

FACTORIZATION THEOREMS FOR DIFFERENT CLASSES OF ANALYTIC FUNCTIONS IN MULTIPLY CONNECTED DOMAINS

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This paper consists of four sections. In the first section we give a survey on the reproducing kernel for harmonic functions in finitely-connected Jordan regions. We also prove a certain version of Fatou's theorem which we will use in the next sections.

In the second part we construct the generalized Schwarz kernel for an arbitrary finitely-connected Jordan domain. This kernel reproduces any continuous single-valued analytic function inside the domain by the boundary values of its real part. Also, we give an explicit formula for the real part of this kernel in terms of the harmonic measures.

In the third section we study the Blaschke products in arbitrary Jordan domains.

The main results are contained in the fourth section. There we prove factorization theorems for the classes N , N_+ , H_p and E_p .

Introduction. It is well known that R. Nevanlinna's and V. I. Smirnov's factorization theorems have been very useful for many problems concerning analytic functions in the unit disc (e.g. see [8], [9], [15], [19]). Unfortunately, a direct attempt to extend these results to multiply connected domains has been unsuccessful. The fact is that the most natural function to play the role of the Schwarz kernel in such domains is not single-valued. Another problem appearing in that case is to define the Blaschke factor. It is obvious that even in an annulus one cannot find a single-valued function $f(z)$ satisfying the following properties: (1) $f(z)$ vanishes only at one given point; (2) $f(z)$ is continuous up to the boundary; (3) $|f(z)|$ is equal to 1 on the whole boundary of the annulus.

Many papers investigating the classes of analytic functions in multiply-connected domains have appeared. We refer the reader to the survey by S. Ya. Havinson and G. C. Tumarkin [10] which is quite detailed. The construction of the "Blaschke products" in finitely connected regions was suggested by V. A. Zmorovič. The convergence theorem for the products of that type has been proved by P. M. Tamrazov in [22].

The first attempts to generalize the Schwarz formula to finitely connected domains had already taken place in the 19th century. (By the

“Schwarz formula” we understand the integral representation of an analytic function inside the domain in terms of the boundary values of its real part.)

The papers [24] and [1] contain many results obtained while trying to solve this problem. In [5–7] L. E. Dunduchenko and S. A. Kas’yanyuk have proved the factorization theorem for functions meromorphic in circular domains, belonging to the Nevanlinna class N there and admitting only finitely many zeroes and poles inside the domain.

They used the analog of the Schwarz kernel which had been suggested by Zmorovič in [23] for the circular domains.

The most convenient construction of the Schwarz kernel for the solution of the factorization problem was given by R. Coifman and G. Weiss in [3]. In that paper they considered domains bounded by analytic curves. For those domains, they have proved the factorization theorem for the analytic functions of Nevanlinna’s class N . Although the authors pointed out the possibility of extending their results to the classes N_+ and H_p , they did not give explicit statements. Some further investigation in this direction is contained in T. S. Kuzina’s paper [17].

If the boundary of the domain is only rectifiable (without assuming any smoothness conditions), or, furthermore, if it consists of arbitrary Jordan curves, the construction of the Schwarz kernel given in [3] cannot be applied. In this case, it is necessary to start out with the general reproducing kernel for harmonic functions. Generally speaking, this kernel is the quotient of the derivatives of the harmonic measures. Such a kernel has often been used in the theory of harmonic functions (see [18], [2]). To make its conjugate function single-valued we follow the procedure given in [3]. As a result, we obtain the factorization representations for the classes N , N_+ and H_p in arbitrary Jordan domains and for the classes E_p in the domains with Jordan rectifiable boundaries.

1. The generalized Green kernel and the Green-Stieltjes integral in finitely connected domains. Let G be a multiply-connected domain with the boundary Γ consisting of n disjoint Jordan curves γ_j . z_0 is a fixed point in G . $\omega(E, z, G)$ denotes the harmonic measure of the set $E \subset \Gamma$ taken at the point $z \in G$. Let $d\mu(\xi)$ be a finite real Baire measure on Γ . According to Koebe’s theorem (see [9]) there exists a function $w = \psi(z)$ mapping G conformally onto the circular domain \mathcal{K} . Note that $\psi(z)$ is bijective and continuous up to the boundary Γ . Let $z = \phi(w) = \psi^{-1}(w)$. $w_0 = \psi(z_0)$. Let $g(z, \zeta)$, $g(w, t)$ be the Green functions in G and \mathcal{K} , respectively.

Consider the following function on $\mathcal{K} \times \partial\mathcal{K}$:

$$\bar{K}(w, t) = \frac{1}{2\pi} \frac{\partial g(w, t)/\partial n_t}{\partial g(w_0, t)/\partial n_t},$$

where $w \in K$, $t \in \partial K$ and $\partial/\partial n_t$ is the derivative in the direction of the inner normal at the point t . Then $\bar{K}(w, t)$ is harmonic as a function of w and continuous as a function of t . Note that the harmonic measure $d\omega(E, w, \mathcal{K})$ is absolutely continuous with respect to Lebesgue measure ds on $\partial\mathcal{K}$, i.e.

$$d\omega(E, w, \mathcal{K}) = \frac{\partial}{\partial n_t} g(w, t) ds.$$

Therefore, the function $\bar{K}(w, t)$ can also be defined as the Radon-Nikodym derivative at the point t of the measure $(1/2\pi)d\omega(E, w, \mathcal{K})$ with respect to the measure $d\omega(E, w_0, \mathcal{K})$. Since harmonic measures are conformal invariants, then we can transfer the function $\bar{K}(w, t)$ into G as follows: Let $z \in G$, $\zeta \in \Gamma$, $\psi(\zeta) = t$, $\psi(z) = w$. Then we define

$$(1.1) \quad K(z, \zeta) = \frac{1}{2\pi} \left. \frac{d\omega(E, z, G)}{d\omega(E, z_0, G)} \right|_{\zeta} = \bar{K}(w, t).$$

In other words, $K(z, \zeta)$ is the Radon-Nikodym derivative at the point ζ of the measure $(1/2\pi)d\omega(E, z, G)$ with respect to the measure $d\omega(E, z_0, G)$. Note that in our case this derivative turns out to be a continuous function of ζ . Also, (1.1) implies that $K(z, \zeta)$ is harmonic in G as a function of z .

Consider the function $u(z)$ defined by

$$(1.2) \quad u(z) = \int_{\Gamma} K(z, \zeta) d\mu(\zeta)$$

It is clear that the function $u(z)$ is harmonic in G .

REMARK 1. Let $d\bar{\mu}(t)$ be the measure on ∂K defined by $\bar{\mu}(E) = \mu(\phi(E))$ for all Baire sets $E \subset \partial K$. Then by (1.1) we have

$$u(z) = \int_{\Gamma} K(z, \zeta) d\mu(\zeta) = \int_{\partial K} \bar{K}(w, t) d\bar{\mu}(t) = u(\psi(z)).$$

We call the integrals (1.2) the *Green-Stieltjes integrals*. We also call $K(z, \zeta)$ the *generalized Green kernel* in G .

