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INTEGRAL COMPARISON THEOREMS FOR RELATIVE HARDY SPACES OF SOLUTIONS OF THE EQUATIONS $\Delta u = Pu$ ON A RIEMANN SURFACE

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We consider two partial differential equations of elliptic type $\Delta u = Pu$ and $\Delta u = Qu$, which are invariantly defined on a Riemann surface R. M. Nakai showed that the Banach spaces PB, QB of bounded solutions on R of these equations are isometrically isomorphic under the condition $\int_R |P - Q| < +\infty$, where it is assumed that R is of hyperbolic type. Let PH_e^p and QH_e^p, $1 < p \leq +\infty$, be the relative Hardy spaces of quotients of solutions of the preceding equations by elliptic measures of R. In this paper we shall prove that the above condition is also sufficient for PH_e^p and QH_e^p to be isometrically isomorphic. For this purpose we shall introduce a mapping between the P-Martin and Q-Martin boundaries of R, and give some properties of this mapping.

1. Introduction. Let R be a hyperbolic Riemann surface and P a density on R, that is, a non-negative Hölder continuous function on R which depends on the local parameter $z = x + iy$ in such a way that the partial differential equation

$$\Delta u = Pu, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2, \tag{1.1}$$

is invariantly defined on R. A real valued function f is said to be a P-harmonic function (or P-solution) on an open set U of R, if f has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on U.

The totality of bounded P-harmonic functions on R is denoted by $PB(R)$. Then, $PB(R)$ is a Banach space with the uniform norm

$$\|f\| = \sup_{z \in R} |f(z)|.$$

H. L. Royden [**10**] considered the pair of differential equations (1.1) and

$$\Delta u = Qu, \tag{1.2}$$

where Q is another density on R, and he proved that, if the densities P and Q satisfy the condition:

$$c^{-1}Q \leq P \leq cQ \tag{1.3}$$

outside a compact set of R, then there exists an isometric isomorphism between the Banach spaces $PB(R)$ and $QB(R)$. On the other hand, concerning this comparison problem M. Nakai [**8**] gave a different criterion from (1.3) for $PB(R)$ and $QB(R)$ to be isomorphic and proved the following theorem: If two densities P and Q on R satisfy the condition

$$(1.4) \qquad \int_R |P(z) - Q(z)| \{G^P(z, w_0) + G^Q(z, w_1)\} \, dx \, dy < +\infty$$

for some points w_0 and w_1 in R, where $G^P(z, w)$ and $G^Q(z, w)$ are Green's functions of R associated with (1.1) and (1.2) respectively, then Banach spaces $PB(R)$ and $QB(R)$ are isomorphic.

A. Lahtinen [**2**] considered the equation (1.1) for densities P which he called acceptable densities. Acceptable densities can also have negative values, and so, P-harmonic functions do not obey the usual maximum principle. He gave generalizations of Nakai's comparison theorem for acceptable densities and also showed, in [**3**], that for non-negative densities Royden's condition (1.3) is a special case of Nakai's condition (1.4).

The P-elliptic measure of R is, by definition, a P-harmonic function on R which takes the constant 1 at the ideal boundary of R and is denoted by e^P. A quotient of a P-harmonic function on R by e^P is called a e-P-harmonic function. Then, the relative Hardy class, denoted by $PH_e^p(R)$, $1 \leq p \leq +\infty$, of e-P-harmonic functions is defined by the way analogous to that of the Hardy class $H^p(R)$ of harmonic functions on R. Recently, the present author [**12**] showed existence of an isometric isomorphism between the relative Hardy spaces $PH_e^p(R)$ and $QH_e^p(R)$, $1 < p \leq +\infty$, related to the equations (1.1) and (1.2) respectively, when the densities P and Q satisfy the above Royden's condition (1.3). And, he also has considered in [**11**] a comparison problem between the relative Hardy space $PH_e^p(R)$, $1 < p \leq +\infty$, and the Hardy space $H^p(R)$ of harmonic functions on R under the Nakai's condition (1.4) in which $Q \equiv 0$.

The purpose of this paper is to extend the theorems cited above to one in which we assume Nakai's condition (1.4) to be valid: if P and Q satisfy the condition (1.4), then the relative Hardy classes $PH_e^p(R)$ and $QH_e^p(R)$, $1 < p \leq +\infty$, are isometrically isomorphic. For the sake of this it is necessary to consider a measurable transformation t^{PQ}: $\Delta_{PQ}^0 \to \Delta_{QP}^0$ between subsets Δ_{PQ}^0 and Δ_{QP}^0 of P-Martin and Q-Martin boundaries of R. In [**12**] we have already constructed the measurable transformation t_{PQ}: $\Delta_{PQ}^- \to \Delta_{QP}^-$ between subsets of P-Martin and Q-Martin boundaries of R, which will be extended to the transformation t^{PQ}: $\Delta_{PQ}^0 \to \Delta_{QP}^0$, that is, $\Delta_{PQ}^0 \supset \Delta_{PQ}^-$, $\Delta_{QP}^0 \supset \Delta_{QP}^-$ and $t^{PQ} = t_{PQ}$ on Δ_{PQ}^-. For fundamental properties of P-harmonic functions we refer to the works of Myrberg [**5**] and Royden [**10**].

2. Reduced functions and mapping of P-solutions to Q-solutions. By a regular region we shall always mean a connected open set in the Riemann surface R whose boundary is composed of at most a countable number of analytic curves clustering nowhere in R. A sequence $\{R_n\}$ of relatively compact regular regions in R is called an exhaustion of R if $\bar{R}_n \subset R_{n+1}$ and $R = \bigcup_{n=1}^{\infty} R_n$.

Let K be a relatively compact regular region and f a continuous function on the boundary ∂K of K. Then, there is a unique continuous function u on the closure $\bar{K}$ of K which is P-harmonic on K and is equal to f on the boundary ∂K of K. This function u is the solution of Dirichlet's problem on K for boundary value f with respect to the equation (1.1), which is denoted by P_f^K. The notation Q_f^K is also understood as above. And, for a lower semi-continuous or upper semi-continuous function f on the boundary of K we can also define P_f^K and Q_f^K by taking a sequence of continuous functions converging to f, which are P-harmonic and Q-harmonic on K respectively.

For two densities P and Q, $G^P(z, w)$ and $G^Q(z, w)$ are Green's functions of R with poles w associated with the equations (1.1) and (1.2) respectively. For a regular region D Green's functions of D with poles w in D associated these equations are denoted by $G^P(D, z, w)$ and $G^Q(D, z, w)$ respectively. We refer to Myrberg [5] for the existence and properties of Green's function of the equation (1.1).

DEFINITION 2.1. Let K be a relatively compact regular region in R. We define transformations $T_{PQ}^K f$ and $T_{QP}^K f$ of real valued bounded continuous function f defined on K as follows:

$$T_{PQ}^K f(z) = f(z) + \frac{1}{2\pi}\int_K (P(w) - Q(w)) G^Q(K, w, z) f(w)\, du\, dv$$

and

$$T_{QP}^K f(z) = f(z) + \frac{1}{2\pi}\int_K (Q(w) - P(w)) G^P(K, w, z) f(w)\, du\, dv,$$

where $w = u + iv$.

The next lemma follows directly from Green's formula (C. F. Nakai [8] and Lahtinen [2]).

