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A REFORMULATION OF THE ARF INVARIANT ONE mod p PROBLEM AND APPLICATIONS TO ATOMIC SPACES

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A (mod p) atomic space is one whose lowest nonvanishing (mod p) homology group has dimension 1 and which has the property that all self-maps which induce isomorphisms on this lowest nonvanishing group are homotopy equivalences. An atomic space cannot be decomposed, up to homotopy, into a produce of other spaces and thus is, in some sense, an atom. In this paper we show that if p is an odd prime and $n > 1$ then $\Omega^3 S^{2n+1}$ and the homotopy-theoretic fibre of the double suspension Σ^2: $S^{2n-1} \to \Omega^2 S^{2n+1}$ are (mod p) atomic. Some indecomposability results are also obtained for the homotopy-theoretic fibre of the degree p map of ΩS^{2n+1}.

Introduction. In homotopy theory we can distinguish between the weak form of the Arf invariant problem which asks if a certain element in the Adams spectral sequence is an infinite cycle and the strong form which asks for a 3-cell complex with a nontrivial Bockstein and Steenrod operation. The strong form implies the weak form and it has been conjectured that they are equivalent. Ravenel's negative solution of the weak form of the problem for $p \geq 5$ shows (somewhat vacuously) that the conjecture holds in this case. (See [**19**].) If $p = 2$, it has been shown that the weak form of the problem is equivalent to the Kervaire problem. (See Browder [**4**].) Barratt and Mahowald have shown that divisibility of a certain Whitehead product by 2 implies the weak form of the (mod 2) Arf invariant problem. (See [**15**], Corollary 2.) In fact it is well-known that divisibility of this Whitehead product by 2 is equivalent to the strong form of the Arf invariant problem, although I have been unable to find all the details in the literature. §1 gives a proof of this equivalence and generalizes the result to odd primes. Throughout this paper, the term Arf invariant will refer to the *strong* form.

§2 proves a technical theorem which gives a sufficient condition for a self-map of a space to be a homotopy equivalence. The main results of the paper are in §3 where the results of §§1 and 2 are applied to show that certain spaces are atomic. In particular (after localizing at an odd prime p)

we obtain the following:

COROLLARY 3.4. *$\Omega S^{2n+1}\{p\}$ is atomic for all n such that $\pi^s_{2n(p-1)-2}$ has no elements of Arf invariant* 1 mod p *(where $S^{2n+1}\{p\}$ is the homotopy-theoretic fibre of the pth power map p: $S^{2n+1} \to S^{2n+1}$).*

COROLLARY 3.5. *If $p \geq 5$ and $n \neq 1$ or p then $\Omega S^{2n+1}\{p\}$ is atomic.*

THEOREM 3.7. *$C(n)$ is atomic for $n > 1$ (where $C(n)$ is the homotopy-theoretic fibre of the double suspension Σ^2: $S^{2n-1} \to \Omega^2 S^{2n+1}$).*

THEOREM 3.8. *$\Omega^3 S^{2n+1}$ is atomic for $n > 1$.*

I. A reformulation of the Arf invariant one problem. Let p be a prime. In this section $H_*(X)$ will denote $H_*(X; Z/pZ)$ and all spaces and maps will be assumed to have been localized at p. Let $P^n(k) = S^{n-1} \cup_k e^n$, $n \geq 2$, where k: $S^{n-1} \to S^{n-1}$ is of degree k. Homotopy with Z/kZ coefficients is defined by $\pi_n(X; Z/kZ) \equiv [P^n(k), X]$. Many of its properties can be found in **[16]**. If g: $X \to Y$, we let C_g denote the homotopy-theoretic cofibre of g.

Given f: $S^{m-1} \to S^0$ in π^s_{m-1}, $m > 1$, since f is torsion we can extend f to $\hat{f}$: $P^m(p^r) \to S^0$ for some r. Of course, $\hat{f}$ is not uniquely determined by f. We say that $\pi^s_{2n(p-1)-2}$ has an element of (strong) Arf invariant 1 mod p if there exists $C_{\hat{f}}$ in which the Steenrod operation P^n (respectively: Sq^{2n}) acts nontrivially, where $f \in \pi^s_{2n(p-1)-2}$.

Let $C(n)$ denote the homotopy-theoretic fibre of Σ^2: $S^{2n-1} \to \Omega^2 S^{2n+1}$. $C(n)$ is $2np - 4$ connected. Let f denote the composite

$$S^{2np-3} \to C(n) \to S^{2n-1}$$

where the first map is a generator of $\pi_{2np-3}(C(n)) \cong Z/pZ$. It is well known that $f = 0 \Leftrightarrow$ there exists an element of Hopf invariant 1 in $\pi^s_{2n(p-1)-1}$. (See **[10]**, Proposition 5.4, p. 300.)

Note. Here, and elsewhere, "equals" means equals within the set of homotopy classes of maps.

Suppose that $n \neq 1$ and that in addition if $p = 2$, $n \neq 2$ or 4 so that $f \neq 0$. Notice that if $p = 2$ then $f = [\iota_{2n-1}, \iota_{2n-1}]$.

THEOREM 1.1. *The following are equivalent*:
(a) *$f = pg$ for some g*
(b) *There exists h: $P^{2np-2}(p) \to \Omega^2 S^{2n+1}$ such that $h_* \neq 0$ on H_{2np-2}.*
(c) *$\pi^s_{2n(p-1)-2}$ has an element of Arf invariant* 1.

Proof. Let $J_k(X)$ denote the kth stage of the James construction on X. That is, $J_k(X) = X^k/\sim$ where

$$(x_1,\ldots,x_{j-1},\ *\ ,x_{j+1},\ldots,x_k) \sim (x_1,\ldots,x_{j-1},x_{j+1},\ *\ ,x_{j+2},\ldots,x_k).$$

Let b' be the statement

(b′) There exists $h\colon P^{2np-2}(p) \to \Omega J_{p-1}(S^{2n})$ such that $h_* \neq 0$ on H_{2np-2}.

b $\Leftrightarrow$ b′: There is a fibration

$$J_{p-1}(S^{2n}) \to \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1}$$

due to James [13] for $p = 2$ and Toda [23] for $p > 2$. So the pair $(\Omega^2 S^{2n+1}, \Omega J_{p-1}(S^{2n}))$ is $2np - 2$ connected and thus b $\Leftrightarrow$ b′.