If $u(z)$ can be represented in the form

$$(1.3) \quad u(z) = \int_{\Gamma} K(z, \zeta) f(\zeta) d\omega(E, z_0, G),$$

where $f(\zeta) \in L^1(d\omega)$, then we shall call $u(z)$ the *Green-Lebesgue integral*.

The following two theorems are well known (see [18, 2, 11]).

THEOREM 1.1. *Let $z = \alpha(\tau)$ be the uniformization mapping of the unit disc \mathfrak{D} onto G . \mathfrak{G} denotes the group of Möbius transformations of \mathfrak{D} corresponding to $\alpha(\tau)$. Let $u(z)$ be a harmonic function in G . Then the following statements are equivalent.*

- (1) $|u(z)|$ has a harmonic majorant in G .
- (2) $u(z) = u^1(z) - u^2(z)$, $u^i(z) \geq 0$ in G , $i = 1, 2$.
- (3) The function $u^*(\tau) = u[\alpha(\tau)]$ is automorphic with respect to \mathfrak{G} and representable by the Poisson-Stieltjes integral in \mathfrak{D} .
- (4)

$$u^+(z) = \begin{cases} u(z), & u(z) \geq 0, \\ 0, & u(z) < 0, \end{cases} \quad \text{has a harmonic majorant in } G.$$

(5) If $\{G^i\}_1^\infty$ is a sequence of domains such that $G^i \subset G^{i+1}$, $\cup_{i=1}^\infty G^i = G$ and $\omega^i(E, z_0, G^i)$ are the harmonic measures on ∂G^i , then

$$\overline{\lim}_{i \rightarrow \infty} \int_{\partial G^i} |u| d\omega^i(E, z_0, G^i) \leq C_u < +\infty.$$

(6)

$$\overline{\lim}_{i \rightarrow \infty} \int_{\partial G^i} u^+ d\omega^i(E, z_0, G^i) \leq C'_u < +\infty.$$

(7) $u(z)$ is representable by the Green-Stieltjes integral in G .

THEOREM 1.2. *Let $u(z)$ be a harmonic function in G and let $\alpha(\tau)$, \mathfrak{G} be the same as in Theorem 1.1. Then the following statements are equivalent.*

- (1) The function $u^*(\tau) = u[\alpha(\tau)]$ is representable by the Poisson-Lebesgue integral in \mathfrak{D} and $u^*(\tau)$ is automorphic with respect to the group \mathfrak{G} .
- (2) If the sequence $\{G^i\}_1^\infty$ is as in Theorem 1.1., then the integrals $\{\int_{\partial G^i} |u| d\omega^i(E, z_0, G^i)\}$ are uniformly absolutely continuous with respect to the harmonic measure (cf. [11]).
- (3) $u(z) = u^1(z) - u^2(z)$ where $u^i(z) \geq 0$, $i = 1, 2$, and there exist nondecreasing sequences of bounded harmonic functions $\{u^i(z)\}_{n=1}^\infty$ converging uniformly to $u^i(z)$ on the compact subsets of G .

- (4) *The least harmonic majorants $u^M(z)$ of the subharmonic functions $[|u(z)| - M]^+$ converge to zero uniformly on the compact subsets of G as $M \rightarrow \infty$.*
- (5) *$u(z)$ is representable by the Green-Lebesgue integral (1.3).*

Let us study the boundary behavior of the Green-Stieltjes integrals.

Let $\zeta \in \Gamma$ and let γ_ζ be a Jordan half-open arc such that $\gamma_\zeta \in G$ and ζ is its endpoint. Let $f(z)$ be any function in G . Assume there exists the limit $A = \lim_{z \rightarrow \zeta; z \in \gamma_\zeta} f(z)$ (finite or infinite). Then following [4] we call A the *asymptotic value* of $f(z)$ at the point ζ along γ_ζ . If $f(z)$ has the asymptotic value A at ζ along a certain curve γ_ζ^0 (it is possible that $A = \pm \infty$) and the asymptotic value along any other curve γ_ζ is either equal to A or does not exist, then A is called the *asymptotic boundary value* of $f(z)$ at ζ . According to the celebrated result of F. Bagemihl, an arbitrary function $f(z)$ has asymptotic boundary values at all points $\zeta \in \Gamma$ except a certain countable set (see [4]). Let us fix arbitrary points $\zeta_i \in \gamma_i$, $i = 1, \dots, n$. Define the generating function $\tilde{\mu}$ for the measure $d\mu$ as follows. Let $\zeta \in \gamma_i$ and $\widehat{\zeta_i \zeta}$ denote the arc of γ_i between ζ_i and ζ oriented in the same directions as γ_i . Then we set

$$\tilde{\mu}(\omega(\widehat{\zeta_i \zeta}, z_0, G)) = \mu(\widehat{\zeta_i \zeta}).$$

It is clear that $\tilde{\mu}$ has bounded variation as a function of the parameter $\omega(\widehat{\zeta_i \zeta}, z_0, G)$.

THEOREM 1.3 (P. Fatou's theorem). *Let us consider all points $\zeta_0 \in \gamma_i$, $i = 1, \dots, n$, such that there exists a finite or infinite derivative*

$$\tilde{\mu}'(\zeta_0) = \lim_{\zeta \rightarrow \zeta_0} \frac{\tilde{\mu}(\omega(\widehat{\zeta_0 \zeta}, z_0, G))}{\omega(\widehat{\zeta_0 \zeta}, z_0, G)}.$$

Then the integral (1.2) has asymptotic boundary values $\tilde{\mu}'(\zeta_0)$ at all such points ζ_0 except, maybe, a countable set.