LEMMA 2.1. *For a continuous function f on the boundary of a relatively compact regular region K, $T_{PQ}^K(P_f^K) = Q_f^K$ on $\bar{K}$.*

LEMMA 2.2. *Let u be a continuous function on the closure of a relatively compact regular region K which is P-harmonic on K. Then, it follows that*

$$T^K_{PQ}u = T^K_{PQ}\left(P^K_{u|\partial K}\right) = Q^K_{(T^K_{PQ}u)|\partial K} \quad on\ K.$$

Proof. From the preceding lemma this lemma follows. □

DEFINITION 2.2. For two densities P and Q on R we denote by $P_Q(R)$ the class of all those P-harmonic functions f on R which satisfy the condition:

$$(2.1) \qquad \int_R |P(z) - Q(z)| G^Q(z, w_0) |f(z)| dx\, dy < +\infty$$

for some point w_0 in R, and by $Q_P(R)$ the class of all those Q-harmonic functions g on R which satisfy the condition:

$$(2.2) \qquad \int_R |Q(z) - P(z)| G^P(z, w_0) |g(z)| dx\, dy < +\infty$$

for some point w_0 in R.

If $f \in P_Q(R)$ and $g \in Q_P(R)$, then (2.1) and (2.2) hold at all points of R by Harnack's inequality. $P_Q(R)$ and $Q_P(R)$ are real linear spaces with respect to the usual definitions of addition and scalar multiplication of real numbers.

DEFINITION 2.3 (Nakai [**8**]). Let f be in $P_Q(R)$. Then, the linear transformation $T_{PQ}f$ of f is defined by

$$(2.3) \quad T_{PQ}f(z) = f(z) + \frac{1}{2\pi}\int_R (P(w) - Q(w)) G^Q(z, w) f(w)\, du\, dv,$$

where $w = u + iv$. For g in $Q_P(R)$ $T_{QP}g$ is defined by

$$(2.4) \quad T_{QP}g(z) = g(z) + \frac{1}{2\pi}\int_R (Q(w) - P(w)) G^P(z, w) g(w)\, du\, dv.$$

An open set D in R is said to be regular whenever its boundary ∂D is composed of at most a countable number of analytic curves clustering nowhere in R. Let $D = \bigcup_{n=1}^{\infty} D^n$ be the decomposition of D into connected components D^n of D, where each D^n is a regular region in R.

Taking a regular open set D of R in place of the surface R in Definition 2.2 we define the linear spaces $P_Q(D)$ and $Q_P(D)$ as follows: $f \in P_Q(D)$ if and only if

$$\int_{D^n} |P(z) - Q(z)| G^Q(D^n, z, w_n) |f(z)| dx\, dy < +\infty$$

for every n, where w_n is some point in D^n. $Q_P(D)$ is also defined by the same way. Then, the linear transformation $T_{PQ}^D f$ of f in $P_Q(D)$ is defined by

$$T_{PQ}^D f \mid D^n = T_{PQ}^{D^n} f, \quad \text{if } P \not\equiv Q \text{ on } D^n;$$
$$T_{PQ}^D f \mid D^n = f, \qquad \text{if } P \equiv Q \text{ on } D^n,$$

where $f \mid E$ is the restriction of f to the set E.

LEMMA 2.3 (*Lahtinen's lemma* [2]). *Let $\{D_i\}$ be an increasing sequence of regular open sets of R such that $\bigcup_{i=1}^{\infty} D_i = R$. For a positive P-solution u on R and a sequence $\{u_i\}$ of P-solutions u_i in $P_Q(D_i)$ such that $\lim_{i \to +\infty} u_i = u$ and there exists v in $P_Q(R)$ with $|u_i| \leq v$, then $T_{PQ}u$ is well defined and*

(i) $\lim_{i \to +\infty} T_{PQ}^{D_i} u_i = T_{PQ} u$,

(ii) *$T_{PQ}u$ is Q-harmonic on R.*

This lemma was given by Lahtinen in the case that the sequence $\{D_i\}$ is an exhaustion of the Riemann surface R, that is, each D_i is relatively compact regular region in R.

Proof. For any z in R we may suppose that z is in D_i. Let D_i' be the connected component of D_i which contains the point z. Since the sequence $\{G^Q(D_i', z, w)\}$ of Green's functions of regular regions D_i' converges increasingly to Green's functions $G^Q(z, w)$ of R, Lebesgue's bounded convergence theorem implies that

$$\lim_{i \to +\infty} \int_{D_i'} (P(w) - Q(w)) G^Q(D_i', w, z) u_i(w)\, du\, dv$$
$$= \int_R (P(w) - Q(w)) G^Q(w, z) u(w)\, du\, dv,$$

from which it follows that

$$\lim_{i \to +\infty} T_{PQ}^{D_i} u_i(z) = \lim_{i \to +\infty} T_{PQ}^{D_i'} u_i(z) = T_{PQ} u(z). \qquad \square$$

Let R^* be a metrizable compactification of the Riemann surface R and denote by Δ the ideal boundary of R in this compatification, that is, $\Delta = R^* - R$. Now, we recall properties of the reduced function of a P-harmonic function on R with respect to a compact subset A of Δ.

A closed subset F of R will be said to be regular if its boundary ∂F consists of at most a countable number of analytic curves clustering nowhere in R. For a positive continuous function u on a regular closed subset F of R, let u_n be a function on the boundary $\partial(R_n - F)$ such that

$$u_n \mid \overline{R}_n \cap \partial F = u \mid \overline{R}_n \cap \partial F$$

and

$$u_n \mid \partial R_n \cap (R - F) = 0.$$

Then, P-harmonic functions $P_{u_n}^{R_n - F}$ form an increasing sequence, whose limit is P-harmonic on $R - F$ and denoted by $P_{u_*}^{R-F}$. For a compact set A in Δ there exists a sequence $\{F_i^*\}$ of closed sets in R^* converging decreasingly to A such that the interior of F_i^* contains A and $R \cap F_i^*$ is a regular closed set in R. In the following, for a set F^* in R^* we shall denote by F the set $F^* \cap R$.

Definition 2.4 (Martin [4], Brelot [1], Nakai [7]). Let u be a positive P-harmonic function on R, and let $\{F_i^*\}$ be a sequence of closed sets in R^* given as above. Then, the sequence $\{P_{u_*}^{R-F_i}\}$ converges decreasingly to a P-harmonic function which is called the reduced function of u relative to the set A in Δ, and which is denoted by $L_A^P u$. Similarly, for positive Q-harmonic function v on R, we have

$$L_A^Q v = \lim_{i \to +\infty} Q_{v_*}^{R-F_i}.$$

Theorem 2.4. *Let A be a compact subset of the ideal boundary Δ of R with respect to the compatification R^*. For a positive P-harmonic function u in $P_Q(R)$ we have*

$$T_{PQ}\left(L_A^P u\right) = L_A^Q\left(T_{PQ} u\right) \quad \textit{on } R.$$

Proof. For a sequence $\{F_i^*\}$ of closed sets in R^* used in Definition 2.4 the sequence of P-harmonic functions $P_{u_*}^{R-F_i}$ converges to $L_A^P u$ as $n \to +\infty$, and $| P_{u_*}^{R-F_i} | \leq u$ for each i, where $F_i = F_i^* \cap R$. Then, Lemma 2.3 gives that

$$T_{PQ}\left(L_A^P u\right) = \lim_{i \to +\infty} T_{PQ}^{R-F_i}\left(P_{u_*}^{R-F_i}\right).$$

And, since for an exhaustion $\{R_n\}$ of R we have

$$\lim_{n\to+\infty} P_{u_n}^{R_n-F_i} = P_{u_*}^{R-R_i}$$

and

$$\left|P_{u_n}^{R_n-F_i}\right| \le u, \qquad u \in P_Q(R),$$

it follows, from Lemmas 2.2 and 2.3, that

$$\begin{aligned}(2.5)\quad T_{PQ}^{R-F_i}\left(P_{u_*}^{R-F_i}\right) &= \lim_{n\to+\infty} T_{PQ}^{R_n-F_i}\left(P_{u_n}^{R_n-F_i}\right)\\ &= \lim_{n\to+\infty} Q^{R_n-F_i} = \lim_{n\to+\infty} Q^{R_n-F_i}_{(T_{PQ}^{R_n-F_i}u)_n} = Q^{R-F_i}_{(T_{PQ}^{R-F_i}u)_*},\end{aligned}$$

where the equality

$$(2.6)\qquad T_{PQ}^{R_n-F_i}u = T_{PQ}^{R-F_i}u = u \quad \text{on } R_n \cap \partial F_i$$

is applied.