a $\Rightarrow$ b′: There is a fibration

$$S^{2n-1} \xrightarrow{i} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$$

due to James [13] for $p = 2$ and Toda [23] for $p > 2$. From the definition of f the composite

$$S^{2np-3} \xrightarrow{f} S^{2n-1} \xrightarrow{\Sigma^2} \Omega^2 S^{2n+1}$$

is null homotopic. Because of the connectivity of the pair $(\Omega^2 S^{2n+1}, \Omega J_{p-1}(S^{2n}))$ it follows that

$$S^{2np-3} \xrightarrow{f} S^{2n-1} \xrightarrow{i} \Omega J_{p-1}(S^{2n})$$

is null homotopic. Therefore $p(ig) = i_\#(pg) = i_\#(f) = 0$. Thus there exists $h\colon P^{2np-2}(p) \to \Omega J_{p-1}(S^{2n})$ such that

$$\begin{array}{ccc} S^{2np-3} & \xrightarrow{j} & P^{2np-2}(p) \\ \downarrow g & & \downarrow h \\ S^{2n-1} & \xrightarrow{i} & \Omega J_{p-1}(S^{2n}) \end{array}$$

is homotopy commutative. We must show that $Th \neq 0$.

Suppose to the contrary that $Th = 0$. Then there exists $h'\colon P^{2np-2}(p) \to S^{2n-1}$ such that $h = ia$. So $i(g - aj) = ig - iaj = ig - hj = 0$. Thus $\Sigma^2(g - aj) = 0$. But

$$\ker \Sigma^2\colon \pi_{2np-3}(S^{2n-1}) \to \pi_{2np-3}(\Omega^2 S^{2n+1})$$

is Z/pZ, generated by f. So $g - aj = \lambda f$ for some λ. Multiplying by p gives $pg - p(hj) = p\lambda f = 0$. Also, $p(aj) = 0$ since multiplication by p kills Im $j^{\#}$. Therefore $pg = 0$. But $pg = f$ and so we have a contradiction.

Thus $Th \neq 0$ in $\pi_{2np-2}(\Omega S^{2np-1}; Z/pZ)$. It follows from the mod p Hurewicz isomorphism (see Neisendorfer [**16**], Theorem 3.8) that $T_* h_* \neq 0$ on H_{2np-2}. Since T_* is an isomorphism on H_{2np-2}, $h_* \neq 0$ on H_{2np-2}.

b$'$ $\Rightarrow$ c: Let h': $P^{2np-1}(p) \to J_{p-1}(S^{2n})$ be the adjoint of h.

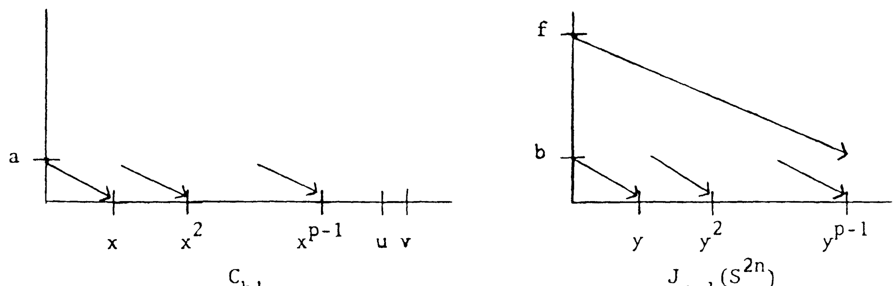
$$H^q(C_{h'}) = \begin{Bmatrix} Z/pZ & q = 2kn, & k \leq p \\ Z/pZ & q = 2np-1 & \\ 0 & \text{otherwise} & \end{Bmatrix}.$$

Let x be a generator of $H^{2n}(C_{h'})$. Then $x^{p-1} \neq 0$. We show $x^p \neq 0$.

The map of homotopy-theoretic fibrations

$$\begin{array}{ccccc} \Omega J_{p-1}(S^{2n}) & \to & EJ_{p-1}(S^{2n}) & \to & J_{p-1}(S^{2n}) \\ \downarrow \Omega\gamma & & \downarrow & & \downarrow \gamma \\ \Omega C_{h'} & \to & EC_{h'} & \to & C_{h'} \end{array}$$

induces a map of cohomology Serre spectral sequences.



$d(x^{p-1} \otimes a) = x^p$. If $x^p = 0$ then there exists $e \in H^{2np-2}(C_{h'})$ such that $d(e) = x^{p-1} \otimes a$. But then diagram chasing shows that $(\Omega\gamma)^*(e) = f$ so $(\Omega\gamma)^*$ is surjective on H^{2np-2}. Therefore $(\Omega\gamma)_*$ is injective on H_{2np-2}. So $(\Omega\gamma)_* h_* \neq 0$ on H_{2np-2}. But $(\Omega\gamma)h$ factors as

$$P^{2np-2}(p) \to \Omega P^{2np-1}(p) \xrightarrow{\Omega h'} \Omega J_{p-1}(S^{2n}) \xrightarrow{\Omega\gamma} \Omega C_{h'}$$

and $\gamma h' = 0$. From this contradiction, we conclude $x^p \neq 0$. So $P^n x = x^p \neq 0$.

Let w be the composite

$$P^{2np}(p) \xrightarrow{\Sigma h'} \Sigma J_{p-1}(S^{2n}) \to S^{2n+1}$$

where the second map is the adjoint of $J_{p-1}(S^{2n}) \to \Omega S^{2n+1}$. We have a map of homotopy-theoretic cofibrations

$$\begin{array}{ccccc} P^{2np}(p) & \to & \Sigma J_{p-1}(S^{2n}) & \to & \Sigma C_{h'} \\ \| & & \downarrow & & \downarrow \\ P^{2np}(p) & \overset{w}{\to} & S^{2n+1} & \to & C_w . \end{array}$$

Diagram chasing shows that P^n acts nontrivially in C_w so that $\pi^s_{2n(p-1)-2}$ has an element of Arf invariant 1.

c $\Rightarrow$ b′: Let w': $P^{2n(p-1)-1}(p^r) \to S^0$ be a stable map such that P^n acts nontrivially in $C_{w'}$. From the commutative square

$$\begin{array}{ccc} S^{m-1} & \overset{k}{\to} & S^{m-1} \\ \| & & \Big| t \\ S^{m-1} & \overset{kt}{\to} & S^{m-1} \end{array}$$

we get an induced map of homotopy-theoretic cofibres $P^m(k) \to P^m(kt)$. Let w'' be the composite

$$P^{2n(p-1)-1}(p) \to P^{2n(p-1)-1}(p^r) \overset{w'}{\to} S^0.$$

We have a map of homotopy-theoretic cofibrations

$$\begin{array}{ccccc} P^{2n(p-1)-1}(p) & \overset{w''}{\to} & S^0 & \to & C_{w''} \\ \downarrow \alpha & & \| & & \downarrow \\ P^{2n(p-1)-1}(p^r) & \overset{w'}{\to} & S^0 & \to & C_{w'} . \end{array}$$