Proof. According to the invariance of harmonic measures and Remark 1 it suffices to prove our theorem for the circular domain \mathcal{K} . At first, we note that if $u(w)$ is representable by the integral (1.2) in \mathcal{K} with the measure $d\bar{\mu}$, then there exists a Baire measure $d\nu$ on $\partial\mathcal{K}$ such that

$$(1.4) \quad u(w) = \frac{1}{2\pi} \int_{\partial\mathcal{K}} \frac{\partial g(w, t)}{\partial n_t} d\nu(t).$$

In reality, the analyticity of $\partial\mathfrak{K}$ implies that $\partial g(w, t)/\partial n_t$ is harmonic in the neighborhood of ∂K and does not vanish in that neighborhood. Then setting

$$\nu(E) = \int_{E \cap \partial K} \frac{1}{\partial g(w_0, t)/\partial n_t} d\bar{\mu}$$

for any Baire set $E \subset \mathbf{R}^2$, we obtain (1.4). Let $\partial\mathfrak{K} = \cup_{j=1}^n l_j$, where l_j are circles. Let $\nu(t)$ be the generating function of the measure $d\nu$. Fix i . Let $t_0 \in l_i$. Assume there exists a finite or infinite derivative $\nu'(t_0)$. Let w tend to t_0 . Then $\partial g(w, t)/\partial n_t \rightarrow 0$ uniformly on all $l_j, j \neq i$. Hence,

$$(1.5) \quad u(w) = \frac{1}{2\pi} \int_{l_i} \frac{\partial}{\partial n_t} g(w, t) d\nu(t) + o(1)$$

if w is sufficiently close to t_0 . Let $g_i(w, t)$ be the Green function of the simply-connected domain \mathfrak{K}_i bounded by l_i and such that $\mathfrak{K}_i \ni \mathfrak{K}$. We define the function $r_w(t)$ as follows:

$$r_w(t) = g(w, t) - g_i(w, t).$$

$r_w(t)$ is harmonic in \mathfrak{K} . As $w \rightarrow t_0$, $r_w(t) \rightarrow 0$ uniformly outside a certain neighborhood of t_0 . Since $r_w(t) \equiv 0$ on l_i , then according to the Riemann-Schwarz symmetry principle we can continue $r_w(t)$ through l_i . Therefore, $r_w(t) \rightarrow 0$ as $w \rightarrow t_0$ uniformly on all the circles l'_j symmetric to $l_j, j \neq i$, with respect to l_i . Hence, $r_w(t) \rightarrow 0$ as $w \rightarrow t_0$ uniformly in the whole domain bounded by $l_j, l'_j, j \neq i$. Then $\partial r_w(t)/\partial n_t \rightarrow 0$ uniformly as $w \rightarrow t_0$. But $\partial g_i(w, t)/\partial n_t, t \in \partial\mathfrak{K}_i$ is the classical Poisson kernel in \mathfrak{K}_i . Thus, from (1.5) we obtain that

$$(1.6) \quad u(w) = \frac{1}{2\pi} \int_{l_i} \frac{\partial}{\partial n_t} g_i(w, t) d\nu(t) + o(1).$$

Assume $w \rightarrow t_0$ inside a certain angle in \mathfrak{K} . Then from (1.6) and the classical Fatou theorem (see [8, 9, 15, 19]) we obtain that $u(w) \rightarrow \nu'(t_0)$. Since

$$\frac{d\bar{\mu}}{d\omega} = \frac{d\bar{\mu}}{d\nu} \cdot \frac{d\nu}{d\omega} = \frac{\partial}{\partial n_t} g(w_0, t) \cdot \frac{d\nu}{(\partial g(w_0, t)/\partial n_t) dt} = \nu'(t_0),$$

then $u(w) \rightarrow \bar{\mu}'(t_0)$ as $w \rightarrow t_0$ inside any angle in \mathfrak{K} . Applying the above-mentioned theorem of Bagemihl, we complete the proof.

COROLLARY 1.1. *The integrals (1.2) have finite asymptotic boundary values almost everywhere with respect to harmonic measure.*

COROLLARY 1.2. *The representing measure $d\mu$ for the harmonic function $u(z)$ in (1.2) is unique.*

Proof. Let us assume there are two measures $d\mu_1, d\mu_2$ such that

$$u(z) = \int_{\Gamma} K(z, \zeta) d\mu_1 = \int_{\Gamma} K(z, \zeta) d\mu_2.$$

Let $d\sigma = d\mu_1 - d\mu_2$. Then

$$(1.7) \quad v(z) = \int_{\Gamma} K(z, \zeta) d\sigma(\zeta) \equiv 0.$$

According to Theorem 1.3 we obtain from (1.7) that $d\sigma$ is singular with respect to $d\omega$. Then, according to the De La Vallée-Poisson theorem, $|d\sigma/d\omega| = \infty$ a.e. with respect to $d\sigma$ (see [21]). In the proof of Theorem 1.3 we have actually shown that at each point where there exists $d\sigma/d\omega$, there always exists an asymptotic value of $v(z)$ equal to $d\sigma/d\omega$. So, a.e. with respect to $d\sigma$ there exist asymptotic values of $v(z)$ equal to $\pm\infty$. This contradicts (1.7).

For the sake of completeness we state and prove the decomposition theorem for Green-Stieltjes integrals. A similar result for Green-Lebesgue integrals was proved in [11].

The following lemma is known (see [12], [14]).

LEMMA 1.1. *Let $\{G^i\}_1^\infty$ be a sequence of domains with smooth boundaries such that $\cup_{i=1}^\infty G^i = G$, $G^i \subset G^{i+1}$, $\partial G^i = \Gamma^i = \cup_{j=1}^n \gamma_j^i$, where γ_j^i is homologous to $\gamma_j \subset \Gamma$. Let G_j^i denote the domain bounded by γ_j^i such that $G_j^i \supset G^i$. Let $g^i(z, \zeta)$, $g_j^i(z, \zeta)$ be the Green functions of G^i and G_j^i , respectively. Then there exists a constant λ such that for all i ,*

$$1 \leq \frac{\partial g_j^i(z, z_0)/\partial n_z}{\partial g^i(z, z_0)/\partial n_z} \leq \lambda, \quad z \in \gamma_j^i, j = 1, \dots, n.$$

THEOREM 1.4. *Let $u(z)$ be representable by the integral (1.2). Let G_j be a domain bounded by γ_j such that $G_j \supset G$. Fix arbitrary $\alpha_j \in \overline{C} \setminus G_j$, $j = 1, \dots, n$. Then*

$$u(z) = \sum_1^n u_j(z),$$

where

$$(1.8) \quad u_j(z) = c_j \ln|z - \alpha_j| + u'_j(z),$$

c_j are constants depending only on $u(z)$. If $\infty \notin G_j$, then $c_j = 0$. For each $j = 1, \dots, n$ $u'_j(z)$ is harmonic in G_j . Moreover, each $u'_j(z)$ is representable by the Green-Stieltjes integral in G_j , $u'_j(\infty) = 0$. The functions $u'_j(z)$, $j = 1, \dots, n$, depend only on $u(z)$ and the choice of α_j .

