from E_1 to E_3, which we denote by $\sigma_2 \circ \sigma_1$. For each natural number n we thus obtain the category **OR(n)** whose objects are sets of cardinality n, and whose morphisms are orientations between these sets, with composition of morphisms defined as just indicated. All morphisms in this category are isomorphisms; if σ is an orientation from E_1 to E_2, then

$$\sigma^{-1} = \{\iota^{-1} \colon \iota \in \sigma\}.$$

Note that if $\#E_1 = \#E_2 = n$, then if $n \geq 2$ there are exactly two orientations from E_1 to E_2, while if $n = 0$ or 1 there is exactly one.

If σ is an orientation from E_1 to E_2, and $\iota\colon E_1 \xrightarrow{\sim} E_2$ a bijection, we write

$$\operatorname{sgn}_\sigma \iota = \begin{cases} 1 & \text{if } \iota \in \sigma, \\ -1 & \text{if } \iota \notin \sigma. \end{cases}$$

If also $\iota_1\colon E_2 \to E_1$ is a bijection, we set $\operatorname{sgn}_\sigma \iota_1$ equal to $\operatorname{sgn}_\sigma \iota_1^{-1} = \operatorname{sgn}_{\sigma^{-1}} \iota_1$.

We next define the category

Fin-n-Sets

of "level n finite sets", by recursion on n, as follows:

Fin-1-Sets is simply the category whose objects are finite sets, and whose morphisms are bijections; a 'level 1 finite set', is simply a finite set. If $n > 1$, a *level n finite set* is defined to be a finite set of pairwise disjoint non-empty level $n - 1$ finite sets; a *level n morphism* between two level n finite sets $\mathfrak{D}_1$, $\mathfrak{D}_2$ is defined to consist of a bijection $\iota\colon \mathfrak{D}_1 \xrightarrow{\sim} \mathfrak{D}_2$, together with the assignment to each $\Delta \in \mathfrak{D}_1$ of a level $n - 1$ morphism ι_Δ from Δ to $\iota(\Delta)$; we denote by **Fin-n-Sets** the category constituted by these level n finite sets and level n morphisms.

Note that level 2 finite sets were called 'partitionings' in ([**2**], Def. 2.5).

If $\mathfrak{D}$ is a level n finite set, we define the relation $\Delta\varepsilon^i\mathfrak{D}$ for all i such that $1 \leq i \leq n$ by recursion on i, as follows:

If $i = 1$, $\Delta\varepsilon^1\mathfrak{D}$ means $\Delta \in \mathfrak{D}$; if $1 < i \leq n$, $\Delta\varepsilon^i\mathfrak{D}$ means there exists $\mathfrak{D}'$ such that $\Delta \in \mathfrak{D}'$ and $\mathfrak{D}'\varepsilon^{n-1}\mathfrak{D}$.

Note that $\Delta\varepsilon^i\mathfrak{D}$ thus implies that Δ is a level $n - i$ finite set (if $i < n$).

$\mathfrak{D}$ being a level n finite set, and $1 \leq i < n$, we denote by $\cup^i\mathfrak{D}$ the set

$$\{\Delta\colon \Delta\varepsilon^{i+1}\mathfrak{D}\};$$

note that this is a level $n - i$ finite set.

We next consider level 2 finite sets in some detail.

Call two finite sequences $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ *order-equivalent* if they contain the same number of elements, and if $\exists\pi \in \mathfrak{S}_n$ such that

$$a_i = b_{\pi i} \qquad (1 \leq i \leq n).$$

We shall denote the order-equivalence class of $(a_1,\ldots,a_n)$ by $\langle a_1,\ldots,a_n\rangle$, and call it an *unordered finite sequence*. In particular, an unordered finite sequence of positive integers, will be called a *numerical partition* (it is convenient to include among the numerical partitions, the 'empty partition' $\langle\ \rangle$).