Since α induces an isomorphism on $H^{2n(p-1)-1}$ diagram chasing in the long exact cohomology sequence shows that P^n also acts nontrivially in $C_{w''}$. Since $\pi_{2np}(S^{2n+1}; Z/pZ)$ is stable, it contains a representative w: $P^{2np}(p) \to S^{2n+1}$ for w''. Let $\tilde{w}$: $P^{2np-1}(p) \to \Omega S^{2n+1}$ be the adjoint of w. For connectivity reasons, $\tilde{w}$ lifts to v: $P^{2np-1}(p) \to J_{p-1}(S^{2n})$. Let F be the homotopy-theoretic fibre of γ: $J_{p-1}(S^{2n}) \to C_v$. Let ϕ denote the mod p Hurewicz homomorphism. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \cdots & \to & \pi_{2np-2}(\Omega F: Z/pZ) & \to & \pi_{2np-2}(\Omega J_{p-1}(S^{2n}); Z/pZ) & \to & \pi_{2np-2}(\Omega C_\iota : Z/pZ) & \to & \pi_{2np-3}(\Omega F; Z/pZ) & \to & \cdot \\ & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \\ 0 & \to & H_{2np-2}(\Omega F) & \to & H_{2np-2}(\Omega J_{p-1}(S^{2n})) & \to & H_{2np-2}(\Omega C_\iota) & \to & H_{2np-3}(\Omega F) & \to & 0 \end{array}$$

where the bottom line is the Serre exact homology sequence. Diagram chasing in the cohomology Serre spectral sequences show that $H^{2np-2}(\Omega C_v) \to H^{2np-2}(\Omega J_{p-1}(S^{2n}))$ is the zero map, so its dual is also zero. According to the mod p Hurewicz Isomorphism Theorem (see Neisendorfer [**16**], Theorem 3.8)
$\phi\colon \pi_{2np-3}(\Omega F; Z/pZ) \to H_{2np-3}(\Omega F)$ is an isomorphism and
$\phi\colon \pi_{2np-2}(\Omega F\colon Z/pZ) \to H_{2np-2}(\Omega F)$ is an epimorphism. So diagram chasing shows that $\phi\colon \pi_{2np-2}(\Omega J_{p-1}(S^{2n}); Z/pZ) \to H_{2np-2}(\Omega J_{p-1}(S^{2n}))$ is an epimorphism. This statement is equivalent to (b′).

b′ ⇒ a: Let $h\colon P^{2np-2} \to \Omega J_{p-1}(S^{2n})$ induce a nonzero map on H_{2np-2}. Since T_* is an isomorphism on H_{2np-2}, $Th \neq 0$. However the composite

$$S^{2np-3} \xrightarrow{j} P^{2np-2}(p) \xrightarrow{h} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$$

must be null homotopic for connectivity reasons. So there exists $g\colon S^{2np-3} \to S^{2n-1}$ such that

$$\begin{array}{ccc} S^{2np-3} & \xrightarrow{j} & P^{2np-2}(p) \\ \downarrow g & & \downarrow h \\ S^{2n-1} & \xrightarrow{i} & \Omega J_{p-1}(S^{2n}) \end{array}$$

is homotopy commutative. Since multiplication by p kills $\operatorname{Im} j^{\#}$, $i_{\#}(pg) = pi_{\#}(g) = pj^{\#}(h) = 0$. Therefore $\Sigma^2(pg) = 0$. But $\ker \Sigma^2\colon \pi_{2np-3}(S^{2n-1}) \to \pi_{2np-3}(\Omega^2 S^{*n+1})$ is Z/pZ generated by f. So $pg = \lambda f$ for some $\lambda \in Z/pZ$. It remains to show that $\lambda \neq 0$. So suppose $\lambda = 0$. Then there exists $\hat{g}\colon P^{2np-2}(p) \to S^{2n-1}$ such that $g = \hat{g}j$. Since $(h - i\hat{g})j = hj - ig = 0$, there exists $e\colon S^{2np-2} \to \Omega J_{p-1}(S^{2n})$ such that $h - ig = ec$ where $c\colon P^{2np-2}(p) \to S^{2np-2}$ is the map which collapses S^{2np-3} to a point. We have $Tec = Th - Tig = Th \neq 0$. Therefore $Te \neq 0$. But this implies that the fibration

$$S^{2n-1} \xrightarrow{i} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$$

has a cross-section, up to homotopy, and so i induces a split monomorphism on homotopy groups. This is a contradiction since $f \neq 0$ but $if = 0$. Therefore $\lambda = 0$. □

II. Self-maps. The purpose of this section is to prove the technical Theorem 2.3 which gives a sufficient condition for a map to be a homotopy equivalence. We begin with some algebraic preliminaries.

LEMMA 2.1. *Let V be either a finite group or a finite dimensional vector space. Let $f: V \to V$. Let $W = \varinjlim_f V$. Let $\theta: V \to W$ be the canonical map. Then θ is onto.*

Proof. $\operatorname{Im} f^{n+1} \subset \operatorname{Im} f^n$ for all n. Since V is finite or finite dimensional, these images stabilize. For convenience, write our direct system

$$V_0 \xrightarrow{f} V_1 \xrightarrow{f} V_2 \to \cdots \to V_n \to \cdots \to W$$

where $V_i = V$ for all i. Let $w \in W$. Find a representative x for w in V_m for some m. Pick N large enough so that $\operatorname{Im} f^{N+k} = \operatorname{Im} f^N$ for all k. Now $f^N x$ belonging to V_{N+m} is another representative of w. Since $f^N x \in \operatorname{Im} f^N = \operatorname{Im} f^{N+m}$, $f^N x = f^{N+m} x'$ for some $x' \in V_0$, and $\theta(x') = w$. □

Given an abelian group G, let $t(G)$ denote its torsion subgroup.

LEMMA 2.2. *Let G be a finitely generated R-module for some $R \subset Q$. Let $f: G \to G$. Let $H = \varinjlim_f G$ and let $\theta: G \to H$ be the canonical map. Then*

(1) coker θ *is divisible.*

(2) coker θ *is a torsion group.*

(3) $t(H) \subset \operatorname{Im} \theta|_{t(G)}$.