Proof. Let the sequence $\{G^i\}_1^\infty$ be the same as in Lemma 1.1. According to Green's theorem applied to G^i , we obtain

$$\begin{aligned} u(z) &= \frac{1}{2\pi} \int_{\Gamma^i} u(\zeta) \left[\frac{\partial}{\partial n} \ln \frac{1}{|z - \zeta|} \right] ds - \frac{1}{2\pi} \int_{\Gamma^i} \ln \frac{1}{|z - \zeta|} \frac{\partial u}{\partial n} ds \\ &= \sum_{j=1}^n u_j(z), \quad z \in G^i, \end{aligned}$$

where

$$(1.9) \quad \begin{aligned} u_j(z) &= \frac{1}{2\pi} \int_{\gamma_j^i} u(\zeta) \left[\frac{\partial}{\partial n} \ln \frac{1}{|z - \zeta|} \right] ds \\ &\quad - \frac{1}{2\pi} \int_{\gamma_j^i} \ln \frac{1}{|z - \zeta|} \frac{\partial u}{\partial n} ds. \end{aligned}$$

It is clear that the $u_j(z)$, $j = 1, \dots, n$, do not depend on i . Also, they are harmonic in G_j , respectively, except, maybe, at infinity (if $\infty \in G_j$). In a neighborhood of ∞ we have

$$(1.10) \quad u_j(z) = c_j \ln|z - \alpha_j| + u'_j(z), \quad j = 1, \dots, n,$$

where

$$c_j = \frac{1}{2\pi} \int_{\gamma_j^i} \frac{\partial u}{\partial n} ds, \quad j = 1, \dots, n,$$

and $u'_j(z)$ are harmonic at ∞ , $j = 1, \dots, n$. Fix j . According to (1.9) and (1.10) there exists the constant M such that the inequality

$$(1.11) \quad |u'_j(z)| \leq |u(z)| + M$$

holds near γ_j^i . Let $d\omega^i(E, z_0, G^i)$, $d\omega_j^i(E, z_0, G_j^i)$ be the harmonic measures on ∂G^i and ∂G_j^i , respectively. Then, by Lemma 1.1, we have

$$(1.12) \quad 1 \leq \frac{d\omega_j^i(E, z_0, G_j^i)}{d\omega^i(E, z_0, G^i)} \leq \lambda.$$

Since $u(z)$ is representable by the integral (1.2) in G , then applying p. 5 of Theorem 1.1 and estimates (1.11) and (1.12), we obtain

$$\int_{\gamma'_j} |u'_j| d\omega'_j(E, z_0, G_j^i) \leq \lambda \int_{\gamma'_j} |u| d\omega^i(E, z_0, G^i) + M \leq \lambda C_u + M < +\infty.$$

Then applying Theorem 1.1 again we obtain that each $u'_j(z)$ is representable by the integral (1.2) in G_j .

In §4 we will make use of the following.

THEOREM 1.5. *Let $u(z)$ be a harmonic function in G . $\{G^i\}_1^\infty$, $d\omega^i(E, z_0, G^i)$ are the same as above. The following statements are equivalent.*

(1) *The integrals $\{\int_{\partial G^i} u^+(\zeta) d\omega^i(E, z_0, G^i)\}$ are uniformly absolutely continuous with respect to the harmonic measure.*

(2) *$u(z)$ is representable by the integral (1.2) with the measure $d\mu(\zeta) = F(\zeta)d\omega + d\nu$, where $d\nu \leq 0$ and $d\nu$ is singular with respect to $d\omega$. $F(\zeta)$ is equal to the asymptotic boundary values of $u(z)$ a.e. with respect to $d\omega$ on Γ .*

(3) *$u(z)$ is representable by the integral (1.2) with the measure $d\mu(\zeta) = \Psi(\zeta)d\omega + d\nu_1$, where $d\nu_1 \leq 0$, $\Psi(\zeta) \geq 0$ and $\Psi(\zeta)$ is equal to the asymptotic boundary values of $u^+(z)$ a.e. with respect to $d\omega$.*

REMARK 2. The class of harmonic functions characterized by Theorem 1.5 as well as the corresponding class of subharmonic functions was introduced by I. I. Privalov in [20] for the unit disc. In [16] these classes were investigated in simply connected domains with rectifiable boundaries.

Proof. (2) \Rightarrow (3). Let $F(\zeta) = F^+(\zeta) - F^-(\zeta)$. Then

$$d\mu = F^+(\zeta)d\omega + d\nu - F^-(\zeta)d\omega = F^+(\zeta)d\omega + d\nu_1,$$

where $d\nu_1 \leq 0$.

(3) \Rightarrow (2). According to the Lebesgue decomposition, we have $d\nu_1 = f(\zeta)d\omega + d\nu$, where $d\nu$ is singular. Since $d\nu_1 \leq 0$, then $f(\zeta) \leq 0$ and $d\nu \leq 0$. Therefore,

$$d\mu(\zeta) = \Psi(\zeta)d\omega + f(\zeta)d\omega + d\nu = F(\zeta)d\omega + d\nu.$$

(2) \Rightarrow (1). Since

$$u(z) = \int_{\Gamma} K(z, \zeta) d\mu(\zeta) \leq \int_{\Gamma} K(z, \zeta) F^+(\zeta) d\omega = v(z),$$

then $u^+(z) \leq v(z)$. According to Theorem 1.2 the integrals $\{\int_{\Gamma^i} v d\omega^i\}$ are uniformly absolutely continuous. Then the integrals $\{\int_{\Gamma^i} u^+ d\omega^i\}$ are also uniformly absolutely continuous. In a similar way one can show that (1) \Rightarrow (2). The theorem is proved.

2. The Green-Schwarz kernel in finitely connected Jordan domains.

We keep the same notation as in §1. Let $\tilde{K}(z, \zeta)$ be the conjugate harmonic function of $K(z, \zeta)$ as $\zeta \in \Gamma$ is fixed. Let $f(z) = u(z) + iv(z)$ be a single-valued analytic function in G such that $u(z)$ is continuous in \bar{G} . Then, according to Theorem 1.2, we have

$$u(z) = \int_{\Gamma} K(z, \zeta)u(\zeta) d\omega(E, z_0, G).$$

Set $p(z, \zeta) = K(z, \zeta) + i\tilde{K}(z, \zeta)$. Then

$$(2.1) \quad f(z) = \int_{\Gamma} u(\zeta)p(z, \zeta) d\omega(E, z_0, G).$$

But if $u(\zeta)$ is an arbitrary real-valued continuous functions on Γ , then the integral (2.1) gives, in general, a multi-valued function in G . We want to construct the kernel $\mathcal{P}(z, \zeta)$ such that for any continuous function $\bar{u}(\zeta)$ on Γ the function

$$f(z) = \int_{\Gamma} \mathcal{P}(z, \zeta)\bar{u}(\zeta) d\omega(E, z_0, G)$$

is analytic and single-valued in G . In the following construction we use the basic ideas of [3].

Let $\Gamma' = \bigcup_{k=1}^n \gamma'_k$, where γ'_k are analytic curves in G and γ'_k is homologous to γ_k , $k = 1, \dots, n$. Let $G' \subset G$ be a domain bounded by Γ' . Let $\omega_k(z)$ be a harmonic measure of γ_k , $k = 1, \dots, n$, i.e. $\omega_k|_{\gamma_k} \equiv 1$, $\omega_k|_{\gamma_j} \equiv 0, j \neq k$, and ω_k is harmonic in G . ω'_k are harmonic measures of γ'_k with respect to G' , $k = 1, \dots, n$.