We may associate to every level 2 finite set $\alpha = \{R_1,\ldots,R_s\}$ the numerical partition $|\alpha| = \langle \#R_1,\ldots, \#R_s\rangle$, and we set

$$\alpha! = (\#R_1)! \cdots (\#R_s)!.$$

Conversely, given any numerical partition

$$\mathcal{Q} = \langle a_1,\ldots,a_s\rangle,\ a_1 \geq \cdots \geq a_s > 0$$

we may associate with it a level 2 finite set, its *Young-Ferrars frame*,

$$F = \{R_1(\mathcal{Q}),\ldots,R_s(\mathcal{Q})\}$$

with

$$R_i(\mathcal{Q}) = \{(i,j)\colon 1 \leq j \leq a_i\} \qquad (1 \leq i \leq s);$$

clearly

$$\mathcal{Q} = |F_{\mathcal{Q}}|\ .$$

Note also that two level 2 finite sets α, α' are isomorphic if and only if $|\alpha| = |\alpha'|$; thus the numerical partitions may be identified with the isomorphism-classes in **Fin-2-Sets**.

DEFINITION 1.1. Let α be a level 2 finite set; we denote by Row(α) the sub-group of $\mathfrak{S}(\cup\alpha)$ consisting of all π in $\mathfrak{S}(\cup\alpha)$ such that

$$b \in R \in \alpha \Rightarrow \pi b \in R,$$

and denote by Alt(α), Sym(α) the elements

$$\sum_{\pi \in \mathrm{Row}(\alpha)} (\mathrm{sgn}\,\pi)\pi, \qquad \sum_{\pi \in \mathrm{Row}(\alpha)} \qquad \text{(respectively)}$$

in $\mathbf{Z}[\mathfrak{S}(\cup\alpha)]$.

We denote by $\mathfrak{S}_{\#}(\alpha)$ the sub-group of $\mathfrak{S}(\alpha)$ consisting of all permutations σ of α (considered simply as a finite set) such that

$$R \in \alpha \Rightarrow \#R = \#(\sigma R);$$

finally, we denote by Aut(α) the group of automorphisms of α in the category **Fin-2-sets**.

REMARK. There is a short exact sequence (natural in α)

$$\{1\} \to \mathrm{Row}(\alpha) \to \mathrm{Aut}(\alpha) \to \mathfrak{S}_{\#}(\alpha) \to \{1\}$$

which splits (but not naturally in α).

Let α be a level 2 finite set, and let I denote $\cup\alpha$. We recall from [**1**] and [**2**] the concept of an "I-indexed function with common domain D taking values in T" (where D and T are any sets) i.e. an element of T^{D^I} (there is a natural left action of $\mathfrak{S}(I)$ on these); recall also the property of having 'Young alternation in α' (defined when T is an Abelian group) possessed by some of these functions (cf. [**2**], Def. 2.4, where α is called a 'partitioning' of I). Denote by $YA_\alpha(D, T)$ the sub-group of T^{D^I} consisting of those I-indexed functions with Young alternation in α.

Such functions with Young alternation in α, may be obtained as follows. A classical construction of Young yields a quasi-idempotent in $\mathbf{Z}[\mathfrak{S}(I)]$, and left multiplication by this projects T^{D^I} into $YA_\alpha(D, T)$ (onto, if T is a $\mathbf{Q}$-module). This quasi-idempotent involves writing the elements of I in a frame $F_{|\alpha|}$, and uses not only the row-structure, but also the column-structure of this frame; in our present terminology, the quasi-idempotent

$$YA\mathrm{lt}(\alpha, <) \in \mathbf{Z}[\mathfrak{S}(I)] \tag{3}$$

in question involves an arbitrary choice of a total ordering $<_R$ on each $R \in \alpha$. [Cf. [**2**], Def. 4.2 for the details; note that there P is used instead of α, and (3) is denoted by $YA(P_<)$]

There are

$$\alpha! = \prod_{R\in\alpha} (\#R)!$$

possible choices for these orderings; thus (when T is a Q-module) we obtain many different projections, $YA\mathrm{lt}(\alpha, <)$, all however onto the same sub-group $YA_\alpha(D, T)$.

The problem thus suggests itself of finding a 'column-free' method of constructing functions with Young alternation in α; we now sketch such a construction. (The proof it works will be left to a later paper, because some ideas to which it leads deserve detailed study in their own right).