By the definitions of T_{PQ} and $T_{PQ}^{R-F_i}$, the inequality

$$G^Q(R-F_i, z, w) \le G^Q(z,w), \qquad z, w \in R - F_i,$$

implies that

$$\left|T_{PQ}u(z) - T_{PQ}^{R-F_i}u(z)\right| \le q(z), \qquad z \in R - F_i,$$

where

$$q(z) = \frac{1}{\pi}\int_R |P(w) - Q(w)|G^Q(z,w)u(w)\,du\,dv.$$

Since $q(z)$ is a potential with kernel $G^Q(z,w)$ of the measure

$$\frac{1}{\pi}|P(w) - Q(w)|u(w)\,du\,dv,$$

as in the case of harmonic Green potentials we can show that $L_A^Q q = 0$ on R (C. F. Brelot [**1**]). Then, we have

$$\left|Q^{R-F_i}_{(T_{PQ}u)_*} - Q^{R-F_i}_{(T_{PQ}^{R-F_i}u)_*}\right| \le Q^{R-F_i}_{q_*},$$

by which the equality

$$\lim_{i\to+\infty} Q^{R-F_i}_{q_*} = L_A^Q q = 0$$

shows that

$$(2.7)\qquad \lim_{i\to+\infty} Q^{R-F_i}_{(T_{PQ}u)_*} = \lim_{i\to+\infty} Q^{R-F_i}_{(T_{PQ}^{R-F_i}u)_*}.$$

Therefore, we have, by (2.5), that

$$T_{PQ}\left(L_A^P u\right) = \lim_{i \to +\infty} Q^{R-F_i}_{(T_{PQ}u)_*} = L_A^Q\left(T_{PQ}u\right). \qquad \square$$

DEFINITION 2.5. We denote by $P'_Q(R)$ the class of all those P-harmonic functions in $P_Q(R)$ whose transformation by T_{PQ} belongs to the class $Q_P(R)$, and by $Q'_P(R)$ the class of all those Q-harmonic functions in $Q_P(R)$ whose transformation by T_{QP} belongs to the class $P_Q(R)$.

DEFINITION 2.6. We denote by $P^0_Q(R)$ the class of all those P-harmonic functions u in $P'_Q(R)$ for which $T_{QP}T_{PQ}u = u$ on R, and by $Q^0_P(R)$ the class of all those Q-harmonic functions v in$Q'_P(R)$ for which $T_{PQ}T_{QP}v = v$ on R.

Later on it will be shown that, if the densities P and Q satisfy Nakai's condition (1.4), then $P^0_Q(R)$ and $Q^0_P(R)$ are not empty.

By the definition, it is evident that $P^0_Q(R) \subset P'_Q(R)$ and $Q^0_P(R) \subset Q'_P(R)$. And, classes $P'_P(R)$, $P^0_Q(R)$ (resp. $Q'_P(R)$, $Q^0_P(R)$) are linear subspaces of $P_Q(R)$ (resp. $Q_P(R)$). It may be shown that T_{PQ} is an isomorphism between linear spaces $P^0_Q(R)$ and $Q^0_P(R)$, and T_{QP} is its inverse.

We recall the definition of the P-elliptic (or P-harmonic) measure of a Riemann surface from the work of H. Royden **[10]**. For R_n in an exhaustion $\{R_n\}$ of R, let e^P_n be the P-harmonic function on R_n continuous on its closure which is identically one on the boundary ∂R_n of R_n. For $P \geq 0$ we have $0 < e^P_n < 1$. Since the maximum principle implies that the functions e^P_n form a monotone decreasing sequence of positive P-harmonic functions, this sequence converges uniformly on each compact set in R to a non-negative P-harmonic function e^P, which is called the P-elliptic (or P-harmonic) measure of R. Similarly the Q-elliptic measure of R is also defined and is denoted by e^Q. The P-elliptic measure e^P is either identically zero or else everywhere positive. In the second case we say that the pair (R, P) is hyperbolic provided $P \not\equiv 0$. The P-elliptic measure e^P of R may be characterized as the largest P-harmonic function on R which is bounded by 1.

The following theorems give a sufficient condition for the P-elliptic measure e^P to belong to the class $P'_Q(R)$ and $P^0_Q(R)$.

THEOREM 2.5. *If the densities P, Q on R satisfy Nakai's condition (cf. Nakai* **[8]**):

$$(2.8) \qquad \int_R |P(z) - Q(z)| \{G^P(z, w_0) + G^Q(z, w_1)\}\, dx\, dy < +\infty$$

for some points w_0, w_1 in R, then the P-elliptic measure e^P of R belongs to the class $P'_Q(R)$ and the Q-elliptic measure e^Q of R belongs to the class $Q'_P(R)$. In this case the transform $T_{PQ}e^P$ is the Q-elliptic measure e^Q and $T_{QP}e^Q$ is the P-elliptic measure e^P.

Proof. By the inequality $e^P < 1$ and Nakai's condition (2.8), Lemmas 2.1 and 2.3 imply that

$$\begin{aligned} T_{PQ}e^P &= \lim_{n\to+\infty} T_{PQ}^{R_n} e_n^P \\ &= \lim_{n\to+\infty} e_n^Q = e^Q \qquad \text{on } R. \end{aligned}$$

And, (2.8) gives that $e^P \in P_Q(R)$ and $e^Q \in Q_P(R)$. Therefore, we have that $e^P \in P'_Q(R)$ and $e^Q \in Q'_P(R)$. □

THEOREM 2.6. *If the densities P, Q on R satisfy the condition* (2.8), *then the P-elliptic measure e^P of R belongs to the class $P_Q^0(R)$ and the Q-elliptic measure e^Q of R belongs to the class $Q_P^0(R)$.*

Proof. This theorem is an immediate consequence of Theorem 2.5. □

3. Relation between the P-Martin and Q-Martin boundary. Nakai [7] studied the Martin theory [4] for the equation (1.1) on a Riemann surface R and showed that the situation was similar to that of harmonic case as was treated by Martin. Let R_P^* be the compatification of R in this sense, which is called the P-Martin compactification of R. Let Δ_P be the ideal boundary $R_P^* - R$. The P-Martin kernel with origin z_0 in R is denoted by $K^P(z, a)$, $(z, a) \in R \times R_P^*$, which satisfies that $K^P(z_0, a) = 1$, $a \in R_P^*$, and is finitely continuous on $R \times \Delta_P$. For points a_1, a_2 in R_P^* the distance between them is given by

$$d_P(a_1, a_2) = \sum_{n=1}^{\infty} 2^{-n} \sup_{z\in R_n} \left| \frac{K^P(z, a_1)}{1 + K^P(z, a_1)} - \frac{K^P(z, a_2)}{1 + K^P(z, a_2)} \right|,$$

where $\{R_n\}$ is an exhaustion of R. P-Martin kernel is also written by K_a^P, that is, $K_a^P(z) = K^P(z, a)$, $(z, a) \in R \times \Delta_P$.