(4) $G \xrightarrow{\theta} \operatorname{Im} \theta$ *is a split epimorphism. Further, this splitting can be chosen to be natural when restricted to* $t(\operatorname{Im} \theta)$.

REMARK. The second statement in (4) means the following:
Let

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \downarrow \alpha & & \downarrow \alpha \\ G' & \xrightarrow{f'} & G' \end{array}$$

be commutative. Then there exists $s: \operatorname{Im} \theta \to G$ and $s': \operatorname{Im} \theta' \to G'$ such that $\theta s = 1$, $\theta' s' = 1$, and

$$\begin{array}{ccc} t(\operatorname{Im} \theta) & \xrightarrow{s} & G \\ \downarrow & & \downarrow \alpha \\ t(\operatorname{Im} \theta') & \xrightarrow{s'} & G' \end{array}$$

commutes.

Proof. Since

$$G \xrightarrow{\theta} H \to \operatorname{coker}\theta \to 0$$

is exact,

$$G \otimes F \xrightarrow{\theta\otimes F} H \otimes F \to \operatorname{coker}\theta \otimes F \to 0$$

is exact for all F. Thus for any field F, $\operatorname{coker}\theta \otimes F = 0$ by Lemma 2.1.

Setting $F = Z/pZ$, we conclude that $p \operatorname{coker}\theta = \operatorname{coker}\theta$. Since this is true for all p, $\operatorname{coker}\theta$ is divisible.

Setting $F = Q$, we get that $\operatorname{coker}\theta$ is a torsion group.

To show (3):
Let $H' = \varinjlim_f t(G)$ and let $H'' = \varinjlim G/t(G)$. Since $\varinjlim$ preserves exactness we get

$$\begin{array}{ccccccccc} 0 & \to & t(G) & \to & G & \to & G/t(G) & \to & 0 \\ & & \downarrow & & \downarrow\theta & & \downarrow & & \\ 0 & \to & H' & \to & H & \to & H'' & \to & 0 \end{array}$$

By Lemma 2.1, $t(G) \to H'$ is onto, so it suffices to show that $t(H)$ goes to zero under $H \to H''$. But this is clear, since H'', being a direct limit of torsion-free groups is torsion-free.

To show (4):
Since G is finitely generated, so is $\operatorname{Im}\theta$. Therefore

$$\operatorname{Im}\theta \cong t(\operatorname{Im}\theta) \oplus F$$

where F is free. So any splitting defined on $t(\operatorname{Im}\theta)$ can be extended to $\operatorname{Im}\theta$. Thus it suffices to do the case where G is a torsion group.

As in the proof of Lemma 2.1, write the direct system as

$$G_0 \xrightarrow{f} G_1 \xrightarrow{f} G_2 \to \cdots \to G_n \to \cdots \to H$$

where $G_i = G$ for all i. Find N such that $\operatorname{Im} f^N = \operatorname{Im} f^{N+k}$ for all k.

$$\begin{array}{ccccc} I \equiv \operatorname{Im} f^N & \to & G_0 & & \\ \downarrow f^N & & \downarrow f^N & \searrow\theta & \\ I' \equiv \operatorname{Im} f^N & \to & G_N & \xrightarrow{j} & H \\ & & \xrightarrow{\cong} & & H \end{array}$$

$H = \operatorname{Im}\theta$ since G is a torsion group. Since $\operatorname{Im} f^N$ is stable, j is an isomorphism. Also $f^k\colon \operatorname{Im} f^N \xrightarrow{\cong} \operatorname{Im} f^{N+k}$. In particular $f^k|_I$ is injective for all k. So $f^N|_I$ is injective. Since I and I' are finite groups of the same order, $f^N\colon I \xrightarrow{\cong} I'$. Thus θ splits.

Observe that our splitting depends upon our choice of N, but is canonical once N has been chosen. Thus given a diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \downarrow \alpha & & \downarrow \alpha \\ G' & \xrightarrow{f'} & G' \end{array}$$

to get naturality, it is merely necessary to use the same N in constructing the two splittings. Of course, N must be chosen large enough so that both systems have stabilized. □

Let p be a prime. Let X be a topological space. Let

$$P^n_*\colon H_q(X; Z/pZ) \to H_{q-2(p-1)n}(X; Z/pZ)$$

be the hom-dual of P^n (respectively: Sq^n_* dual to Sq^n). This defines a (left) A_*-module structure on $H_*(X; Z/pZ)$ where A_* is the opposite algebra of the mod p Steenrod Algebra A. Let $\hat{A}$ denote the subalgebra of A generated by $\{P^n\}_{n=1}^{\infty}$ (respectively: generated by $\{\mathrm{Sq}^n\}_{n=2}^{\infty}$) and let $\hat{A}_*$ denote its opposite algebra. Let $E^r(X)$ denote the mod p homology Bockstein Spectral Sequence of X and let $\beta^{(r)}$ be the rth Bockstein. Let ϕ be the Hurewicz homomorphism and let

$$r\colon \pi_*(X) \to \pi_*(X; Z/pZ) \quad \big(\text{respectively: } H_*(X) \to H_*(X; Z/pZ)\big)$$

denote reduction mod p.

We now define some subspaces of $PH_*(X; Z/pZ)$, the primitives in the homology of X. Let

$$\operatorname{Ann} H_*(X; Z/pZ) = \big\{x \in PH_*(X; Z/pZ) \mid x \in \ker P^I_* \text{ for all } P^I_* \in A_*\big\}.$$

Let

$$\widehat{\operatorname{Ann}}\, H_*(X; Z/pZ) = \big\{x \in PH_*(X; Z/pZ) \mid x \in \ker P^I_* \text{ for all } P^I_* \in \hat{A}_*\big\}.$$

Let

$$SH_*(X; Z/pZ) = \big\{x \in H_*(X; Z/pZ) \mid x = f_*(\iota_n) \text{ for some } f\colon S^n \to X\big\}.$$

Let

$$MH_*(X; Z/pZ)$$

$$= \{x \in \text{Ann } H_*(X; Z/pZ) \mid \beta^{(r)}x = 0 \text{ for all } r \text{ and either}$$

$$\begin{aligned} &(1) \quad x = 0; \text{ or} \\ &(2) \quad x \text{ represents a nonzero class in } E^\infty(X); \text{ or} \\ &(3) \quad x \text{ represents a nonzero class in } E^r(X), \text{ but} \\ &\qquad x \in \text{Im } \beta^{(r)}(\text{Ann } H_*(X; Z/pZ)) \mid \text{for some } r\}. \end{aligned}$$

Let

$$MSH_*(X; Z/pZ) = MH_*(X; Z/pZ) \cap SH_*(X; Z/pZ).$$

The main result of this section is:

THEOREM 2.3. *Let X be a simply connected space having the homotopy type of a CW complex of finite type. Let f: $X \to X$. Suppose that f_*^N restricted to $MSH_*(X; Z/pZ)$ is an injection for all N. Then $f_{(p)}$ is a homotopy equivalence.*

Proof. Following Cohen, Moore, and Neisendorfer [**9**], §4, let $Y = \varinjlim_f X$, the infinite mapping telescope of f. Then $\pi_*(Y) = \varinjlim_{f_\#} \pi_*(X)$ and $H_*(X) = \varinjlim_{f_*} H_*(X)$. Similar statements hold for mod p homotopy and homology.