PROPOSITION AND DEFINITION 1.2. *Let α be a level 2 finite set, and let $\pi \in \mathfrak{S}(\cup\alpha)$; then there exists an integer $c_\alpha(\pi)$, the Young index of π with respect to α, uniquely characterized by the following property*:

If D is any set, T any Abelian group, and f any function indexed by $\cup\alpha$, with common domain D and taking values in T, which has Young alternation in α, then

$$\mathrm{Alt}(\alpha)\pi f = c(\pi) f. \tag{4}$$

We then define $Y\mathrm{Alt}(\alpha)$ to be the element

$$\sum \{c(\pi)\pi \colon \pi \in \mathfrak{S}(\cup\alpha)\}$$

in $\mathbf{Z}[\mathfrak{S}(\cup\alpha)]$; this is a quasi-idempotent, left-multiplication by which maps $T^{D^{(\cup\alpha)}}$ into $YA_\alpha(D, T)$ (onto, if T is a $\mathbf{Q}$-module).

Note. If we modify the hypotheses on f, under which (4) holds, by requiring instead that f have Young symmetry in α(Cf. [**2**], Def. 4.2) then we have, instead,

$$\mathrm{Sym}(\alpha)\pi f = (\mathrm{sgn}\,\pi) c(\pi) f,$$

and the quasi-idempotent

$$Y\mathrm{Sym}(\pi) = \sum \left\{(\mathrm{sgn}\,\pi) c(\pi)\pi \colon \pi \in \mathfrak{S}\left(\cup^2\alpha\right)\right\}$$

maps into the group of such functions.

DEFINITION 1.3. Let α, α' be level 2 finite sets, with

$$\#(\cup\alpha) = \#(\cup\beta).$$

By a *reflection of α' in α* will be meant a level 1 isomorphism ι of α' with a level 2 set $\bar{\alpha}$, such that $\cup\alpha = \cup\bar{\alpha}$. By an *exchange-matrix from α to α'* will be meant a map

$$M: \alpha \times \alpha' \to \text{(set of non-negative integers)}$$

satisfying the two following conditions:

(i) For all R in α, $\#R = \Sigma_{R' \in \alpha'} M(R, R')$.

(ii) for all R' in α', $\#R' = \Sigma_{R \in \alpha} M(R, R')$.

We then set

$$M! = \Pi\{[M(R, R')]!: R \in \alpha, R' \in \alpha'\}.$$

If $\iota = \alpha' \to \bar{\alpha}$ is a reflection of α' in α, we denote by M^ι the exchange-matrix defined by

$$M^\iota(R, R') = \#(R \cap \iota(R')) \qquad (R \in \alpha, R' \in \alpha')$$

Given a bijection ϕ: $\cup\alpha' \xrightarrow{\sim} \cup\alpha$, we denote by ι^ϕ the reflection of α' in α defined by

$$\iota^\phi(R') = \phi(R') \qquad (R' \in \alpha')$$

and by M^ϕ the exchange-matrix defined by

$$M^\phi(R, R') = \#(R \cap \phi(R')).$$

Note. If $\iota = \iota^\phi$ then $M^\iota = M^\phi$. Given a reflection ι or exchange-matrix M, there exists a bijection ϕ with $\iota = \iota^\phi$ or $M = M^\phi$ respectively.

PROPOSITION AND DEFINITION 1.3. *Let α be a level* 2 *finite set, M an exchange-matrix from α to α; then by the Young index $c(M)$ of M will be meant the common value of* $(\operatorname{sgn}\phi)c(\phi)$ *for all ϕ in $\mathfrak{S}(\cup\alpha)$ such that $M = M^\phi$.*

2. Some module-theoretic constructions. On the purely module-theoretic level, the constructions next to be defined are contained in [**1**]; the purpose of this section is to clarify the functorial dependence of these constructions on the level n finite sets involved ($n = 1, 2, 3$) as a preliminary to the constructions in Section Three. Throughout this section, A will denote a fixed commutative, associative ring with 1.