Let $u(z)$ be a harmonic function in G and let $v(z)$ be its conjugate. According to the Cauchy-Riemann equations and Green's formula the conditions for $v(z)$ to be single-valued in G can be written in the following form:

$$\Delta_{\gamma_k} v = \int_{\gamma'_k} \frac{\partial v}{\partial s} ds = - \int_{\gamma'_k} \frac{\partial u}{\partial n} ds = - \int_{\Gamma'} u \frac{\partial \omega'_k}{\partial n} ds = 0, \quad k = 1, \dots, n - 1.$$

From now on we assume $\gamma_1, \dots, \gamma_{n-1}$ lie inside of γ_n . Let

$$(2.3) \quad \pi_{kj} = \int_{\gamma'_j} \frac{\partial \tilde{\omega}_k}{\partial s} ds = - \int_{\gamma'_j} \frac{\partial \omega_k}{\partial n} ds, \quad k, j = 1, \dots, n - 1.$$

Here, $\bar{\omega}_k$ is the conjugate function of ω_k . As it is known (see [3]), the matrix $\|\pi_{kj}\|_1^{j-1}$ is symmetric and positive definite. Hence, $\det\|\pi_{kj}\| \neq 0$. Therefore, for any harmonic function $u(z)$ in G there exist real numbers $\lambda_1, \dots, \lambda_{n-1}$ such that the conjugate function of $u(z) - \sum_1^{n-1} \lambda_j \omega_j(z)$ is single-valued in G . Those $\lambda_1, \dots, \lambda_{n-1}$ are uniquely defined by $u(z)$. By (2.2) and (2.3) it is also clear that $\lambda_j, j = 1, \dots, n - 1$, satisfy the following equations:

$$(2.4) \quad \sum_{j=1}^{n-1} \lambda_j \pi_{jK} = - \int_{\gamma_k} \frac{\partial u}{\partial n} ds, \quad k = 1, \dots, n - 1.$$

THEOREM 2.1. *There exists a unique function $\mathcal{P}(z, \zeta)$ continuous on $G \times \Gamma$ and satisfying the following properties: (1) If $\zeta \in \Gamma$ is fixed, then $\mathcal{P}(z, \zeta)$ is single-valued and analytic in G . (2) If $z \in G$ is fixed, then*

$$\int_{\gamma_k} \mathcal{P}(z, \zeta) d\omega(E, z_0, G) = 0, \quad k = 1, \dots, n - 1;$$

$$\int_{\gamma_n} \mathcal{P}(z, \zeta) d\omega(E, z_0, G) = 1.$$

(3) *If $f(z) = u(z) + iv(z)$ is a single-valued analytic function in G and $u(z)$ is continuous in \bar{G} , then*

$$f(z) = \int_{\Gamma} \mathcal{P}(z, \zeta) u(\zeta) d\omega(E, z_0, G) + iv(z_0).$$

We shall call $\mathcal{P}(z, \zeta)$ the Green-Schwarz kernel in G .

Proof. (1) Fix $\zeta \in \Gamma$. Choose the numbers $\Lambda_1(\zeta), \dots, \Lambda_{n-1}(\zeta)$ such that the conjugate function $\tilde{R}(z, \zeta)$ of the function

$$R(z, \zeta) = K(z, \zeta) - \sum_1^{n-1} \Lambda_j(\zeta) \omega_j(z)$$

is single-valued in G and $\tilde{R}(z_0, \zeta) = 0$. In case of $u(z) = K(z, \zeta)$, from (2.4) it follows that $\Lambda_1(\zeta), \dots, \Lambda_{n-1}(\zeta)$ are continuous functions of ζ . Setting $\mathcal{P}(z, \zeta) = R(z, \zeta) + i\tilde{R}(z, \zeta)$, we complete the proof of (1).

(2) Let $\bar{u}(\zeta)$ be an arbitrary real-valued continuous function on Γ . Consider the following integrals:

$$(2.5) \quad u(z) = \int_{\Gamma} K(z, \zeta) \bar{u}(\zeta) d\omega(E, z_0, G),$$

$$u_1(z) = \int_{\Gamma} R(z, \zeta) \bar{u}(\zeta) d\omega(E, z_0, G) = u(z) - \sum_1^{n-1} \lambda_j \omega_j(z);$$

where

$$(2.6) \quad \lambda_j = \int_{\Gamma} \Lambda_j(\zeta) \bar{u}(\zeta) d\omega(E, z_0, G);$$

$$(2.7) \quad \mathfrak{F}(z) = \int_{\Gamma} \mathfrak{P}(z, \zeta) \bar{u}(\zeta) d\omega(E, z_0, G) = u_1(z) + iv_1(z),$$

where v_1 is the conjugate of u_1 . Then $\mathfrak{F}(z)$ is single-valued. Take $\bar{u}(\zeta) = \omega_n(\zeta)$. Then $u(z) = \omega_n(z)$. Put $\lambda_1 = \dots = \lambda_{n-1} = -1$. Since system (2.4) has a unique solution, $u_1(z) \equiv 1$. Hence,

$$\int_{\Gamma} \mathfrak{P}(z, \zeta) \omega_n(\zeta) d\omega(E, z_0, G) = \int_{\gamma_n} \mathfrak{P}(z, \zeta) d\omega(E, z_0, G) \equiv 1 + ic.$$

But $\tilde{R}(z_0, \zeta) = 0$. So $c = 0$. Letting $\bar{u}(\zeta) = \omega_k(\zeta)$, $k < n$, and putting $\lambda_1 = 0, \dots, \lambda_k = -1, \lambda_{k+1} = 0, \dots, \lambda_{n-1} = 0$, one can easily verify that

$$\int_{\gamma_k} \mathfrak{P}(z, \zeta) d\omega(E, z_0, G) = 0.$$

(3) Let $f(z) = u(z) + iv(z)$ satisfy our hypothesis. Then system (2.4) has only a trivial solution. If we plug $\bar{u}(\zeta) = u(\zeta)$ into (2.5)–(2.7), then we obtain that $u_1(z) \equiv u(z)$ and $f(z) = \mathfrak{F}(z) + iv(z_0)$.