We have a canonical map θ: $X \to Y$ inducing the obvious maps on homotopy and homology. Let F be the homotopy-theoretic fibre of θ: $X \to Y$. Suppose $F_{(p)}$ is not contractible. Find n such that $F_{(p)}$ is $(n-1)$ connected but not n connected. We show that there exists a nonzero x in $MSH_*(X; Z/pZ)$ such that $x \in \text{Im } i_*$, where i: $F \to X$. Given such an x, the hypothesis implies that $\theta_* x \neq 0$. But this is impossible since $x \in \text{Im } i_*$. Thus $F_{(p)}$ is contractible and so $\theta_{(p)}$ is a homotopy equivalence. Therefore $f_{(p)}$ is a homotopy equivalence. So it suffices to show the existence of such an x.

Case 1. $\pi_n(F; Z/pZ) \neq 0$.

By the mod p Hurewicz isomorphism, $H_n(F; Z/pZ) \cong \pi_n(F; Z/pZ) \neq 0$. Let $x = i_*(w)$ for some nonzero w in $H_n(F; Z/pZ)$. By the Serre exact homology sequence and Lemma 2.1,

$$i_*: H_n(F; Z/pZ) \to H_n(X/ Z/pZ)$$

is injective so $x \neq 0$. Since

$$w \in SH_*(F; Z/pZ) \cap \text{Ann } H_*(F; Z/pZ)$$

and $\beta^{(s)}w = 0$ for all s, x has these properties also. It remains to show that x satisfies either condition (2) or condition (3) in the definition of $MH_*(X; Z/pZ)$.

If x persists to a nonzero element in $E^\infty(X)$ we are finished, so suppose not. Then for some m, x is nonzero in $E^m(X)$, but $x \in \text{Im } \beta^{(m)}$. We must show that $x \in \beta^{(m)}(\widehat{\text{Ann}}\, H_*(X; Z/pZ))$.

Find $b \in H_n(X)$ such that order $b = p^m$ and $rb = x$. We adjust b so that it lies in $\text{Im } i_*$ as follows:

By Lemma 2.2, find splittings such that

$$\begin{array}{ccccc} H_n(Y) & \supset & t(\text{Im } \theta_*) & \overset{s}{\to} & H_n(X) \\ & & \downarrow r & & \downarrow r \\ H_n(Y; Z/pZ) & \supset & \text{Im } \theta_* & \overset{s}{\to} & H_n(X; Z/pZ) \end{array}$$

commutes. Since $b \in t(H_n(X))$, $\theta_* b \in t(\text{Im } \theta)$ so $s\theta_* b$ is defined. Let $b' = b - s\theta_* b$. We have

$$rb' = rb - rs\theta_* b = x - sr\theta_* b = x - s\theta_* rb = x - s\theta_* x = x$$

since $\theta_* x = \theta_* i_* w = 0$. Because $p^m b = 0$, it follows that $p^m b' = 0$. Since $rb' = x$ and x is nonzero in $E^m(X)$, order $b' = p^m$. Finally,

$$\theta_* b' = \theta_* b - \theta_* s\theta_* b = \theta_* b - \theta_* b = 0$$

so $b' = i_* a$ for some $a \in H_n(F)$ by the Serre exact homology sequence

$$\cdots \to H_{n+1}(X) \overset{\theta_*}{\to} H_{n+1}(Y) \overset{\partial}{\to} H_n(F) \overset{i_*}{\to} H_n(X) \overset{\theta_*}{\to} H_n(Y) \to \cdots.$$

Next we adjust a so that order a = order b'.

Since $i_*(p^m a) = p^m b' = 0$, $p^m a = \partial y$ for some y belonging to $\text{coker } \theta_*$: $H_{n+1}(X) \to H_{n+1}(Y)$. By Lemma 2.2, $\text{coker } \theta_*$ is divisible so $y = p^m y'$ for some y' belonging to $\text{coker } \theta_*$. Let $a' = a - \partial y'$. Then $i_* a' = i_* a - i_* \partial y' = b'$ and $p^m a' = p^m a - p^m \partial y' = p^m a - \partial y = 0$.

Let g be the image of a' under

$$H_n(F) \to H_n(F_{(p)}) \underset{\cong}{\overset{\theta^{-1}}{\to}} \pi_n(F_{(p)}).$$

Since $p^m g = 0$, g extends to $\hat{g}$: $P^{n+1}(p^m) \to F_{(p)}$.

Let u_n and v_{n+1} be a basis for $\tilde{H}_*(P^{n+1}(p^m);\ Z/pZ)$ such that $\beta^{(m)}v = u$. Then under the composite

$$H_*(P^{n+1}(p^m);\ Z/pZ) \xrightarrow{g_*} H_*(F_{(p)};\ Z/pZ) \xrightarrow{i_*} H_*(X_{(p)};\ Z/pZ) \cong H_*(X;\ Z/pZ)$$

u goes to x. So if we let x' be the image of v, then

$$x' \in \widehat{\mathrm{Ann}}\, H_*(X;\ Z/pZ) \quad \text{and} \quad \beta^{(m)}x' = x.$$

Case 2. $\pi_n(F;\ Z/pZ) = 0$.