Let E be an A-module, D a finite set with n elements. For the class of questions under discussion, there is some advantage in replacing the usual

n-fold tensor product $E^{\otimes n}$, spanned by elements

$$e_1 \otimes \cdots \otimes e_n \qquad (e\text{'s in } E)$$

by the module $\otimes_D E = \otimes_D^A E$, spanned by elements

$$\bigotimes_{d \in D} e(d) \qquad (e \text{ any map } D \to E).$$

Of course there is an A-isomorphism, natural in E, between $E^{\otimes n}$ and $\otimes_D E$: the point is that this isomorphism is not natural in D (in the category of finite sets and bijections), since it depends on the choice of a particular ordering for D. Similarly, we shall replace the usual nth exterior power $\wedge^n E$, spanned by elements

$$e_1 \wedge \cdots \wedge e_n \qquad (e\text{'s in } E)$$

by the module $\wedge^D E = \wedge_A^D E$, spanned by elements

$$\bigwedge_{d \in E} e(d) \qquad (e \text{ any map } D \to E)$$

Here again, although the latter module is A-isomorphic (naturally in E) to $\wedge^n E$, this isomorphism cannot be chosen naturally in D (unless $n = 0$ or 1); there is rather (if $\#D \geq 2$) an arbitrary choice between two such isomorphisms, corresponding to the two orientation-classes of bijections

$$D \xrightarrow{\sim} \{1, \ldots, n\}$$

We thus regard $\otimes_D E$, $\wedge^D E$ as functors

$$\text{(4)} \qquad \textbf{Fin-1-Sets} \times \textbf{Mod}_A \to \textbf{Mod}_A$$

in two variables; the functorial dependence on the first variable D is specified as follows: a bijection $\sigma: D \to D'$ induces the isomorphisms

$$\bigotimes_\sigma E: \bigotimes_{D'} E \xrightarrow{\sim} \bigotimes_D E, \qquad \bigotimes_{d \in D'} e(d') \to \bigotimes_{d \in D} e(\sigma(d)),$$

$$\bigwedge_\sigma E: \wedge^{D'} E \to \wedge^D E, \qquad \bigwedge_{d \in D'} e(d') \to \bigwedge_{d \in D} e(\sigma(d)).$$

(We thus obtain a right action of $\mathfrak{S}(D)$ on $\otimes_D E$, yielding the usual right action of $\mathfrak{S}_n$ on $E^{\otimes n}$ if $E = \{1, \ldots, n\}$.)

We next modify similarly the functor $\wedge^{a_1, \ldots, a} E$ constructed in [**1**]: we define the functor

$$\text{(5)} \qquad \wedge = \wedge_A: \textbf{Fin-2-Sets} \times \textbf{Mod}_A \to \textbf{Mod}_A, \qquad (\alpha, E) \to \wedge_A^\alpha E,$$

contravariant in the first variable and covariant in the second, as follows.

DEFINITION 2.1 Let α be a level two finite set, E an A-module; then by

$$\bigwedge^{\alpha} E = \bigwedge^{\alpha}_{A} E$$

will be meant the R-module, defined by generators and relations as follows:

For each map e: $\bigcup \alpha \to E$ we assign a generating element for $\bigwedge^{\alpha}_{A} E$, which we shall denote by

$$\prod_{R \in \alpha} \bigwedge_{b \in R} e(b); \tag{6}$$

these are to generate $\bigwedge^{\alpha}_{R} E$ over A, with relations next to be described.

Let

$$\omega^{\alpha}(E, A) = \omega^{\alpha} \in \left(\bigwedge^{\alpha}_{A} E\right)^{E^{(\bigcup \alpha)}}$$

be the $(\bigcup \alpha)$-indexed function which assigns to each map e: $\bigcup \alpha \to E$ the generator (6); the relations over A on these generators, are then to be those generated over A by the requirement that ω^{α} have Young alternation in α.

If σ: $\alpha' \to \alpha$ is a level 2 morphism, then $\bigwedge(\sigma)$: $\bigwedge^{\alpha} E \to \bigwedge^{\alpha'} E$ is well-defined by the requirement that it map (6) into

$$\prod_{R' \in \alpha'} \bigwedge_{b' \in R'} e(\sigma(b')).$$

REMARK. If $|\alpha| = \langle a_1, \ldots, a_s \rangle$ then $\bigwedge^{\alpha} E$ is isomorphic to $\bigwedge^{a_1, \ldots, a_s} E$, naturally in E, but not in α.