To prove the uniqueness of $\mathfrak{P}(z, \zeta)$, let us assume there exists another function $\mathfrak{P}_1(z, \zeta)$ satisfying (1)–(3). Take an arbitrary real-valued function $\bar{u}(\zeta)$ continuous on Γ . Define u , u_1 and \mathfrak{F} by (2.5), (2.6) and (2.7), respectively. According to (2) and (3), we have

$$\begin{aligned} \mathfrak{F}(z) &= \int_{\Gamma} \mathfrak{P}_1(z, \zeta) u_1(\zeta) d\omega = \int_{\Gamma} \mathfrak{P}_1(z, \zeta) \left(\bar{u}(\zeta) - \sum_1^{n-1} \lambda_j \omega_j(\zeta) \right) d\omega \\ &= \int_{\Gamma} \mathfrak{P}_1(z, \zeta) \bar{u}(\zeta) d\omega = \int_{\Gamma} \mathfrak{P}(z, \zeta) \bar{u}(\zeta) d\omega \end{aligned}$$

for all $z \in G$. Since $\bar{u}(\zeta)$ was an arbitrary continuous function, we obtain $\mathfrak{P}_1(z, \zeta) \equiv \mathfrak{P}(z, \zeta)$. The proof is complete.

To compute $\text{Re } \mathfrak{P}(z, \zeta)$ we have to introduce more notation.

At first, let us consider the circular domain \mathfrak{K} . Let $\partial\mathfrak{K} = \bigcup_{j=1}^n l_j$, and let ω_j be a harmonic measure of l_j . We define the functions $S_j(t)$ on $\partial\mathfrak{K}$ as follows:

$$S_j(t, w_0) = S_j(t) = \frac{\partial \omega_j(t) / \partial n_t}{\partial g(t, w_0) / \partial n_t}, \quad t \in \mathfrak{K}; j = 1, \dots, n - 1.$$

It is clear that $S_j(t)$ are continuous functions on Γ . Let $\Gamma' = \bigcup_{k=1}^n l'_k$ be defined in the same way as above. Let $u(z)$ be a harmonic function in \mathfrak{K} continuous in $\overline{\mathfrak{K}}$. Let $v(z)$ be its conjugate. Then, according to (2.2), we have

$$\Delta_{l'_k} v = - \int_{\Gamma'} u \frac{\partial \omega'_k}{\partial n} ds = - \int_{\Gamma'} u(\tau) \frac{\partial \omega'_k(\tau)/\partial n_\tau}{\partial g(t, w_0)/\partial n_\tau} \cdot \frac{\partial}{\partial n_\tau} g(\tau, w_0) ds.$$

As $\Gamma' \rightarrow \partial \mathfrak{K}$ it can be easily seen that

$$\int_{\Gamma'} u(\tau) \frac{\partial \omega'_k(\tau)/\partial n_\tau}{\partial g(\tau, w_0)/\partial n_\tau} \cdot \frac{\partial}{\partial n_\tau} g(\tau, w_0) ds \rightarrow \int_{\partial \mathfrak{K}} \bar{u}(t) S_k(t) d\omega.$$

Therefore,

$$(2.8) \quad \Delta_{\gamma_k} v = - \int_{\partial \mathfrak{K}} u(t) S_k(t) d\omega(E, w_0, \mathfrak{K}).$$

Let G be an arbitrary n -connected Jordan domain. $w = \psi(z)$ is as in §1. We put

$$S_j(\zeta, z_0) = S_j(\zeta) = S_j(\psi(\zeta)), \quad \zeta \in \Gamma.$$

Since both sides in (2.8) are invariant with respect to conformal maps, then (2.8) still holds for G ($u(z) = u(\psi(z))$), i.e.

$$(2.9) \quad \Delta_{\gamma_k} v = - \int_{\Gamma} u(\zeta) S_k(\zeta) d\omega(E, z_0, G).$$

THEOREM 2.2. Let $\|q_{jk}\|_1^{n-1} = (\|\pi_{jk}\|_1^{n-1})^{-1}$. Let $\mathfrak{P}(z, \zeta)$ be the Green-Schwarz kernel in G . Then

$$\operatorname{Re} \mathfrak{P}(z, \zeta) \stackrel{\text{def}}{=} R(z, \zeta) = K(z, \zeta) - \sum_{j=1}^{n-1} \Lambda_j(\zeta) \omega_j(z),$$

where

$$(2.10) \quad \Lambda_j(\zeta) = - \sum_{k=1}^{n-1} q_{jk} S_k(\zeta).$$

Proof. Let $\bar{u}(\zeta)$ be an arbitrary real-valued continuous function on Γ . Define $u(z)$ in G by (2.5). From (2.4) and (2.9) we obtain

$$\lambda_j = - \sum_{k=1}^{n-1} q_{jk} \int_{\Gamma} S_k(\zeta) \bar{u}(\zeta) d\omega(E, z_0, G).$$

At the same time, by (2.6), we have

$$\lambda_j = \int_{\Gamma} \Lambda_j(\zeta) \bar{u}(\zeta) d\omega(E, z_0, G).$$

Therefore,

$$\int_{\Gamma} \left[\Lambda_j(\zeta) + \sum_{k=1}^{n-1} q_{jk} S_k(\zeta) \right] \bar{u}(\zeta) d\omega(E, z_0, G) \equiv 0$$

for any \bar{u} . From this (2.10) follows.

3. The generalized Blaschke products. In the following theorem we put together the necessary information (most of which is known) concerning zeroes of analytic functions. We keep the same notation as in §§1, 2.

THEOREM 3.1. *Let $f(z)$ be a single-valued analytic function in G . Let $\{z_k\}_1^\infty$ be the sequence of its zeroes in G . The following statements are equivalent:*

- (1) $\ln |f(z)|$ has a harmonic majorant in G .
 - (2) The series $\sum_1^\infty g(z_k, z)$ converges uniformly on compact subsets of $G \setminus \{z_k\}_1^\infty$.
 - (3) Let $\{z_k^j\}_{k=1}^\infty, j = 1, \dots, n$, be the subsequences of $\{z_k\}_{k=1}^\infty$ such that all cluster points of $\{z_k^j\}$ belong to γ_j ; $\{z_k\}_1^\infty = \bigcup_{j=1}^n \{z_k^j\}_{k=1}^\infty$, $\{z_k^i\}_{k=1}^\infty \cap \{z_k^j\}_{k=1}^\infty = \emptyset$ as $i \neq j$. Then the series $\sum_{k=1}^\infty g_j(z_k^j, z), j = 1, \dots, n$, converge uniformly on the compact subsets of $G_j \setminus \{z_k^j\}_{k=1}^\infty$.
 - (4) The series $\sum_{k=1}^\infty |\omega_i(z_k^j) - \delta_{ij}|, i = 1, \dots, n, j = 1, \dots, n$, converge. (δ_{ij} is the Kronecker symbol.)
- If $\Gamma = \bigcup_{j=1}^n \gamma_j$ is analytic and ζ_j denotes the closest point to z_j on Γ , then (1)–(4) are equivalent to the following:

$$(5) \quad \sum_{j=1}^\infty |z_j - \zeta_j| < +\infty.$$

Proof (1) \Leftrightarrow (2) follows immediately from the Green-Jensen formula. (2) \Leftrightarrow (3). From Lemma 1.1 it easily follows that the following inequalities hold near each $\gamma_j, j = 1, \dots, n$:

$$1 \leq g_j(z, z_0)/g(z, z_0) \leq C_j,$$

where C_j are certain constants. This implies that (2) \Leftrightarrow (3).