According to the Universal Coefficient Theorem, (see Neisendorfer [**16**], Proposition 1.4), there is a short exact sequence

$$0 \to \pi_n(F_{(p)}) \otimes Z/pZ \to \pi_n(F_{(p)};\ Z/pZ) \to \mathrm{Tor}(\pi_{n-1}(F_{(p)});\ Z/pZ) \to 0.$$

So

$$\pi_n(F_{(p)}) \otimes Z/pZ \cong \pi_n(F_{(p)};\ Z/pZ) \cong \pi_n(F;\ Z/pZ) = 0.$$

Therefore $\pi_n(F_{(p)})$ is divisible. Since $\pi_n(X_{(p)})$ is a finitely generated $Z_{(p)}$-module, it follows that $i_\#\colon \pi_n(F_{(p)}) \to \pi_n(X_{(p)})$ is the zero map. Thus

$$\pi_n(F_{(p)}) \cong \mathrm{coker}\,\theta_\#\colon \pi_{n+1}(X_{(p)}) \to \pi_{n+1}(Y_{(p)}).$$

Since $F_{(p)}$ is $(n-1)$ connected but not n connected we can find a nonzero a in $\pi_n(F_{(p)})$. By Lemma 2.2, $\pi_n(F_{(p)})$ is a torsion group so $p^s a = 0$ for some s. By replacing a by $p^{s-1}a$, we may assume that $pa = 0$.

Because $i_\#(a) = 0$, $a = \partial y$ for some y belonging to $\pi_{n+1}(Y_{(p)})$. Since $\partial(py) = pa = 0$, $py = \theta_\#(g)$ for some g in $\pi_{n+1}(X_{(p)})$. Let x be the image of g under

$$\pi_{n+1}(X_{(p)}) \xrightarrow{r} \pi_{n+1}(X_{(p)};\ Z/pZ) \xrightarrow{\phi} H_{n+1}(X_{(p)};\ Z/pZ) \cong H_{n+1}(X;\ Z/pZ).$$

By construction $x \in SH_*(X;\ Z/pZ)$. We next show that $x \in \mathrm{Im}\, i_*$ and $x \neq 0$.

Let $S^n\{p\}$ denote the homotopy-theoretic fibre of $p\colon S^n \to S^n$. Let $k\colon P^n(p) \to S^n\{p\}$ be the inclusion of the n-skeleton into $S^n\{p\}$. Let g' and

y' be the adjoints of g and y respectively. From the homotopy commutative square

$$\begin{array}{ccc} S^n & \xrightarrow{g'} & \Omega X_{(p)} \\ \downarrow p & & \downarrow \Omega\theta \\ S^n & \xrightarrow{y'} & \Omega Y_{(p)} \end{array}$$

we get an induced map of homotopy-theoretic fibres $b\colon S^n\{p\} \to \Omega F_{(p)}$.

We have a homotopy commutative diagram

$$(1)\qquad \begin{array}{ccccccc} S^{n-1} & \searrow & & & & & \\ \downarrow j & & \Omega S^n & \xrightarrow{\Omega y'} & \Omega^2 Y_{(p)} & \xrightarrow{\partial} & \Omega F_{(p)} \\ P^n(p) & & \downarrow & & \downarrow & & \| \\ & \overset{k}{\searrow} & S^n\{p\} & \xrightarrow{b} & \Omega F_{(p)} & = & \Omega F_{(p)} \\ & \underset{r}{\searrow} & \downarrow & & \downarrow \Omega i & & \downarrow \\ & & S^n & \xrightarrow{g'} & \Omega X_{(p)} & \to & EF_{(p)} \\ & & \downarrow p & & \downarrow & & \downarrow \\ & & S^n & \xrightarrow{y'} & \Omega Y_{(p)} & \xrightarrow{\partial} & F_{(p)} \end{array}$$

where the columns are homotopy-theoretic fibrations and the collapse map, denoted r here, induces reduction mod p on homotopy groups.

(2)

$$\begin{array}{ccccc} \pi_{n+1}(F_{(p)}; Z/pZ) & \xrightarrow{i_\#} & \pi_{n+1}(X_{(p)}; Z/pZ) & \cong & \pi_{n+1}(X; Z/pZ) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ H_{n+1}(F_{(p)}; Z/pZ) & \xrightarrow{i_*} & H_{n+1}(X_{(p)}; Z/pZ) & \cong & H_{n+1}(X; Z/pZ) \end{array}$$

From Lemma 2.1, the long exact homotopy sequence, and the Serre exact homology sequence, $i_\#$ and i_* are injective. Also, since we are doing Case 2, the leftmost map is an isomorphism by the mod p Hurewicz Isomorphism Theorem.

From (1) we see that $r^{\#}(g') = (\Omega i)_{\#}(bk)$. In other words, in $\pi_{n+1}(X_{(p)};\ Z/pZ)$, $rg = i_{\#}$(adjoint of bk). Therefore $x \in \operatorname{Im} i_*$ and to show $x \neq 0$, by diagram chasing from (2), it suffices to show $bk \neq 0$. But following (1) across the top, bkj is the adjoint of a, which is nonzero. Thus $x \neq 0$.

To show $x \in MH_*(X;\ Z/pZ)$ it now suffices to show that x represents a nonzero element in $E^{\infty}(X)$. Suppose this is not true. Then there exists z belonging to $H_{n+1}(X_{(p)})$ such that $rz = x$ and order $z < \infty$. Let s be the splitting of Lemma 2.2 chosen so that

$$\begin{array}{ccccc} H_{n+1}(Y_{(p)}) & \supset & t(\operatorname{Im}\theta_*) & \xrightarrow{s} & H_{n+1}(X_{(p)}) \\ \downarrow r & & \downarrow r & & \downarrow r \\ H_{n+1}(Y_{(p)};\ Z/pZ) & \supset & \operatorname{Im}\theta_* & \xrightarrow{s} & H_{n+1}(X_{(p)};\ Z/pZ) \end{array}$$

commutes. Since $z \in t(H_{n+1}(X_{(p)}))$, $s\theta_* z$ is defined. Let $z' = z - s\theta_* z$. Since $x \in \operatorname{Im} i_*$, $\theta_* x = 0$ and so

$$rz' = rz - rs\theta_* z = x - s\theta_* x = x.$$

Also

$$\theta_* z' = \theta_* z - \theta_* s\theta_* z = \theta_* z - \theta_* z = 0$$

so $z' = i_*(w)$ for some $w \in H_{n+1}(F_{(p)})$.