DEFINITION 2.2. Let $\mathfrak{D}$ be a level 3 finite set, E an A-module; then by

$$\bigwedge^{\mathfrak{D}}_{A} E = \bigwedge^{\mathfrak{D}} E$$

will be meant the A-module

$$\bigotimes_{\alpha \in \mathfrak{D}} \bigwedge^{\alpha}_{A} E.$$

If σ: $\mathfrak{D}' \to \mathfrak{D}$ is a level 3 morphism, the A-isomorphism $\bigwedge(\sigma)$: $\bigwedge^{\mathfrak{D}} E \to \bigwedge^{\mathfrak{D}'} E$ is well-defined by the requirement that it map

$$\bigotimes_{\alpha \in \mathfrak{D}} \prod_{R \in \alpha} \bigwedge_{b \in R} e(b) \qquad \left(e \text{ any map } \bigcup{}^2 \mathfrak{D} \to E\right)$$

into

$$\bigotimes_{\alpha' \in \mathfrak{D}'} \prod_{R' \in \alpha'} \bigwedge_{b' \in R'} e(\sigma(b')).$$

REMARK. We thus have a functor

$$\bigwedge = \bigwedge_A : \textbf{Fin-3-Sets} \times \textbf{Mod}_A \to \textbf{Mod}_A, \tag{7}$$

contravariant in the first variable and covariant in the second. (There is an abuse of notation involved in using the same symbol $\bigwedge_A$ for the three functors (4), (5), (7)). Since all morphisms in **Fin-n-Sets** are isomorphisms, the fact $\bigwedge$ is contravariant in the first variable must be regarded as a choice of convention rather than as a fact of life; one could make it covariant by replacing $\bigwedge(\sigma)$ by $\bigwedge(\sigma^{-1})$.)

DEFINITION 2.3. If α is a level 2 finite set, we denote by $\mathbb{S}\alpha$ the level 3 finite set, whose elements are the singleton sets $\{R\}$ containing the elements R of α.

REMARK. Thus, if $|\alpha| = \langle a_1, \ldots, a \rangle$, then $\bigwedge^{\mathbb{S}\alpha} E$ is isomorphic (naturally in E, but not in α) to $\bigwedge^{a_1} E \otimes \cdots \otimes \bigwedge^{a} E$.

DEFINITION 2.4. Let α be a level 2 finite set, A any commutative ring. Denote by $A \cdot (\cup \alpha)$ the free A-module on the set $\cup\alpha$; then by the *Specht-Young A-module associated to* α will be meant the sub-A-module $SY_A(\alpha)$ of

$$\bigwedge_A^{\alpha}(A \cdot (\cup \alpha))$$

generated over A by the set of all

$$\prod_{R \in \alpha} \bigwedge_{b \in R} \pi b \qquad (b \in \mathfrak{S}(\cup \alpha)).$$

REMARK. It follows easily from results in [**1**] that SY_A is a functor from Fin-2-Sets to the category of free A-modules, and that there is a natural isomorphism

$$SY_A(\alpha) \approx SY_{\mathbf{Z}}(\alpha) \otimes_{\mathbf{Z}} A.$$

3. Exchange-transformations.

DEFINITION 3.1. By a level 2 oriented pair will be meant an ordered triple $\delta = (\alpha, \varepsilon, \alpha')$ where α, α' are level 2 finite sets and ε is an orientation from $\cup\alpha$ to $\cup\alpha'$ (note this implies $\# \cup \alpha = \# \cup \alpha'$).

PROPOSITION AND DEFINITION 3.2. *Let $\delta = (\alpha, \varepsilon, \alpha')$ be a level 2 oriented pair, and let ι be a reflection of α' in α; let E be an A-module.*

Then the following A-homomorphism has the same value for all bijections $\sigma\colon \bigcup\alpha' \xrightarrow{\sim} \bigcup \alpha$, such that $\iota = \iota^\sigma$, and will be called the interchange-transformation $\mathrm{INT}^\delta(\iota)$ *associated to ι and δ:*

$$\bigotimes_{\bigcup\alpha} E \to \wedge^{\alpha'} E, \qquad \bigotimes_{b\in\bigcup\alpha} e(b) \mapsto (\mathrm{sgn}_\varepsilon\,\sigma) \bigotimes_{R'\in\alpha'} \bigwedge_{b'\in R'} e(\sigma b').$$

PROPOSITION AND DEFINITION 3.3. *Let $\delta = (\alpha, \varepsilon, \beta)$ be a level 2 oriented pair, and let M be an exchange-matrix from α to α'. Let E be a module over the commutative ring A.*