(2) \Leftrightarrow (4). Let us transfer everything into the circular domain \mathcal{H} . Let $g_j(w, t)$ be the Green functions of $\mathcal{H}_j, j = 1, \dots, n$, and $w_k = \psi(z_k)$. According to the invariance of Green's functions we have

$$\sum_1^\infty g(z_k, z_0) = \sum_1^\infty g(w_k, w_0).$$

Since we have already proved that (2) \Leftrightarrow (3), then all series $\sum_{k=1}^{\infty} g_j(w_k^j, w_0)$, $j = 1, \dots, n$, $w_k^j = \psi(z_k^j)$, converge as soon as $\sum_1^{\infty} g(w_k, w_0) < +\infty$. According to the equivalence of (1) and (2), and the well-known Blaschke theorem for the unit disc, we obtain the convergence of the following series: $\sum_1^{\infty} |w_k^j - t_k^j|$, $j = 1, \dots, n$. Here, t_k^j is the closest point to w_k^j on $\partial\mathcal{H}$. It is clear that there exists a constant M such that $|\omega_j(w') - \omega_j(w'')| \leq M|w' - w''|$ for all $w', w'' \in \overline{\mathcal{H}}$. From this and according to the fact that all ω_j are invariant with respect to the conformal mappings, we obtain our statement. We note that in the last argument we only used the analyticity of $\partial\mathcal{H}$. Therefore, if ∂G is analytic we obtain that (1)–(4) are equivalent to (5).

Let $a \in G, a' \in \Gamma$. Following [3] we define the functions

$$(3.1) \quad \mathfrak{B}(z, a) = (z - a) \exp \left\{ - \int_{\Gamma} \mathfrak{P}(z, \zeta) \ln |\zeta - a| d\omega(E, z_0, G) \right\}.$$

$$(3.2) \quad \mathfrak{B}(z, a') = (z - a') \exp \left\{ - \int_{\Gamma} \mathfrak{P}(z, \zeta) \ln |\zeta - a'| d\omega(E, z_0, G) \right\}.$$

In the same way as it has been done in [3], one can show that the function (3.1) conformally maps G onto the unit disc with slits along circular arcs centered at the origin. Similarly, the function (3.2) conformally maps G onto the annulus with slits along circular arcs around the origin.

The proof of the following theorem is almost the same as that for the corresponding result in [3], so we omit it.

We recall that $\|q_{jk}\|_1^{n-1} = (\|\pi_{jk}\|_1^{n-1})^{-1}$ (see §2).

THEOREM 3.2. *Let the sequence $\{z_k\}_1^{\infty}$ satisfy (1)–(4) of Theorem 3.1. Let $\zeta_k^j = \psi^{-1}(t_k^j)$, where t_k^j are the same as in the proof of Theorem 3.1 ($k = 1, \dots, j; j = 1, \dots, n$). Then the product*

$$\mathfrak{B}_0(z) = \prod_{j=1}^n \prod_{k=1}^{\infty} \frac{\mathfrak{B}(z, z_k^j)}{\mathfrak{B}(z, \zeta_k^j)}$$

converges absolutely and uniformly on the compact subsets of G . $\mathfrak{B}_0(z) = 0$ if and only if $z \in \{z_k\}_1^{\infty}$. Moreover, if the sequence $\{G^i\}_{i=1}^{\infty}$ is the same as in §1, then $(\Gamma^i = \partial G^i = \cup_{k=1}^n \gamma_k^i)$

$$\lim_{i \rightarrow \infty} \int_{\gamma_k^i} |\ln |\mathfrak{B}_0(z)|| - C_k | d\omega^i(E, z_0, G^i) = 0,$$

where

$$C_k = \sum_{i=1}^{\infty} \sum_{l=1}^n \sum_{j=1}^{n-1} q_{jkl} (\delta_{jl} - \omega_j(z_i^l)), \quad k = 1, \dots, n-1; \quad C_n = 0.$$

Finally, $|\mathfrak{B}_0(\zeta)|_{\gamma_k} = \exp(C_k)$ a.e. with respect to $d\omega$, $k = 1, \dots, n$.

We shall call $\mathfrak{B}_0(z)$ the generalized Blaschke product.

For the reader's convenience let us recall the definitions of the basic classes of analytic functions in multiply connected domains. (Details can be found in [10].)

Let $f(z)$ be a single-valued analytic function in G . $\{G^i\}$, $\partial G^i = \Gamma^i$ are as above.

The function $f(z)$ belongs to $N(G)$ (Nevanlinna's class) if

$$\overline{\lim}_{i \rightarrow \infty} \int_{\Gamma^i} \ln^+ |f| d\omega^i(E, z_0, G^i) \leq \text{const} < +\infty.$$

We say that $f(z)$ belongs to $N_+(G)$ if the integrals

$$\left\{ \int_{\Gamma^i} \ln^+ |f| d\omega^i(E, z_0, G^i) \right\}$$

are uniformly absolutely continuous with respect to the harmonic measure. The function $f(z)$ belongs to $H_p(G)$, $p > 0$ (Hardy's classes), if

$$\overline{\lim}_{i \rightarrow \infty} \int_{\Gamma^i} |f|^p d\omega^i(E, z_0, G^i) \leq \text{const} < +\infty.$$

Finally, the function $f(z)$ belongs to $E_p(G)$, $p > 0$ (Smirnov's classes), if there exists the sequence $\{G^i\}$ as above such that $\Gamma^i = \partial G^i$ are rectifiable and

$$\overline{\lim}_{i \rightarrow \infty} \int_{\Gamma^i} |f|^p ds \leq \text{const} < +\infty.$$

It is known (see [8–10, 15, 19]) that $N \supset N_+ \supset H_p$ and $N \supset E_p$ for all $p > 0$.

COROLLARY 3.1. *If $f(z) \in N(G)$, then the Blaschke product corresponding to the zeroes of $f(z)$ converges.*

4. Factorization theorems in finitely connected domains. Let G be an n -connected Jordan domain.

LEMMA 4.1. *Let $f(z) \in N(G)$, $f(z) \in N_+(G)$ or $f(z) \in H_p(G)$. Let $\mathfrak{B}_0(z)$ be the Blaschke product corresponding to the zeroes of $f(z)$. Then the function $\mathfrak{F}(z) = f(z)/\mathfrak{B}_0(z)$ belongs to the same class as $f(z)$.*

Proof. For the classes N , N_+ the lemma follows directly from the definitions and Theorem 3.1. Let $f(z) \in H_p(G)$. Then according to the results in [10], $f(z) \in N_+(G)$. So $\mathfrak{F}(z) \in N_+(G)$. Moreover, the condition

$$\int_{\Gamma} |\mathfrak{F}|^p d\omega \leq \text{const} < +\infty$$

follows from the fact that

$$\int_{\Gamma} |f|^p d\omega \leq \text{const} < +\infty,$$

since $|B_0(\zeta)|_{\gamma_k} \equiv \text{const}$ for all $k = 1, \dots, n$. Therefore from the generalization of the Polubarinova-Kochina theorem obtained in [10] it follows that $\mathfrak{F}(z) \in H_p$. The lemma is proved.