We have $i_* rw = rz' = x = i_*\phi(bk)$. Since i_* is a monomorphism, this implies $rw = \phi(bk)$. Therefore diagram chasing from

$$\begin{array}{ccccccccc} \cdots & \to & \pi_{n+1}(F_{(p)}) & \xrightarrow{r} & \pi_{n+1}(F_{(p)};\ Z/pZ) & \xrightarrow{j^{\#}} & \pi_n(F_{(p)}) & \to & \cdots \\ & & \downarrow\phi & & \cong\downarrow\phi & & \cong\downarrow\phi & & \\ \cdots & \to & H_{n+1}(F_{(p)}) & \xrightarrow{r} & H_{n+1}(F_{(p)};\ Z/pZ) & \to & H_n(F_{(p)}) & \to & \cdots \end{array}$$

shows that $j^{\#}(bk) = 0$. But as noted earlier, $j^{\#}(bk)$ is the adjoint of a, which is nonzero. This is a contradiction, so x must be nonzero in $E^{\infty}(X)$. □

III. Some atomic spaces. In this section homology is assumed to be with Z/pZ coefficients unless stated otherwise.

DEFINITION 3.1. Let $X_{(p)}$ be $(n-1)$ connected. Then X is called *mod p atomic* if:

(1) $H_n(X) = Z/pZ$

(2) $f\colon X \to X$ such that f induces an isomorphism on $H_n(X)$ implies that $f_{(p)}$ is a homotopy equivalence.

X is called *atomic* if it is mod p atomic for all p.

Clearly X atomic implies that X is indecomposable in the sense that if $X \approx Y \times Z$ then either $Y \approx \{\text{pt.}\}$ or $Z \approx \{\text{pt.}\}$.

Some trivial examples of atomic spaces are S^n and ΩS^{2n+1}. A nontrivial example is given by:

THEOREM 3.2 (*Cohen and Mahowald*). *$\Omega^2 S^{2n+1}$ is atomic for $n > 1$.*

Proof. See [**5**].

REMARK 1. The corresponding statement for $n = 1$ is that $\Omega^2 S^3\langle 3\rangle$ is atomic. This is also proved in [**5**].

REMARK 2. If $p > 2$, $\Omega^2 S^{2n}$ cannot be mod p atomic since after localization at p,

$$\Omega S^{2n} \approx S^{2n-1} \times \Omega S^{4n-1}.$$

From now on we shall assume that p is odd and we shall take the term "atomic" to mean mod p atomic for all odd primes p.

THEOREM 3.3. *Assume $n > 1$. Then $MSH_q(\Omega S^{2n+1}\{p\}) = 0$ for $q > 2n - 1$ if and only if there are no elements of Arf invariant $1 \bmod p$ in $\pi^s_{2n(p-1)-2}$.*

COROLLARY 3.4. *$\Omega S^{2n+1}\{p\}$ is atomic for all n such that $\pi^s_{2n(p-1)-2}$ has no elements of Arf invariant $1 \bmod p$. In particular if $n \neq p^k$ for some k then $\Omega S^{2n+1}\{p\}$ is atomic.* □

Applying the theorem of Ravenel ([**19**]) gives:

COROLLARY 3.5. *If $p \geq 5$ and $n \neq 1$ or p then $\Omega S^{2n+1}\{p\}$ is atomic.* □

Proof of Theorem 3.3. As a Hopf Algebra over the Steenrod Algebra, $H_*(\Omega S^{2n+1}\{p\})$ is given by the following (see [**6**]):

$$H_*(\Omega S^{2n+1}\{p\}) = \bigotimes_{k=0}^{\infty} E[a_{2np^k-1}] \otimes \bigotimes_{k=1}^{\infty} Z/pZ[b_{2np^k-2}] \otimes Z/pZ[c_{2n}]$$

with the generators primitive and

$$\begin{aligned} \beta c_{2n} &= a_{2n-1}, && \\ \beta a_{2np^k-1} &= b_{2np^k-2}, && k \geq 1, \\ P^1_* a_{2np^k-1} &= P^1_* b_{2np-2} = P^1_* c_{2n} = 0, && \\ P^1_* b_{2np^k-2} &= -\left(b_{2np^{k-1}-2}\right)^p, && k \geq 2, \\ P^{p^r}_* &\equiv 0, && r \geq 1. \end{aligned}$$

From this description we see that a_{2n-1} and b_{2np-2} form a basis for $MH_*(\Omega S^{2n+1}\{p\})$. It remains to show that $b_{2np-2} \in SH_*(\Omega S^{2n+1}\{p\})$ if and only if there exist elements of Arf invariant 1 mod p in $\pi^s_{2n(p-1)-2}$.

Suppose first that there exists h: $S^{2np-2} \to \Omega S^{2n+1}\{p\}$ such that $h_*(\iota_{2np-2}) = b_{2np-2}$. Let h': $S^{2np-3} \to \Omega^2 S^{2n+1}\{p\}$ be the adjoint of h. Let σ_* denote the homology suspension. Then $\sigma_* h'_*(\iota_{2np-3}) = h_*(\iota_{2np-2}) \neq 0$, so $h'_*(\iota_{2np-3}) \neq 0$.

Localize at p and let π: $\Omega^2 S^{2n+1} \to S^{2n-1}$ be the map constructed by Cohen, Moore, and Neisendorfer in [7]. According to Cohen, Moore, and Neisendorfer ([8], Theorem 1.1) $\pi \circ \Sigma^2 = p$: $S^{2n-1} \to S^{2n-1}$ and $\Sigma^2 \circ \pi = p$: $\Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$. Thus we get a map of homotopy-theoretic fibrations

$$\begin{array}{ccccccc} \Omega^3 S^{2n+1} & \to & \Omega^2 S^{2n+1}\{p\} & \xrightarrow{k} & \Omega^2 S^{2n+1} & \xrightarrow{p} & \Omega^2 S^{2n+1} \\ \| & & \downarrow i & & \downarrow \pi & & \| \\ \Omega^3 S^{2n+1} & \to & C(n) & \xrightarrow{j} & S^{2n-1} & \xrightarrow{\Sigma^2} & \Omega^2 S^{2n+1}. \end{array} \qquad *$$

From the far left square we can see that i_* induces an isomorphism on H_{2np-3} so $i_* h'_* \neq 0$ on H_{2np-3}. Therefore by the Hurewicz isomorphism, ih' is a generator of $\pi_{2np-3}(C(n))$. Let $f = jih'_*$. Since Σ^2: $\pi_{2np-3}(S^{2n-1}) \to \pi_{2np-3}(\Omega^2 S^{2n+1})$ is onto, $kh' = \Sigma^2 g$ for some g belonging to $\pi_{2np-3}(S^{2n-1})$. Thus $f' = jih' = \pi kh' = \pi\Sigma^2 g = pg$. So there exist elements of Arf invariant 1 mod p in $\pi^s_{2n(p-1)-2}$ by Theorem 1.1.