Then, by the associated switch-transformation

$$SW^\delta(M)\colon \wedge^{\mathcal{S}\alpha} E \to \wedge^{\mathcal{S}\beta} E \tag{8}$$

will be meant the unique A-homomorphism whose composite with the canonical projection

$$\bigotimes_{\bigcup\alpha} E \to \wedge^{\mathcal{S}\alpha} E, \qquad \bigotimes_{b\in^2\alpha} e(b) \mapsto \bigotimes_{R\in\alpha} \bigwedge_{b\in R} e(b)$$

is the A-homomorphism

$$\sum \{\mathrm{INT}^\delta(\iota)\colon M = M^\iota\}\colon \bigotimes_{\bigcup\alpha} E \to \wedge^\beta E$$

the sum being extended over the set of all $\alpha!/M!$ reflections ι of β in α such that $M = M^\iota$. If $\alpha = \beta$ and ε is the orientation of the identity map, we write also $SW^\alpha(M)$ for (8).

PROPOSITION AND DEFINITION 3.4. *Let E be an A-module, and let α be a level 2 finite set.*

We denote by $\mathcal{G}^\alpha = \mathcal{G}^\alpha(E)$ the natural projection

$$\wedge^{\mathcal{S}\alpha} E \to \wedge^\alpha E, \qquad \bigotimes_{R\in\alpha} \bigwedge_{b\in R} e(b) \mapsto \prod_{R\in\alpha} \bigwedge_{b\in R} e(b)$$

and by $\mathcal{YQ}^\alpha = \mathcal{YQ}^\alpha(E)$ the natural transformation

$$\mathcal{YQ}^\alpha = \sum_M M!c(M)SW^\alpha(M)\colon \wedge^{\mathcal{S}\alpha} E \to \wedge^{\mathcal{S}\alpha} E$$

(*the sum being taken over all exchange-matrices M from α to α, and $c(M)$ denoting the Young index of* Def. 1.3)

Finally, we denote by $\mathcal{P}^\alpha = \mathcal{P}^\alpha(E)$ *the unique A-homomorphism which makes the following diagram commute*:

$$\begin{array}{ccccc} \wedge^{\mathcal{S}\alpha}E & & \xrightarrow{\mathcal{YA}^\alpha} & & \wedge^{\mathcal{S}\alpha}E \\ & \mathcal{P}^\alpha\searrow & & \nearrow\mathcal{J}^\alpha & \\ & & \wedge^\alpha E & & \end{array} \tag{9}$$

Definition 3.5. Let $\mathcal{D}$ be a level 3 finite set; then we define the natural transformations in the commuting diagram.

$$\begin{array}{ccccc} \wedge^{\mathcal{S}\cup\mathcal{D}} & & \xrightarrow{\mathcal{YA}^\mathcal{D}} & & \wedge^{\mathcal{S}\cup\mathcal{D}} \\ & \mathcal{P}^\mathcal{D}\searrow & & \nearrow\mathcal{J}^\mathcal{D} & \\ & & \wedge^\mathcal{D} & & \end{array} \tag{10}$$

as follow:

$$\mathcal{YA}^\mathcal{D} = \bigotimes_{\alpha\in\mathcal{D}} \mathcal{YA}^\mathcal{D}, \qquad \mathcal{P}^\mathcal{D} = \bigotimes_{\alpha\in\mathcal{D}} \mathcal{P}^\alpha, \qquad \mathcal{J}^\mathcal{D} = \bigotimes_{\alpha\in\mathcal{D}} \mathcal{J}^\alpha$$

Definition 3.6. By a *level* 3 *oriented pair* will be meant an ordered triple

$$\delta = (\mathcal{D}_1, \varepsilon, \mathcal{D}_2)$$

with $\mathcal{D}_1$, $\mathcal{D}_2$ level 3 finite sets, and ε an orientation from $\cup^2\mathcal{D}_1$ to $\cup^2\mathcal{D}_2$; if M is then an exchange matrix from $\cup\mathcal{D}_1$ to $\cup\mathcal{D}_2$, we define the associated *exchange-transformation* $EX^\delta(M)$ to be the natural transformation

$$EX^\delta(M) = \mathcal{P}^{\mathcal{D}_2} \circ \mathrm{INT}^{\delta'}(M) \circ \mathcal{J}^{\mathcal{D}_1}\colon \wedge^{\mathcal{D}_1} \to \wedge^{\mathcal{D}_2}$$

where $\delta' = (\cup\mathcal{D}_1, \varepsilon, \cup\mathcal{D}_2)$.

