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ON RANDOM SOLUTIONS OF VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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This paper is concerned with the existence, uniqueness and boundedness of random solutions of a random nonlinear mixed integral equation of Volterra-Fredholm type. The main tools for the study are the theory of admissibility of integral operators and the theory of random contractors.

1. Introduction. The mathematical description of various processes in physical, biological and engineering sciences gives rise to random or stochastic operator equations. Many applied problems leading to operator equations require the existence and continuity of the inverse operator. In [**1**], Altman has introduced the notions of inverse differentiability and contractors, which are very useful tools for solving deterministic operator equations in Banach spaces. The subject of random integral operator equations has been the object of numerous investigations in recent years and we refer the reader to the works of Bharucha-Reid [**2**] and Tsokos and Padgett [**12**]. Lee and Padgett [**5**], have developed the theory of random contractors, extending the work of Altman [**1**] and employed it in the study of random nonlinear Volterra integral equations.

In this paper, we obtain theorems on the existence, uniqueness, boundedness and stability of solutions of the following nonlinear stochastic integral equation of Volterra-Fredholm type

$$(1.1)\qquad x(t;\omega)=f(t;\omega)+\int_0^t a(t,s;\omega)g(s,x(s;\omega))\,ds + \int_0^\infty b(t,s;\omega)h(s,x(s;\omega))\,ds,\qquad t\geq 0,$$

where $\omega\in\Omega$, the supporting set of the probability measure space (Ω, A, P), $x(t;\omega)$ is the unknown random variable for each $t\in R_+=[0,\infty)$, $f(t;\omega)$ is the stochastic free term defined for $t\in R_+$, a and b are stochastic kernels and g and h are scalar functions defined on $R_+\times R$, with $R=(-\infty,\infty)$. The tools used are admissibility theory of integral operators (cf. [**4**]) and contractor theory.

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We note that the equation (1.1) is a generalization of the equations discussed in [**3**, **5**, **6**, **7**,**11**]. Our notation and terminology are fairly standard and can be understood by referring to [**5**, **6**] and [**12**]. However, in order to make the paper more self contained, we state briefly the basic definitions that are needed for the main results.

Let (Ω, A, P) be a complete probability measure space. $C \equiv C(R_+, L_2(\Omega, A, P))$ denotes the space of all continuous functions from R_+ into $L_2(\Omega, A, P) \equiv L_2$, with the compact open topology. Notice that C is a Fréchet space and contains the second order mean square continuous stochastic processes. For a positive continuous function u on R_+, we define the space C_u, to be the space of all continuous functions $x(t; \omega)$ from R_+ into L_2 satisfying

$$\|x(t;\omega)\|_{L_2} = \left(\int_\Omega |x(t;\omega)|^2 dP(\omega)\right)^{1/2} \leq Ku(t), \qquad t \geq 0,$$

for some positive constant K. Note that C_u is a Banach space, with the norm defined by

$$\|x\|_{C_u} = \sup_{t\geq 0} \left\{\|x(t;\omega)\|_{L_2}/u(t)\right\}.$$

The space C_1 for $u \equiv 1$, includes all the bounded continuous mean square processes on R_+. We say that a pair (B, D) of Banach spaces in C, such that the convergence in B or D implies the convergence in C, is said to be admissible with respect to an integral operator T, if $T(B) \subset D$.

We shall adopt the following definition from [**6**].

DEFINITION 1.1. Let $f(t, x; \omega)$ be a real valued function for $t \in R_+$, $x \in R$ and $\omega \in \Omega$ such that $f(t, x(t; \omega); \omega) \in B$ whenever $x(t; \omega) \in D$. The function f is said to have a bounded integral contractor Γ with respect to (B, D) if

(i) for each $t \in R_+$ and $x \in R$ there exists a bounded linear operator Γ from D to B such that $\|\Gamma(t, x)\|$ is continuous in (t, x) and $\|\Gamma(t, x)\| \leq Q$ for some positive constant Q; and

(ii) for each $t \in R_+$ and $x, y \in D$, there exists $\alpha > 0$ such that

$$\begin{aligned}\|f(t, x(t;\omega) + y(t;\omega) + [T\Gamma(t, x(t;\omega))y](t;\omega);\omega)\\ -f(t, x(t;\omega);\omega) - [\Gamma(t, x(t;\omega))y](t;\omega)\|_B\\ \leq \alpha\|y(t;\omega)\|_D.\end{aligned}$$

The constant α is called the contractor constant.