Conversely if there exist elements of Arf invariant 1 mod p in $\pi^s_{2n(p-1)-2}$ then there exists g in $\pi_{2np-3}(S^{2n-1})$ such that $pg = js$ where s is a generator of $\pi_{2np-3}(C(n))$. The middle square of $(*)$ is a homotopy pullback and $js = pg = \pi\Sigma^2 g$, so s and $\Sigma^2 g$ can be used to define a map $h'\colon S^{2np-3} \to \Omega^2 S^{2n+1}\{p\}$. It is easy to see that if we let $h\colon S^{2np-2} \to \Omega S^{2n+1}\{p\}$ be the adjoint of h' then $h_*(\iota_{2np-2}) = b_{2np-2}$ so that $b_{2np-2} \in SH_*(\Omega S^{2n+1}\{p\})$. □

Assuming $n > 1$, as a Hopf algebra

$$H_*(\Omega^3 S^{2n+1}) \cong \bigotimes_{\substack{k\geq 1\\ j\geq 0}} E\left[a_{2(np^k-1)p^j-1}\right] \otimes \bigotimes_{\substack{k\geq 1\\ j\geq 1}} Z/pZ\left[b_{2(np^k-1)p^j-2}\right] \otimes \bigotimes_{k\geq 0} Z/pZ\left[c_{2np^k-2}\right]$$

with the generators primitive. (See [**6**]). For convenience we will write these generators as $a_n(j,k)$, $b_n(j,k)$, and $c_n(k)$ respectively and when no confusion is possible, we will drop the n.

From the Nishida relations the actions of β and P^1_* are as follows:

$$\begin{aligned}
\beta a(j,k) &= b(j,k), && j\geq 1,\\
\beta c(k) &= a(0,k), && k\geq 1,\\
\beta a(0,k) &= \beta b(j,k) = \beta c(0) = 0, &&\\
P^1_* b(j,k) &= (b(j-1),k)^p, && j\geq 2,\\
P^1_* c(k) &= -(c(k-1))^p, && k\geq 2,\\
P^1_* c(1) &= (n-1)(c(0))^p, &&\\
P^1_* a(j,k) &= P^1_* b(1,k) = P^1_* c(0) = 0. &&
\end{aligned}$$

Since $H_*(\Omega^3 S^{2n+1}) \to H_*(C(n))$ simply projects off of $c(0)$, we can deduce the action of β and P^1_* in $H_*(C(n))$. We easily calculate that $MH_*(\Omega^3 S^{2n+1})$ must be contained in the subspace generated by the elements

(1) $a(0,1)$,
(2) $b(1,k)$, $k \geq 1$,
(3) $(c(0))^{p^t}$, $t \geq 0$.

Similarly $MH_*(C(n))$ is contained in the subspace generated by

(1) $a(0,1)$,
(2) $b(1,k)$, $k \geq 1$.

LEMMA 3.6. *Assume $n > 1$. Then for all k, $b(k,1)$ does not belong to $SH_*(\Omega^3 S^{2n+1})$ and its image does not belong to $SH_*(C(n))$.*

Proof. If $b(1, k)$ were spherical in $SH_*(\Omega^3S^{2n+1})$ then its image in $H_*(C(n))$ would be spherical also, so it suffices to prove the second statement.

Toda ([**22**]) constructs a homotopy-theoretic fibration

$$\Omega S^{2np-1}\{p\} \to C(n) \overset{H}{\to} C(np).$$

(See also [**20**], Theorem 13.) Since $H_*(b_n(1, k)) = b_{np}(1, k-1)$ for $k \geq 2$, if $b_n(1, k) \in SH_*(C(n))$ then $b_{np}(1, k-1) \in SH_*(C(np))$. So it suffices to prove the lemma for $k = 1$.

Suppose that $b(1, 1) \in SH_*(C(n))$. Because $|b(1, 1)| < 2np^2 - 4$ which is the connectivity of $C(np)$, $b(1, 1)$ lifts to an element in $SH_{2mp-2}(\Omega S^{2m+1}\{p\})$ for $m = np - 1$. But this contradicts Theorem 3.3, since m is not a power of p. Therefore $b(1, 1) \notin SH_*(C(n))$. □

THEOREM 3.7. *$C(n)$ is atomic for $n > 1$.*

Proof. Using Lemma 3.6 we see that $MSH_q(C(n)) = 0$ for $q > 2np - 3$ so this is immediate from Theorem 2.3. □

THEOREM 3.8. *Ω^3S^{2n+1} is atomic for $n > 1$.*

Proof. Let f: $\Omega^3S^{2n+1} \to \Omega^3S^{2n+1}$ such that f induces an isomorphism on $H_{2n-2}(\Omega^3S^{2n+1})$. It is well known that $a(0, 1) \notin SH_*(\Omega^3S^{2n+1})$ by the non-existence of elements of Hopf invariant 1 in $\pi_{2n(p-1)-1}$, the argument being similar to the proof of the $b' \Rightarrow c$ step in the proof of Theorem 1.1. Thus using Lemma 3.6, $MSH_*(\Omega^3S^{2n+1})$ is contained in the subspace generated by $\{(c(0)^{p^t})\}_{t=0}^{\infty}$. So by Theorem 2.3, to show that f is a homotopy equivalence it suffices to show that $f_*^N((c(0))^{p^t}) \neq 0$ for all t and for all N. But this is easy to see by considering the action of f^* on cohomology with $Z_{(p)}$ coefficients. □

REMARK. F. Cohen, F. Peterson, and the author have recently shown that Ω^3S^{2n+1} is also mod 2 atomic for $n > 1$, using different techniques.

For our final application, suppose $n > 1$ and let $D(n)$ denote the homotopy-theoretic fibre of π: $\Omega^2S^{2n+1} \to S^{2n-1}$. Using [**8**], Lemma 2.1 and the fact that

$$\Sigma^2 \circ \pi = p\colon \Omega^2S^{2n+1} \to \Omega^2S^{2n+1}$$

we can construct a homotopy-theoretic fibration

$$D(n) \to \Omega^2 S^{2n+1}\{p\} \to C(n).$$

The main theorem of [**20**] asserts that this fibration splits when $n = p$.

THEOREM 3.9. *The homotopy-theoretic fibration*

$$D(n) \to \Omega^2 S^{2n+1}\{p\} \to C(n)$$

cannot split unless $\pi^s_{2n(p-1)-2}$ *contains an element of Arf invariant* 1.

Proof. If the fibration splits then the generator of $\pi_{2np-3}(C(n))$ lifts to a map from S^{2np-3} to $\Omega^2 S^{2n+1}\{p\}$ and this results in the same situation as occurred in the proof of Theorem 3.3 □

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