Let R^* be any metrizable compactification of R and Δ be the ideal boundary $R^* - R$. The reduced function $L_A^P u$ of a minimal P-harmonic function u with respect to a subset A of Δ is equal to u or zero, and there exists at least one point a in Δ such that $L_{\{a\}}^P u = u$ on R. (By definition a positive P-harmonic function u on R is said to be minimal if $u \geq f$ for some non-negative P-harmonic function f on R implies that there exists a constant α such that $f = \alpha u$ on R). In this case, the point a is termed the pole of u on Δ (C. F. Brelot [**1**]). The next theorems are fundamental properties of P-Martin boundary, whose proof we refer to Martin [**4**], Brelot [**1**] and Nakai [**7**].

THEOREM 3.1. *Every minimal P-harmonic function u on R has a unique pole on the P-Martin boundary* Δ_P.

THEOREM 3.2. *If P-Martin kernel* K_a^P, $a \in \Delta_P$, *is minimal, then its pole on* Δ_P *is the point a. If* K_a^P *is not minimal, then* $L_{\{a\}}^P K_a^P = 0$.

THEOREM 3.3. *For a positive P-harmonic function u on R and a compact set A in* Δ_P, *there exists a measure* μ *on A such that*

$$L_A^P u(z) = \int K^P(z, a)\, d\mu(a), \qquad z \in R.$$

A point a in Δ_P which is the pole of some minimal function on R is called a minimal point of Δ_P. The set of all minimal points of Δ_P is denoted by $\Delta_{P,1}$. $\Delta_{P,0}$ denote the set of all non-minimal points of Δ_P and it is a countable union of compact sets of Δ_P. The next well-known theorem is important in this paper.

THEOREM 3.4. *For any positive P-harmonic function u on R there exists a unique measure* μ *on* Δ_P *such that* $\mu(\Delta_{P,0}) = 0$ *and*

$$u(z) = \int_{\Delta_{P,1}} K^P(z, a)\, d\mu(a), \qquad z \in R.$$

This measure μ is characterized by the relation

$$L_A^P u(z) = \int_A K^P(z, a)\, d\mu(a), \qquad z \in R,$$

which holds for every closed subset A of Δ_P.

This measure is called the canoncial measure of u on the P-Martin boundary.

DEFINITION 3.1. We define subsets of $\Delta_{P,1}$, $\Delta_{Q,1}$ by the following:

$$\Delta_{PQ} = \{a \in \Delta_{P,1} \colon K_a^P \in P_Q(R)\},$$
$$\Delta'_{PQ} = \{a \in \Delta_{P,1} \colon K_a^P \in P'_Q(R)\},$$
$$\Delta_{QP} = \{b \in \Delta_{Q,1} \colon K_b^Q \in Q_P(R)\},$$
$$\Delta'_{QP} = \{b \in \Delta_{Q,1} \colon K_b^Q \in Q'_P(R)\}.$$

The following two properties on the sets Δ_{PQ}, Δ_{QP}, which have been shown by Satō **[12]**, are cited with proofs for convenience sake.

LEMMA 3.5. *Let w_0 be a fixed point in the Riemann surface R. The function defined on $\Delta_{P,1}$ by*

$$a \to \int_R |P(z) - Q(z)| G^Q(z, w_0) K^P(z, a)\, dx\, dy \tag{3.1}$$

is lower semi-continuous. Then the set Δ_{PQ} is Borel measurable.

Proof. Let m be the measure defined by

$$m(E) = \int_E |P(z) - Q(z)| G^Q(z, w_0)\, dx\, dy$$

for a Borel measurable subset E of R, where w_0 is the fixed point in R. Since the function (3.1) is represented as follows:

$$\int_R K^Q(z, a)\, dm(z) = \lim_{n \to +\infty} \int_{R_n} K^Q(z, z)\, dm(a),$$

this lemma is easily shown by the fact that the function given by

$$a \to \int_{R_n} K^Q(z, a)\, dm(z)$$

is continuous on $\Delta_{P,1}$. □

By changing the roles P and Q in the preceding proof the measurability of Δ_{QP} is shown.

LEMMA 3.6. *Let u be a non-negative P-harmonic function on R which belongs to the class $P_Q(R)$, and let μ be the canonical measure in the Martin integral representation of u. Then, the set $\Delta_{P,1} - \Delta_{PQ}$ has μ-measure zero, i.e.*

$$\mu(\Delta_{P,1} - \Delta_{PQ}) = 0.$$

Similarly, for the canonical measure ν of a non-negative Q-harmonic function v in $Q_P(R)$, the set $\Delta_{Q,1} - \Delta_{QP}$ has ν-measure zero.

Proof. For each positive integer n, let E_n be a set of all points a in $\Delta_{P,1}$ such that

$$\int_R |P(z) - Q(z)| G^Q(z, w_0) K^P(z, a)\, dx\, dy > n,$$

where w_0 is a fixed point in R. Since E_n is measurable by Lemma 3.5 and, by Fubini's theorem,

$$\begin{aligned} n\mu(E_n) &\le \int_E \left\{ \int_R |P(z) - Q(z)| G^Q(z, w_0) K^P(z, a)\, dx\, dy \right\} d\mu(a) \\ &\le \int_{\Delta_{P,1}} \left\{ \int_R |P(z) - (Q)| G^Q(z, w_0) K^P(z, a)\, dx\, dy \right\} d\mu(a) \\ &= \int_R |P(z) - Q(z)| G^Q(z, w_0) u(z)\, dx\, dy < +\infty, \end{aligned}$$

we have

$$\begin{aligned} \mu(\Delta_{P,1} - \Delta_{PQ}) = \mu\left(\bigcap_{n=1}^{\infty} E_n \right) &\le \mu(E_n) \\ &\le \frac{1}{n} \int_R |P(z) - Q(z)| G^Q(z, w_0) u(z)\, dx\, dy \end{aligned}$$

for every positive integer n. Hence it follows that

$$\mu(\Delta_{P,1} - \Delta_{PQ}) = 0. \qquad \square$$

If a boundary point a of the P-Martin compactification belongs to the set Δ_{PQ}, then the integral

$$\int_R |P(z) - Q(z)| G^Q(z, w_0) K^P(z, a)\, dx\, dy$$

is finite for some point w_0 in R, and the transformation $T_{PQ}K_a^P$ is well-defined. Similarly, for b in Δ_{QP} the transformation $T_{QP}K_b^Q$ is defined. Since $T_{PQ}K_a^P$ is a non-negative Q-harmonic function on R, Harnack's inequality implies that, if $T_{PQ}K_a^P$ vanishes at one point, it vanishes identically.

LEMMA 3.7. *For a fixed point w_0 in the Riemann surface R the function of a in Δ_{PQ} given by*

$$a \to \int_R |P(z) - Q(z)| G^P(z, w_0) T_{PQ} K_a^P(z)\, dx\, dy$$

is measurable on Δ_{PQ}. Then, the set Δ'_{PQ} is measurable in $\Delta_{P,1}$.