Remark. If A is a $\mathbf{Q}$-algebra, these exchange-transformations yield the generating set over A, promised in the introduction, for the A-module

$$\mathrm{Nat\,Tsf}_A(\mathcal{D}_1, \mathcal{D}_2)$$

of natural transformations from the functor

$$\wedge_A^{\mathcal{D}_1}\colon \mathbf{Mod}_A \to \mathbf{Mod}_A$$

into the functor $\wedge_A^{\mathcal{D}_2}$; the set-theoretical structure on $\mathcal{D}_1$, $\mathcal{D}_2$ needed for this construction is simply that of level 2 finite sets (together with an orientation ε needed to eliminate ambiguity of signs). On the contrary, some additional set-theoretic structure (involving arbitrary choice of orderings) is needed to select a basis for $\operatorname{Nat}\operatorname{Tsf}_A(\mathfrak{D}_1, \mathfrak{D}_2)$, consisting of the 'standard' exchange-transformations, as given by the two following defintions. We note a possible modification of this final step in the construction: once the arbitrary choices involved in Def. 3.7. have been made, the 'column-free' method of constructing functions with Young alternation given by Def. 1.1 and Def. 3.4 may be replaced by the procedure (cf. [**1**] and [**2**]) involving the usual Young quasi-idempotent; it turns out the combinatorial requirements of Def. 3.8 work *without modification* if the exchange-transformations are modified in this way.

DEFINITION 3.7. By a *level* 3 *ordering* $<$ of a level 3 finite set, will be meant a total ordering $<_{\mathfrak{D}}$ of the set $\mathfrak{D}$, together with the assignment to each $\alpha \in \mathfrak{D}$ of a total ordering $<_\alpha$ of α, and the assignment to each $R \in^2 \mathfrak{D}$ of a total ordering $<_R$ of R, subject to the requirement that $\alpha \in \mathfrak{D}$, R_1 and $R_2 \in \alpha$, $R_1 <_\alpha R_2 \Rightarrow \#R_1 \geq \#R_2$.

REMARKS. Thus, a level 3 finite set $\mathfrak{D}$, together with a level 3 ordering $<$, may be thought of as an ordered set of Young-Ferrars frames. Note that $<$ then induces a total ordering $\ll$ on $\cup\mathfrak{D}$, defined by:

(11) $R \ll R'$ if either:

$$R \in \alpha \in \mathfrak{D}, \quad R' \in \alpha' \in \mathfrak{D}, \quad \alpha <_{\mathfrak{D}} \alpha',$$

or

$$R \text{ and } R' \in \alpha \in \mathfrak{D}, \quad R <_\alpha R'.$$

DEFINITION 3.8. Let $(\mathfrak{D}, \alpha, \mathfrak{D}')$ be a level 3 oriented pair; let $<$, $<'$ be level 3 orderings for $\mathfrak{D}$, $\mathfrak{D}'$ respectively; then an exchange-matrix M from $\cup\mathfrak{D}$ to $\cup\mathfrak{D}'$ will be called *standard* with respect to $<$, $<'$ if it satisfies the two following conditions (where $\ll$ is given by (11), and $\ll'$ is defined similarly in terms of $<'$):

(i) If $\alpha \in \mathfrak{D}$, $R <_\alpha R_1$ (so $R \in \alpha$, $R_1 \in \alpha$), $S \in^2 \mathfrak{D}'$ then

$$\sum_{S' \ll R} M(R, S') \geq \sum_{S' \ll S} M(R_1, S').$$

(ii) If $\alpha \in \mathfrak{D}$, $S <_{\alpha'} S_1$, $R \in^2 \mathfrak{D}$ then

$$\sum_{R' \ll R} M(R', S) \geq \sum_{R' \ll R} M(R', S_1);$$

the corresponding exchange-transformation $EX^\delta(M)$ will then be called *standard* with respect to $<$ and $<'$.

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