2. Existence, uniqueness and boundedness of random solutions. In this section, we obtain results on the existence, uniqueness and boundedness of random solutions of (1.1). By a random solution of a stochastic integral equation such as equation (1.1), we shall mean that for each $t \in R_+$, $x(t; \omega)$ satisfies the equation almost surely. We note that the results of this section are extensions of **[6]** to (1.1) and are more general than those of **[10]**.

The following hypotheses will be used in the subsequent discussion. Let B and D be Banach subspaces of C.

(H_1) $f(t; \omega) \in D$.

(H_2) The pair (B, D) is admissible with respect to the operator T: $C \to C$, defined by

$$(Tx)(t; \omega) = \int_0^t a(t, s; \omega)x(s; \omega)ds + \int_0^\infty b(t, s; \omega)x(s; \omega)ds, \qquad t \geq 0.$$

(H_3) $x(t; \omega) \to g(t; x(t; \omega))$ are continuous mappings from C into C, such that $x(t; \omega) \in D$ implies $g(t, x(t; \omega))$ and $h(t, x(t; \omega)) \in B$.

(H_4) For each $\lambda > 0$, there exists a $\delta_1 > 0$ such that

$$\|g(t, x(t; \omega)) - g(t, y(t; \omega))\|_B \leq \lambda \|x(t; \omega) - y(t; \omega)\|_D$$

whenever $\|x(t; \omega)\|_D$, $\|y(t; \omega)\|_D \leq \delta_1$.

(H_5) For each $\eta > 0$, there exists a $\delta_2 > 0$ such that

$$\|h(t, x(t; \omega)) - h(t, y(t; \omega))\|_B \leq \eta \|x(t; \omega) - y(t; \omega)\|_D$$

whenever $\|x(t; \omega)\|_D$, $\|y(t; \omega)\|_D \leq \delta_2$.

THEOREM 2.1. *Let* (H_1), (H_2) *and* (H_3) *hold. Also suppose that the functions g and h have bounded integral contractors* Γ_1 *and* Γ_2 *with contractor constants* α *and* β *respectively. Then there exists a random solution to* (1.1) *provided* $K(\alpha + \beta) < 1$ *where* $K = \|T\|$. *Furthermore, the random solution is unique, if the integral equation*

$$y(t; \omega) + \int_0^t a(t, s; \omega)\Gamma_1(s, x(s; \omega))y(s; \omega)ds + \int_0^\infty b(t, s; \omega)\Gamma_2(s, x(s; \omega))y(s; \omega)ds = z(t; \omega)$$

where $x(t; \omega)$, $z(t; \omega) \in D$, *has a solution in D.*

THEOREM 2.2. *Let* (H_1) *and* (H_2) *hold, in which* $B = D = C_u$ *and* u *and* G *are positive continuous functions on* R_+, *such that*

$$\int_0^t [\|a(t,s;\omega)\| + \|b(t,s;\omega)\|] u(s)\,ds + \int_t^\infty \|b(t,s;\omega)\| u(s)\,ds \le MG(t),$$

for $t \in R_+$, *where* M *is a positive constant. In addition, assume that for each* $t \in R_+$ *and* $x, y \in C_u$, *there exist bounded linear operators* $\Gamma_1(t,x) \equiv \Gamma_1$ *and* $\Gamma_2(t,x) \equiv \Gamma_2$ *such that*

$$\Big\| g\Big(t, x(t;\omega) + y(t;\omega) + \int_0^t a(t,s;\omega)(\Gamma_1 y)(s;\omega)\,ds\Big) - g(t, x(t;\omega)) - (\Gamma_1 y)(t;\omega)\Big\|_{L_2} \le \alpha \|y(t;\omega)\|_{L_2}$$

and similarly

$$\Big\| h\Big(t, x(t;\omega) + y(t;\omega) + \int_0^\infty b(t,s;\omega)(\Gamma_2 y)(s;\omega)\,ds\Big) - h(t, x(t;\omega)) - (\Gamma_2 y)(t;\omega)\Big\|_{L_2} \le \beta \|y(t;\omega)\|_{L_2}.$$

Then, there exists a random solution $x(t;\omega)$ *of* (1.1) *such that* $E|x(t;\omega)|^2 \le \lambda^2 G^2(t)$, $t \ge 0$, *where* $\lambda > 0$ *is a constant, provided* $K(\alpha + \beta) < 1$.