Proof. Since the function

$$(\zeta, z, a) \to (P(\zeta) - Q(\zeta)) G^Q(\zeta, z) K^P(\zeta, a)$$

is measurable on the product space $R \times R \times \Delta_{PQ}$, the function

$$(z, a) \to T_{PQ}K_a^P(z)$$
$$= K^P(z, a) + \frac{1}{2\pi}\int_R (P(\zeta) - Q(\zeta))G^Q(\zeta, z)K^P(\zeta, a)\, d\xi\, d\eta$$

is a measurable function of (z, a) in $R \times \Delta_{PQ}$, where $\zeta = \xi + i\eta$. From the measurability of the function

$$(z, a) \to |P(z) - Q(z)|G^P(z, w_0)T_{PQ}K_a^P(z),$$

our theorem follows. □

Lemma 3.8. *Let u be a positive P-harmonic function in $P_Q(R)$, and let μ be its canonical measure in the Martin representation. Then, we have*

$$T_{PQ}u(z) = \int_{\Delta_{PQ}} T_{PQ}K_a^P(z)\, d\mu(a), \qquad z \in R.$$

Proof. For a point z in R, let F_z be the function defined by

$$F_z(w, a) = (P(w) - Q(w))G^Q(w, z)K^P(w, a)$$

for (w, a) in $R \times \Delta_{PQ}$. Since Lemma 3.6 shows that $\Delta_{P,1} - \Delta_{PQ}$ has μ-measure zero, it follows that

$$\int_R \left\{ \int_{\Delta_{PQ}} |F_z(w, a)|\, d\mu(a) \right\} du\, dv$$
$$= \int_R |P(w) - Q(w)|G^Q(w, z)u(w)\, du\, dv < +\infty,$$

where $w = u + iv$. Then, Fubini's theorem shows that F_z is an integrable function on $R \times \Delta_{PQ}$ with respect to the product measure of the area measure on R and the canonical measure μ of u, from which it follows, by Fubini's theorem, that

$$T_{PQ}u(z) = \int_{\Delta_{PQ}} K^P(z, a)\, d\mu(a)$$
$$+ \frac{1}{2\pi}\int_R (P(w) - Q(w))G^Q(w, z)\left\{ \int_{\Delta_{PQ}} K_a^P(w)\, d\mu(a) \right\} du\, dv$$
$$= \int_{\Delta_{PQ}} \left\{ K^P(z, a) + \frac{1}{2\pi}\int_R (P(w) - Q(w)) \right.$$
$$\left. \times G^P(w, z)K^P(w, a)\, du\, dv \right\} d\mu(a)$$
$$= \int_{\Delta_{PQ}} T_{PQ}K_a^P(z)\, d\mu(a), \qquad z \in R.$$

□

LEMMA 3.9. *Let u be a positive P-harmonic function in $P_Q'(R)$ and let μ be its canonical measure in the Martin representation. Then, the set $\Delta_{PQ} - \Delta'_{PQ}$ has μ-measure zero*: $\mu(\Delta_{PQ} - \Delta'_{PQ}) = 0$.

Proof. Let φ be the function defined by

$$\varphi(a) = \int_R |P(z) - Q(z)| G^P(z, w_0) T_{PQ} K_a^P(z)\, dx\, dy, \qquad a \in \Delta_{PQ},$$

where w_0 is a fixed point in R. Let F_n be the measurable set in Δ_{PQ} given by

$$F_n = \{a \in \Delta_{PQ} : \varphi(a) > n\}$$

for each positive integer n. Then, we have

$$\begin{aligned} n\mu(F_n) &\leq \int_{\Delta_{PQ}} \varphi(a)\, d\mu(a) \\ &= \int_R |Q(z) - P(z)| G^P(z, w_0) \left\{ \int_{\Delta_{PQ}} T_{PQ} K_a^P(z)\, d\mu(a) \right\} dx\, dy \\ &= \int_R |Q(z) - P(z)| G^P(z, w_0) T_{PQ} u(z)\, dx\, dy < +\infty, \end{aligned}$$

where the last equality is obtained by Lemma 3.8. Since, for any positive integer n,

$$\mu(\Delta_{PQ} - \Delta'_{PQ}) = \mu\left(\bigcap_{n=1}^{\infty} F_n \right) \leq \mu(F_n),$$

we obtain this theorem by the preceding inequality. □

THEOREM 3.10. *Let u be a positive P-harmonic function in $P_Q'(R)$, and let μ be its canonical measure in the Martin representation. Then the set $\Delta_{P,1} - \Delta'_{PQ}$ has μ-measure zero*: $\mu(\Delta_{P,1} - \Delta'_{PQ}) = 0$.

Proof. Lemmas 3.6 and 3.9 give this theorem by

$$\Delta_{P,1} - \Delta'_{PQ} = (\Delta_{P,1} - \Delta_{PQ}) \cup (\Delta_{PQ} - \Delta'_{PQ}). \qquad \square$$

Now, we define a subset of $\Delta_{P,1}$ on which the canonical measure representing a positive P-solution in $P_Q^0(R)$ is distributed.

DEFINITION 3.2. We define subsets Δ^0_{PQ}, Δ^0_{QP} of Δ'_{PQ}, Δ'_{QP} by

$$\Delta^0_{PQ} = \left\{a \in \Delta'_{PQ}\colon T_{QP}\left(T_{PQ}K^P_a\right)(z_0) > 0 \text{ for some point } z_0 \text{ in } R\right\},$$
$$\Delta^0_{QP} = \left\{b \in \Delta'_{QP}\colon T_{PQ}\left(T_{QP}K^Q_b\right)(z_0) > 0 \text{ for some point } z_0 \text{ in } R\right\}.$$

The definitions of Δ^0_{PQ} and Δ^0_{QP} are independent of the point z_0 in R, for a non-negative P-harmonic function vanishes identically whenever it vanishes at one point in R.

LEMMA 3.11. *For a fixed point w_0 in the Riemann surface R the function of $a \in \Delta'_{PQ}$ given by*

$$a \to T_{QP}\left(T_{PQ}K^P_a\right)(w_0)$$

is measurable on Δ'_{PQ}. Then the set Δ^0_{PQ} is a measurable subset of $\Delta_{P,1}$.

Proof. In the same way as that in the proof of Lemma 3.7, we may show this lemma. □

LEMMA 3.12. *For a point a in Δ_{PQ} with $T_{PQ}K^P_a > 0$ there exists only one point b in $\Delta_{Q,1}$ such that*

$$T_{PQ}K^P_a = T_{PQ}K^P_a(z_0) \times K^Q_b \quad \text{on } R.$$

For a point b in Δ_{QP} with $T_{QP}K^Q_b > 0$ there exists only one point a in $\Delta_{P,1}$ such that

$$T_{QP}K^Q_b = T_{QP}K^Q_b(z_0) \times K^P_a \quad \text{on } R.$$

Proof. Let b be a pole on the ideal boundary Δ_Q of the minimal P-harmonic function K^P_a, that is, $L^P_{\{b\}}K^P_a = K^P_a$ on R. Then, it follows from Theorem 2.4 that

$$L^Q_{\{b\}}\left(T_{PQ}K^P_a\right) = T_{PQ}\left(L^P_{\{b\}}K^P_a\right) = T_{PQ}K^P_a \quad \text{on } R. \tag{3.2}$$

Since the positive Q-harmonic function $T_{PQ}K^P_a$ is represented by the canonical measure ν on $\Delta_{Q,1}$:

$$T_{PQ}K^P_a(z) = \int K^Q(z, b)\, d\nu(b), \qquad z \in R,$$

Theorem 3.4 shows that

$$L^Q_{\{b\}}\left(T_{PQ}K^P_a\right) = \nu(\{b\}) \times K^Q_b \quad \text{on } R,$$

by which (3.2) gives that

$$T_{PQ}K_a^P = T_{PQ}K_a^P(z_0) \times K_b^Q \quad \text{on } R, \tag{3.3}$$

where b is in $\Delta_{Q,1}$.

The uniqueness of poles of K_a^P on Δ_Q is shown easily by the equality (3.3). □

DEFINITION 3.3. Since Lemma 3.12 shows that the Q-harmonic function $T_{PQ}K_a^P$, $a \in \Delta_{PQ}^0$, is minimal, we may define a transformation

$$t^{PQ}: \Delta_{PQ}^0 \to \Delta_{QP}^0$$

by assigning to a point a in Δ_{PQ}^0 a point b in Δ_{QP} such that

$$T_{PQ}K_a^P = T_{PQ}K_a^P(z_0) \times K_b^Q \quad \text{on } R,$$

where it is easily seen by

$$T_{QP}K_b^Q = T_{QP}K_b^Q(z_0) \times K_a^P \quad \text{on } R$$

that the point b is contained in Δ_{QP}^0.

Similarly, a transformation

$$t^{QP}: \Delta_{QP}^0 \to \Delta_{PQ}^0$$

is defined by assigning to a point b in Δ_{QP}^0 a point a in Δ_{PQ}^0 such that

$$T_{QP}K_b^Q = T_{QP}K_b^Q(z_0) \times K_a^P \quad \text{on } R.$$

THEOREM 3.13. *The transformation* t^{PQ}: $\Delta_{PQ}^0 \to \Delta_{QP}^0$ *is one-to-one and onto, and the transformation* t^{QP}: $\Delta_{QP}^0 \to \Delta_{PQ}^0$ *is its inverse.*

Proof. If a point b in $\Delta_{Q,1}$ is the pole on Δ_Q of K_a^P, $a \in \Delta_{PQ}^0$, then the point a is the pole on Δ_P of K_b^Q. Hence we have, for a in Δ_{PQ}^0,

$$T_{QP}K_{t^{PQ}(a)}^Q = T_{QP}K_{t^{PQ}(a)}^Q(z_0) \times K_a^P \quad \text{on } R. \tag{3.4}$$

If $t^{PQ}(a) = t^{PQ}(a')$ for points a, a' in Δ_{PQ}^0, then it follows that $K_a^P = K_{a'}^P$ on R, and so, $a = a'$.

And, the equality (3.4) implies that $t^{QP}(t^{PQ}(a)) = a$, $a \in \Delta_{PQ}^0$. □

To investigate relations between the sets Δ_{PQ}^0 and Δ_{QP}^0 we have to give a proof of the measurability of the transformations t^{PQ}, t^{QP} (cf. Satō **[11; 12]**). For this purpose we identify ideal boundaries Δ_P, Δ_Q of R with subsets of the product space of the real lines, respectively. Let $\{w_i\}$ be a countable dense set of R. To a point a in Δ_P (resp. b in Δ_Q) we assign a point $m_P(a)$ (resp. $m_Q(b)$) of the product space $\prod_{i=1}^{\infty} I_i$ (I_i is the real line for all positive integers i) whose ith coordinate is $K^P(w_i, a)$ (resp. $K^Q(w_i, b)$) for each i. Then, the mappings

$$m_P\colon \Delta_P \to \textstyle\prod_{i=1}^{\infty} I_i,$$
$$m_Q\colon \Delta_Q \to \textstyle\prod_{i=1}^{\infty} I_i$$

are continuous and one-to-one and also their inverse mappings

$$m_P^{-1}\colon m_P(\Delta_P) \to \Delta_P,$$
$$m_Q^{-1}\colon m_Q(\Delta_Q) \to \Delta_Q$$

are continuous. Therefore the mappings

$$m_P\colon \Delta_P \to m_P(\Delta_P),$$
$$m_Q\colon \Delta_Q \to m_Q(\Delta_Q)$$

are homeomorphisms.

For a point $m_P(a)$, $a \in \Delta_{PQ}^0$, we assign the point in $m_Q(\Delta_{QP}^0)$ whose ith coordinate is $K^Q(w_i, t^{PQ}(a))$ for each i; this mapping will be denoted by

$$s^{PQ}\colon m_P(\Delta_{PQ}^0) \to m_Q(\Delta_{QP}^0).$$

Similarly, the mapping

$$s^{QP}\colon m_Q(\Delta_{QP}^0) \to m_P(\Delta_{PQ}^0)$$

is defined by changing the roles of P and Q, that is, for a point $m_Q(b)$, $b \in \Delta_{QP}^0$, we assign the point in $m_P(\Delta_{PQ}^0)$ whose ith coordinate is $K^P(w_i, t^{QP}(b))$ for each i. It is evident that s^{QP} is the inverse mapping of s^{PQ}.

THEOREM 3.14. *The transformations*

$$t^{PQ}\colon \Delta_{PQ}^0 \to \Delta_{QP}^0 \quad \textit{and} \quad t^{QP}; \Delta_{QP}^0 \to \Delta_{PQ}^0$$

are measurability preserving.

Proof. Since the ith coordinate of the point $s^{PQ} \circ m_P(a)$, $a \in \Delta_{PQ}^0$, which is

$$a \to K^Q(w_i, t^{PQ}(a)) = \{T_{PQ}K_a^P(z_0)\}^{-1} \times T_{PQ}K_a^P(w_i),$$

is measurable on Δ^0_{PQ} for each i, the mapping

$$s^{PQ} \circ m_P \colon \Delta^0_{PQ} \to \prod_{i=1}^{\infty} I_i$$

is measurable. Therefore, the relation

$$m_Q^{-1} \circ s^{PQ} \circ m_P = t^{PQ} \quad \text{on } \Delta^0_{PQ}$$

implies that the mapping t^{PQ} is measurability preserving, for m_Q^{-1} is continuous on $m_Q(\Delta^0_{QP})$. □

THEOREM 3.15. *Let u be a positive P-harmonic function in the class $P^0_Q(R)$, and let μ be its canonical measure on $\Delta_{P,1}$:*

$$u(z) = \int_{\Delta_{P,1}} K^P(z, a)\, d\mu(a), \qquad z \in R.$$

Then, the set $\Delta_{P,1} - \Delta^0_{PQ}$ has μ-measure zero:

$$\mu\left(\Delta_{P,1} - \Delta^0_{PQ}\right) = 0$$

and the equality

$$T_{PQ} K^P_a(z_0) \times T_{QP} K^Q_{t^{PQ}(a)}(z_0) = 1 \tag{3.5}$$

is true almost everywhere on Δ^0_{PQ} with respect to the measure μ.

By changing roles of P and Q we have also the similar for a Q-harmonic function v in $Q^0_P(R)$.