The proofs of Theorems 2.1 and 2.2 may be easily constructed, following those of Theorems 3.1, 3.2 and 3.3 of **[6]**. We omit the details.

THEOREM 2.3. *Suppose that the conditions* (H_1)–(H_5) *hold. Then, if there exists a number* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \le \varepsilon_0$,

$$\|f(t;\omega)\|_D + K\|g(t,0)\|_B + K\|h(t,0)\|_B < \delta\varepsilon,$$

for some fixed $\delta \in (0,1)$ *and* $K = \|T\|$, *then there exists a unique random solution* $x(t;\omega)$ *of* (1.1) *in* D *and further* $\|x(t;\omega)\| \le \varepsilon$.

Theorem 2.3 extends Theorem 2.2 **[10**, p. 314] by dropping the condition $g(t,0) \equiv h(t,0) \equiv 0$ and its proof is a direct application of Banach's contraction mapping principle. Also it may be viewed as a stochastic stability result of solutions of (1.1). For, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x(t;\omega)\|_D < \varepsilon$ for all $t \ge 0$, with probability one, whenever $\|f(t;\omega)\|_D$ is sufficiently small with probability one.

THEOREM 2.4. *Assume the following*:

(i) $f(t;\omega) \in C_1$.

(ii) (C_1, C_1) *is admissible with respect to the nonlinear random operator defined by*

$$(Nx)(t;\omega) = \int_0^t a(t,s;\omega)g(s,x(s;\omega))ds + \int_0^\infty b(t,s;\omega)h(s,x(s;\omega))ds.$$

(iii) *There exist positive numbers* A_0 *and* B_0 *such that*

$$\sup_{t\geq 0}\int_0^t \|a(t,s;\omega)\|ds \leq A_0 < \infty \quad \textit{and} \quad \sup_{t\geq 0}\int_0^\infty \|b(t,s;\omega)\|ds \leq B_0 < \infty.$$

(iv) *On the set* $S_\gamma = \{x(t;\omega) \in C_1 \colon \|x(t;\omega)\|_{C_1} \leq \gamma\}$, *g and h satisfy Lipschitz conditions in* L_2*-norm with constants* λ *and* μ, *respectively.*

Then there exists a unique random solution $x^*(t;\omega)$ of (1.1) in C_1 such that $E\,|\,x^*(t;\omega)\,|^2 \leq \gamma^2$ for all $t \geq 0$, provided $(\lambda A_0 + \mu B_0) < 1$ and $\| f(t;\omega)\|_{C_1} \leq \gamma[1 - (\lambda A_0 + \mu B_0)]$.

It is easy to see that Theorem 2.4 is a combination of the results on random Volterra equations **[11]** and on random Fredholm equations **[8]**. However in these papers the hypothesis (ii) is replaced by (H_2) in which the operator is linear. We may also apply Theorem 3.4 of **[5]**, by noticing that the identity random operator $I(\omega)$ is a bounded random contractor for the operator $U\colon S_\gamma \to C_1$ given by $(U(\omega)x)(t;\omega) = x(t;\omega) - f(t;\omega) - (N(\omega)x)(t;\omega)$. Also, the hypothesis on the Volterra kernel in Theorem 2.4 is weaker than the corresponding condition (i) of Theorem 3.4 **[6]**.

3. Boundedness results and examples. In this section, assuming existence, some additional theorems and examples on the boundedness of solutions under different conditions than those in §2, are presented. These theorems extend some of the results of [3] and [7].

We consider the following equation

$$x(t;\omega) = f(t;\omega) + \int_0^t a(t,s;\omega)g(x(s;\omega))ds + \int_0^\infty b(t,s;\omega)h(x(s;\omega))ds, \qquad t \geq 0, \tag{3.1}$$

and list the following conditions:

(H_6) $f(t;\omega)$ is bounded on $R_+ \times \Omega$.

(H_7) $\sup_{t\geq 0} \int_0^t | a(t, s; \omega) | \, ds \leq A_0 < \infty$ with probability one, and with probability one, $\sup_{t\geq 0} \int_0^\infty | b(t, s; \omega) | \, ds \leq B_0 < \infty$.

(H_8) $g, h \in \mathcal{C}(-\infty, \infty)$, where $\mathcal{C}$ denotes the class of continuous functions from $R \to R$.

(H_9) With probability one, $a(t; \omega) \leq 0$ for $0 \leq t < \infty$ and $a'(t; \omega) \geq 0$ on $0 < t < \infty$.

(H_{10}) $\int_0^\infty |f'(t; \omega)| \, dt < \infty$, with probability one.

In (H_9) and (H_{10}) "$'$" denotes the derivative with respect to t, for each ω.

(H_{11}) $g \in \mathcal{C}(-\infty, \infty)$, $G(x) = \int_0^x g(u)du \to \infty$ as $|x| \to \infty$, and $| g(x) | \leq K[1 + G(x)]$, $| x | < \infty$, for some constant $K > 0$.

THEOREM 3.1. *Let* (H_6), (H_7) *and* (H_8) *hold. Further suppose that with probability one*

$$(3.2) \quad \left(\sup_{t\geq 0} \int_0^t |a(t, s; \omega)| ds \right) \limsup_{|x| \to \infty} \left| \frac{g(x)}{x} \right| < 1, \text{ and with probability one}$$

$$\left(\sup_{t\geq 0} \int_0^\infty |b(t, s; \omega)| ds \right) \limsup_{|x| \to \infty} \left| \frac{h(x)}{x} \right| < 1.$$

Then every random solution of (3.1) *is bounded on* $0 \leq t < \infty$, *with probability one.*

Proof. Let $x(t; \omega)$ be a random solution of (3.1) for $t \geq 0$. In view of the condition (3.2), we can choose K_1, K_2 sufficiently large, and $\rho_1, \rho_2 > 0$ with $\rho_1 + \rho_2 < 1$ and such that, with probability one

$$(3.3) \quad \left(\sup_{t\geq 0} \int_0^t |a(t, s; \omega)| ds \right) |g(x(t; \omega))| < \rho_1 |x(t; \omega)|,$$

for $| x(t; \omega) | \geq K_1$ and with probability one,

$$\left(\sup_{t\geq 0} \int_0^\infty |b(t, s; \omega)| ds \right) |h(x(t; \omega))| < \rho_2 |x(t; \omega)|,$$

for $| x(t; \omega) | \geq K_2$.

Let $K = \max\{K_1, K_2\}$.

Define

$$I_1 = \{u \colon |x(u; \omega)| > K\} \quad \text{and} \quad I_2 = \{u \colon |x(u; \omega)| \leq K\}, \qquad \omega \in \Omega.$$

From (H_6), there exists $M > 0$ such that $|f(t; \omega)| \leq M$ on $0 \leq t < \infty$, $\omega \in \Omega$. Since g and h are continuous, there exist L_1, $L_2 > 0$ such that

$|g(x(u;\omega))| \leq L_1$ and $|h(x(u;\omega))| \leq L_2$ for $|x(u;\omega)| \leq K$, that is for $u \in I_2$. Now from (3.1) and (3.3), we have

$$x(t;\omega) = f(t;\omega) + \int_{I_1} a(t,s;\omega) g(x(s;\omega))\,ds$$

$$+ \int_{I_2} a(t,s;\omega) g(x(s;\omega))\,ds + \int_{I_1} b(t,s;\omega) h(x(s;\omega))\,ds$$

$$+ \int_{I_2} b(t,s;\omega) h(x(s;\omega))\,ds.$$

That is,

$$|x(t;\omega)| \leq M + \left(\rho_1 / \sup_{t\geq 0} \int_0^t |a(t,s;\omega)|\,ds\right) \int_{I_1} |a(t,s;\omega)|\,|x(s;\omega)|\,ds$$

$$+ L_1 \int_{I_2} |a(t,s;\omega)|\,ds + \left(\rho_2 / \sup_{t\geq 0} \int_0^\infty |b(t,s;\omega)|\,ds\right)$$

$$\times \int_{I_1} |b(t,s;\omega)|\,|x(s;\omega)|\,ds + L_2 \int_{I_2} |b(t,s;\omega)|\,ds,$$

$$\leq M + \rho_1 \sup_{s\in I_1, s\leq t} |x(s;\omega)| + L_1 A_0 + \rho_2 \sup_{s\in I_1, s\leq t} |x(s;\omega)| + L_2 B_0.$$