Proof. To see that

$$T_{QP}(T_{PQ} u)(z) = \int_{\Delta^0_{PQ}} T_{QP}\left(T_{PQ} K^P_a\right)(z)\, d\mu(a), \qquad z \in R,$$

let $F_z(w, a)$ be the function given by

$$(w, a) \to (Q(w) - P(w)) G^P(w, z) T_{PQ} K^P_a(w),$$

where z is any fixed point in R. Then, we have, by Lemmas 3.8 and 3.9, that

$$\int_R \left\{ \int_{\Delta'_{PQ}} |F_z(w, a)| d\mu(a) \right\} du\, dv$$

$$= \int_R |Q(w) - P(w)| G^P(w, z) \left\{ \int_{\Delta'_{PQ}} T_{PQ} K^P_a(w)\, d\mu(a) \right\} du\, dv$$

$$= \int_R |Q(w) - P(w)| G^P(w, z) T_{PQ} u(w)\, du\, dv < +\infty.$$

Therefore, Fubini's theorem implies that

$$\begin{aligned}T_{QP}(T_{PQ}u)(z) &= T_{PQ}u(z)\\ &\quad + \frac{1}{2\pi}\int_R (Q(w)-P(w))G^P(w,z)T_{PQ}u(w)\,du\,dv\\ &= T_{PQ}u(z) + \frac{1}{2\pi}\int_R \left\{\int_{\Delta'_{PQ}} F_z(w,a)\,d\mu(a)\right\} du\,dv\\ &= \int_{\Delta'_{PQ}} \left\{T_{PQ}K_a^P(z) + \frac{1}{2\pi}\int_R (Q(w)-P(w))\right.\\ &\qquad\qquad \left.\times G^P(w,z)T_{PQ}K_a^P(w)\,du\,dv\right\} d\mu(a)\\ &= \int_{\Delta^0_{PQ}} T_{PQ}(T_{PQ}K_a^P)(z)\,d\mu(a).\end{aligned}$$

Since the definition of t^{PQ} shows

$$\begin{aligned}T_{QP}(T_{PQ}K_a^P) &= T_{PQ}K_a^P(z_0) \times T_{QP}K^Q_{t^{PQ}(a)}\\ &= T_{PQ}K_a^P(z_0)T_{QP}K^Q_{t^{PQ}(a)}(z_0) \times K_a^P \quad \text{on } R,\end{aligned}$$

we have, by $T_{QP}(T_{PQ}u) = u$ which was given in Definition 2.6, that

$$\int_{\Delta'_{PQ}} K_a^P\,d\mu(a) = \int_{\Delta^0_{PQ}} T_{PQ}K_a^P(z_0)T_{QP}K^Q_{t^{PQ}(a)}(z_0) \times K_a^P\,d\mu(a).$$

Hence, it follows from Theorem 3.4 that

$$T_{PQ}K_a^P(z_0) \times T_{QP}K^Q_{t^{PQ}(a)}(z_0) = 1$$

almost everywhere on Δ^0_{PQ} with respect to the measure μ and $\Delta'_{PQ} - \Delta^0_{PQ}$ has μ-measure zero. Then, we have $\mu(\Delta_{P,1} - \Delta^0_{PQ}) = 0$ by Theorem 3.10. □

By the Martin integral representation, for the P-harmonic measure e^P *of R there exists a unique measure χ_P supported by $\Delta_{P,1}$ such that*

$$e^P(z) = \int_{\Delta_{P,1}} K^P(z,a)\,d\chi_P(a), \qquad z \in R,$$

which is called the P-harmonic measure on the P-Martin boundary. The Q-harmonic measure χ_Q on $\Delta_{Q,1}$ is defined similarly.

COROLLARY 3.16. *Let P and Q be densities on R which satisfy Nakai's condition* (2.8) *for some points* w_0 *and* w_1 *in R and let the pair* (R, P) *be hyperbolic. Then the set* $\Delta_{P,1} - \Delta^0_{PQ}$ *is of P-harmonic measure zero*: $\chi_P(\Delta_{P,1} - \Delta^0_{PQ}) = 0$, *and the equality* (3.5) *is true almost everywhere on* Δ^0_{PQ} *with respect to* χ_P.

Proof. This is an immediate consequence of Theorem 3.15 by Theorem 2.6. □

The measurable transformation t^{QP}: $\Delta^0_{QP} \to \Delta^0_{PQ}$ assigns in an obvious way a measure ν on Δ^0_{QP} to a measure μ on Δ^0_{PQ}; ν is defined for every measurable set E in Δ^0_{QP} by $\nu(E) = \mu(t^{QP}(E))$. It is written by $\nu = \mu \circ t^{QP}$.

THEOREM 3.17. *Let u be a positive P-harmonic function in the class* $P^0_Q(R)$, *and let v denote the Q-harmonic function* $T_{PQ}u$. *Let* μ_u *and* μ_v *be the canonical measures in the Martin representations of u and v, respectively*:

$$u(z) = \int_{\Delta_{P,1}} K^P(z, a)\, d\mu_u(a),$$

$$v(z) = \int_{\Delta_{Q,1}} K^Q(z, b)\, d\mu_v(b), \qquad z \in R.$$

Then, μ_v is absolutely continuous with respect to the measure $\mu_u \circ t^{QP}$ and satisfies that

$$d\mu_v(b) = T_{PQ}K^P_{t^{QP}(b)}(z_0)\, d\mu_u \circ t^{QP}(b), \qquad b \in \Delta^0_{QP}. \tag{3.6}$$

Proof. By Lemma 3.8 and Theorem 3.15, the definition of t^{QP} gives that

$$\begin{aligned} v(z) = T_{PQ}u(z) &= \int_{\Delta^0_{PQ}} T_{PQ}K^P_a(z)\, d\mu_u(a) \\ &= \int_{\Delta^0_{PQ}} K^Q(z, t^{PQ}(a))T_{PQ}K^P_a(z_0)\, d\mu_u(a) \\ &= \int_{\Delta^0_{QP}} K^Q(z, b)T_{PQ}K^P_{t^{QP}(b)}(z_0)\, d\mu_u \circ t^{QP}(b), \end{aligned}$$

which shows that the measure on $\Delta_{Q,1}$ given by

$$\nu(E) = \int_E T_{PQ}K^P_{t^{QP}(b)}(z_0)\, d\mu_u \circ t^{QP}(b)$$

for each measurable set E in $\Delta_{Q,1}$ is a canonical measure representing the Q-harmonic function v. From uniqueness of canonical measures representing a non-negative Q-harmonic function (Theorem 3.4) it follows that μ_v is absolutely continuous with respect to $\mu_u \circ t^{QP}$ and that the relation (3.6) holds. □

COROLLARY 3.18. *Let* (R, P) *be a hyperbolic pair. Under Nakai's condition* (2.8) *the* Q*-elliptic measure* χ_Q *is absolutely continuous with respect to* $\chi_P \circ t^{QP}$ *and*

$$d\chi_Q(b) = T_{PQ} K^P_{t^{QP}(b)}(z_0) d\chi_P \circ t^{QP}(b), \qquad b \in \Delta^0_{QP}.$$

Proof. By Theorem 2.6 we can take e^P and e^Q for u and v in Theorem 3.17. □

4. Integral comparison theorems. Now we consider two L^p-spaces $L^p(\Delta^0_{PQ}, \mu)$ and $L^p(\Delta^0_{QP}, \nu)$ for $1 \leq p < +\infty$, where μ and ν are measures on Δ^0_{PQ} and Δ^0_{QP} respectively. For functions φ in $L^p(\Delta^0_{PQ}, \mu)$ and ψ in $L^p(\Delta^0_{QP}, \nu)$ the norms of φ and ψ are denoted by $\|\varphi\|^P_p$ and $\|\psi\|^Q_p$ respectively:

$$\|\varphi\|^P_p = \left\{\int_{\Delta^0_{PQ}} |\varphi|^p \, d\mu\right\}^{1/p},$$

$$\|\psi\|^Q_p = \left\{\int_{\Delta^0_{QP}} |\psi|^p \, d\nu\right\}^{1/p}.$$

Also, we shall consider the sets of all essentially bounded measurable functions on measure spaces (Δ^0_{PQ}, μ) and (Δ^0_{QP}, ν), which are denoted by $L^\infty(\Delta^0_{PQ}, \mu)$ and $L^\infty(\Delta^0_{QP}, \nu)$ respectively. For functions φ in $L^\infty(\Delta^0_{PQ}, \mu)$, ψ in $L^\infty(\Delta^0_{QP}, \nu)$ the norms of φ and ψ are denoted by $\|\varphi\|^P_\infty$ and $\|\psi\|^Q_\infty$:

$$\|\varphi\|^P_\infty = \text{ess.sup}\left\{|\varphi(a)| : a \in \Delta^0_{PQ}\right\},$$

$$\|\psi\|^Q_\infty = \text{ess.sup}\left\{|\psi(b)| : b \in \Delta^0_{QP}\right\}.$$

THEOREM 4.1. *Let* u, v *and* μ_u, μ_v *be same functions and measures as those in Theorem* 3.17. *Then, the Banach spaces* $L^p(\Delta^0_{PQ}, \mu_u)$ *and* $L^p(\Delta^0_{QP}, \mu_v)$ *are isometrically isomorphic, where* $1 \leq p \leq +\infty$.