Since this is valid for every $t \geq 0$, we have, with probability one

$$\sup_{0\leq t\leq T} |x(t;\omega)| \leq M + L_1 A_0 + L_2 B_0 + (\rho_1 + \rho_2) \sup_{0\leq s\leq T} |x(s;\omega)|,$$

for every $T > 0$ and since $0 < \rho_1 + \rho_2 < 1$, we see that

$$\sup_{0\leq t\leq T} |x(t;\omega)| \leq \frac{M + L_1 A_0 + L_2 B_0}{1 - (\rho_1 + \rho_2)}.$$

Since this bound is independent of t, we see that

$$\sup_{0\leq t\leq \infty} |x(t;\omega)| \leq \frac{M + L_1 A_0 + L_2 B_0}{1 - (\rho_1 + \rho_2)}$$

with probability one and this completes the proof of the theorem.

COROLLARY 3.2. *Assume* (H_6), (H_7) *and* (H_8) *and that*

$$\limsup_{|x|\to\infty} \left|\frac{g(x)}{x}\right| = \limsup_{|x|\to\infty} \left|\frac{h(x)}{x}\right| = 0. \tag{3.4}$$

Then, every solution of (3.1) *is bounded on* R_+.

Since the hypothesis (3.4) implies (3.2), the proof follows from Theorem 3.1.

The next result is also a boundedness result and is specialized to the random integral equation of Volterra type, given by

$$x(t;\omega) = f(t;\omega) + \int_0^t a(t-s;\omega)g(x(s;\omega))ds, \qquad t \geq 0. \tag{3.5}$$

THEOREM 3.3. *Assume* (H_9), (H_{10}) *and* (H_{11}). *Then every solution of* (3.5) *satisfies* $\sup_{0 \leq t < \infty} |x(t;\omega)| < \infty$, *with probability one.*

The deterministic analogue of (3.5) has been considered by Londen [7]. Following the proof of Theorem 2 [7] and using the Lemma 6.1 [**2**, p. 192], the proof of Theorem 3.3 may be completed.

Now consider the equation (3.1), in which $b(t, s; \omega) = 0$ for $0 \leq s$, $t < \infty$ and $\omega \in \Omega$ and $a(t, s; \omega) \equiv a(t-s; \omega)$. Then Theorems 3.1 and 3.3 provide sufficient conditions for the boundedness of all solutions of (3.5). It is interesting to note that the two sets of conditions are in general different, which can be seen from the following examples.

EXAMPLE 3.4. Consider the following equation

$$x(t;\omega) = f(t;\omega) + \int_0^t a(t, s;\omega)g(x(s;\omega))ds, \tag{3.6}$$

$t \geq 0$ and $\omega \in \Omega \subset R_+$, where

$$f(t;\omega) = e^{-t/3-\omega}, \quad t \geq 0, \quad \omega \in \Omega, \quad a(t, s;\omega) = e^{-(t-s+2\omega/3)},$$

$$0 \leq s \leq t < \infty, \quad \omega \in \Omega \quad \text{and} \quad g(x) = -\frac{2}{3}x^{1/3} \quad \text{for } x \in R.$$

Now by Theorem 3.1 all solutions of (3.6) are bounded on R_+. Indeed, the random function $x(t;\omega) = e^{-(t+\omega)}$, $0 \leq t < \infty$ and $\omega \in \Omega$ is a bounded solution of (3.6). It may be noted that Theorem 3.3 cannot be applied to (3.6) as all its hypotheses are not satisfied.

EXAMPLE 3.5. All the solutions of the nonlinear random equation

$$x(t;\omega) = f(t;\omega) + \int_0^t a(t-s;\omega)g(x(s;\omega))ds \tag{3.7}$$

$t \geq 0$, and $\omega \in \Omega \subset R_+$, where $f(t;\omega) = 1/(1+\omega)$, $a(t;\omega) \equiv -1$, for $t \in R_+$ and $\omega \in \Omega$, and $g(x) = x^2$ on R, are bounded by Theorem 3.3. In particular $x(t; \omega) = 1/(1 + t + \omega)$, $0 \le t < \infty$, and $\omega \in \Omega$ is a bounded solution of (3.7). It is easy to see that the conditions (H_7) and (3.2) of Theorem 3.1 are not satisfied.

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