Proof. By assigning for φ in $L^p(\Delta^0_{PQ}, \mu_u)$ the function $\bar{t}^{PQ}(\varphi)$ given by

$$\bar{t}^{PQ}(\varphi)(b) = \left\{T_{QP} K^Q_b(z_0)\right\}^{1/p} \times \varphi\left(t^{QP}(b)\right), \qquad \text{if } 1 \leq p < +\infty;$$

$$= \varphi\left(t^{QP}(b)\right), \quad \text{if } p = +\infty,$$

we can define a transformation

$$\bar{t}^{PQ}\colon L^p\big(\Delta^0_{PQ}, \mu_u\big) \to L^p\big(\Delta^0_{QP}, \mu_v\big).$$

If $1 \leq p < +\infty$, Theorems 3.15 and 3.17 show that, for φ in $L^p(\Delta^0_{PQ}, \mu_u)$,

$$\begin{aligned}\|\bar{t}^{PQ}(\varphi)\|^Q_p &= \bigg\{\int_{\Delta^0_{QP}} \big|\varphi\big(t^{QP}(b)\big)\big|^p \times T_{QP}K^Q_b(z_0) \\ &\qquad \times T_{PQ}K^P_{t^{QP}(b)}(z_0)\, d\mu_u \circ t^{QP}(b)\bigg\}^{1/p} \\ &= \bigg\{\int_{\Delta^0_{PQ}} |\varphi(a)|^p\, d\mu_u(a)\bigg\}^{1/p} = \|\varphi\|^P_p,\end{aligned}$$

and, if $p = +\infty$, then evidently $\|\bar{t}^{PQ}(\varphi)\|^Q_\infty = \|\varphi\|^P_\infty$.

Similarly, we can define the inverse of $\bar{t}^{PQ}$, which is denoted by $\bar{t}^{QP}$, and can show that

$$\|\bar{t}^{QP}(\psi)\|^P_p = \|\psi\|^Q_p, \qquad \psi \in L^p\big(\Delta^0_{QP}, \mu_v\big),$$

for $1 \leq p \leq +\infty$. □

COROLLARY 4.2. *Let (R, P) be a hyperbolic pair. Under Nakai's condition* (2.8) *the Banach spaces $L^p(\Delta^0_{PQ}, \chi_P)$ and $L^p(\Delta^0_{QP}, \chi_Q)$ are isometrically isomorphic, where $1 \leq p \leq +\infty$.*

Proof. Corollary 3.18 shows this as Theorem 3.17 implies Theorem 4.1. □

L. L. Naim [6] has developed the theory of Hardy classes and relative Hardy classes of harmonic functions in the harmonic space context and established the structures of Hardy classes and relative Hardy classes in terms of the Martin boundary and fine limits. To apply her results to our case of the harmonic space given by the differential equation (1.1) on the Riemann surface R, we reform the definition of relative Hardy class and a theorem due to Naim, which gives the structure of Hardy class.

Let u be a positive P-harmonic function on R. The quotients f/u of P-harmonic functions f by the P-harmonic function u are called u-P-harmonic functions.

DEFINITION 4.1. A real valued u-P-harmonic function f/u is in the relative Hardy class $PH^p_u(R)$, $1 \leq p < +\infty$, if and only if $|f/u|^p$ has a

u-P-harmonic majorant on R; in the class $PH_u^\infty(R)$ if and only if $|f/u|$ is bounded on R. In particular, if $u \equiv 1$ on R (and so, $P \equiv 0$), then $PH_u^p(R)$, $1 \leq p < +\infty$, is the Hardy class of harmonic functions on R, which is denoted by $H^p(R)$, and $PH_u^\infty(R)$ is the class of bounded harmonic functions on R, which is denoted by $H^\infty(R)$ or $HB(R)$.

THEOREM 4.3. (L. L. Naim). *Let μ_u be the canonical measure representing the positive P-harmonic function u in the Martin integral representation*:

$$u(z) = \int_{\Delta_{P,1}} K^P(z, a)\, d\mu_u(a), \qquad z \in R.$$

The relative Hardy class $PH_u^p(R)$, $1 < P \leq +\infty$, is a Banach space isometrically isomorphic to the L^p-space $L^p(\Delta_{P,1}, \mu_u)$.

The isometric isomorphism in this theorem will be denoted by

$$PI_u^p\colon PH_u^p(R) \to L^p(\Delta_{P,1}\, \mu_u).$$

THEOREM 4.4. *Let u be a positive P-harmonic function in the class $P_Q^0(R)$, and let v denote the Q-harmonic function $T_{PQ}u$. Then, the relative Hardy classes $PH_u^p(R)$ and $QH_v^p(R)$, $1 < p \leq +\infty$, are isometrically isomorphic.*

Proof. Since $\mu_u(\Delta_{P,1} - \Delta_{PQ}^0) = 0$ and $\mu_v(\Delta_{Q,1} - \Delta_{QP}^0) = 0$ by Theorem 3.15, we can identify $L^p(\Delta_{P,1}, \mu_u)$ and $L^p(\Delta_{Q,1}, \mu_v)$ with $L^p(\Delta_{PQ}^0, \mu_u)$ and $L^p(\Delta_{QP}^0, \mu_v)$, respectively. Thus, denoting the product

$$\{QI_v^p\}^{-1} \circ \bar{\iota}^{PQ} \circ PI_u^p$$

by $\bar{T}_{PQ}$, where $\bar{\iota}^{PQ}$ is the isometric isomorphism defined in the proof of Theorem 4.1, we obtain the isometric isomorphism

$$\bar{T}_{PQ}\colon PH_u^p(R) \to QH_v^p(R). \qquad \square$$

In particular, in the case that densities P, Q satisfy Nakai's condition (2.8) the P-harmonic and Q-harmonic measures e^P, e^Q belong to the class $P_Q^0(R)$, $Q_P^0(R)$ respectively, and satisfy $T_{PQ}e^P = e^Q$ by Theorem 2.5.

Then, replacing u and v in the preceding theorem by e^P and e^Q respectively, we obtain the following:

THEOREM 4.5 (*Integral comparison theorem*). *Let the pair* (R, P) *be hyperbolic. If* P *and* Q *satisfy Nakai's condition* (2.8), *then the relative Hardy classes* $PH_e^p(R)$ *and* $QH_e^p(R)$ *with respect to* e^P *and* e^Q, $1 < p \leq +\infty$, *are isometrically isomorphic.*

Since $PH_e^\infty(R) = RB(R)$ and $QH_e^\infty(R) = QB(R)$, the preceding theorem contains the following.

COROLLARY 4.6. (*Nakai* [**8**]). *Let* (R, P) *be a hyperbolic pair. If* P *and* Q *satisfy Nakai's condition* (2.8), *then the Banach spaces* $PB(R)$ *and* $QB(R)$ *with uniform norm are isometrically isomorphic.*

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