The following theorem extends the result of R. Coifman and G. Weiss to Jordan domains.

THEOREM 4.1. *Let $f(z) \in N(G)$. Then $f(z)$ can be represented in the following form:*

$$(4.1) \quad f(z) = Q(z)B_0(z)\exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta) d\mu(\zeta)\right\},$$

where $B_0(z)$ is the Blaschke product corresponding to the zeroes of $f(z)$; $d\mu(\zeta)$ is a real Baire measure on Γ ;

$$Q(z) = \exp\left(\sum_1^{n-1} \lambda_j w_j(z)\right), \quad w_j(z) = \omega_j(z) + i\tilde{\omega}_j(z), j = 1, \dots, n-1,$$

where λ_j , $j = 1, \dots, n-1$, are real numbers such that the numbers $(1/2\pi)\Delta_{\gamma_k} \arg Q(z)$, $k = 1, \dots, n$, are integers. The converse is also true, namely, if (4.1) holds with λ_j , $j = 1, \dots, n$, such that $(1/2\pi)\Delta_{\gamma_k} \arg Q(z)$ are integers, then $f(z) \in N(G)$.

Proof. Let $f(z) \in N(G)$, and let $B_0(z)$ be its Blaschke product. According to Lemma 4.1 $\mathfrak{F}(z) = f(z)/B_0(z) \in N(G)$. Then, by Theorem 1.1, $\ln|\mathfrak{F}(z)|$ is representable in the form (1.2), i.e.

$$\ln|\mathfrak{F}(z)| = \int_{\Gamma} K(z, \zeta) d\mu(\zeta),$$

where $d\mu$ is a real Baire measure on Γ . Consider the function $\int_{\Gamma} \mathfrak{P}(z, \zeta) d\mu(\zeta)$. According to Theorem 2.1 we have

$$u(z) \stackrel{\text{def}}{=} \operatorname{Re} \int_{\Gamma} \mathfrak{P}(z, \zeta) d\mu = \ln|\mathfrak{F}(z)| - \sum_1^{n-1} \lambda_j \omega_j(z).$$

Therefore,

$$\mathfrak{F}(z) = \exp\left(\sum_1^{n-1} \lambda_j \omega_j(z)\right) \exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta) d\mu(\zeta)\right\}.$$

From this (4.1) follows. Since $\mathfrak{F}(z)$ is single-valued, then

$$(1/2\pi) \Delta_{\gamma_k} \arg Q(z), \quad k = 1, \dots, n,$$

are integers.

Assume (4.1) holds. Then the first two factors are bounded functions in G . Let us denote the third factor by $\phi(z)$. We have

$$\begin{aligned} \ln|\phi(z)| &= \int_{\Gamma} K(z, \zeta) d\mu(\zeta) - \sum_1^{n-1} \lambda_j \omega_j(z) \\ &= \int_{\Gamma} K(z, \zeta) \left[d\mu(\zeta) - \sum_1^{n-1} \lambda_j \omega_j(\zeta) d\omega(E, z_0, G) \right]. \end{aligned}$$

So $\ln|\phi(z)|$ is representable by the Green-Stieltjes integral. Applying Theorem 1.1 we obtain that $\phi(z) \in N(G)$. Therefore, $f(z) \in N(G)$.

COROLLARY 4.1. *The factor $Q(z)$ can also be represented as*

$$Q(z) = \prod_1^{n-1} (B(z, a'_j))^{-m_k},$$

where a'_1, \dots, a'_{n-1} are arbitrary points on $\lambda_1, \dots, \lambda_{n-1}$, respectively, and $m_k, k = 1, \dots, n - 1$, are integers defined by

$$m_k = (1/2\pi) \Delta_{\gamma_k} \arg [f(z)/B_0(z)].$$

Proof. Since $\mathfrak{F}(z) = f(z)/B_0(z)$ is single-valued, then $m_k = (1/2\pi) \Delta_{\gamma_k} \arg \mathfrak{F}(z), k = 1, \dots, n - 1$, are integers. From (4.1) it follows

that $\Delta_{\gamma_k} \arg \mathfrak{F} = \Delta_{\gamma_k} \arg Q, k = 1, \dots, n - 1$, i.e.

$$m_k = \sum_{j=1}^{n-1} \pi_{kj} \lambda_j.$$

Hence, $\lambda_j = \sum_{k=1}^{n-1} q_{jk} m_k$. Fix arbitrary points $a'_k \in \gamma_k, k = 1, \dots, n - 1$. According to the result in [3], we have

$$|B(z, a'_k)| = \exp \left\{ - \sum_{j=1}^{n-1} q_{jk} \omega_j(z) \right\}, \quad k = 1, \dots, n - 1.$$

So

$$\omega_j(z) = - \sum_{\nu=1}^{n-1} \pi_{j\nu} \ln |B(z, a'_\nu)|.$$

Hence,

$$\begin{aligned} \sum_1^{n-1} \lambda_j \omega_j(z) &= - \sum_{k, \nu, j} m_k q_{kj} \pi_{j\nu} \ln |B(z, a'_\nu)| \\ &= - \sum_{k, \nu} m_k \ln |B(z, a'_\nu)| \sum_{j=1}^{n-1} q_{kj} \pi_{j\nu} \\ &= - \sum_{k=1}^{n-1} m_k \ln |B(z, a'_k)|. \end{aligned}$$

From this we obtain our corollary.

THEOREM 4.2. $f(z) \in N_+(G)$ if and only if (4.1) holds with $d\mu = \ln |f(\xi)| d\omega(E, z_0, G) + d\nu$, where $f(\xi)$ is the asymptotic boundary values of $f(z)$, $d\nu \leq 0$ and $d\nu$ is singular with respect to $d\omega$.

Proof. Let $f(z) \in N_+(G)$. Then (4.1) holds with $d\mu$ taken from the representation of $\ln |\mathfrak{F}(z)|$ by the Green-Stieltjes integral. Since, according to Lemma 4.1, $\mathfrak{F}(z) \in N_+(G)$, then the integrals $\{ \int_\Gamma \ln^+ |\mathfrak{F}| d\omega^i \}$ are uniformly absolutely continuous with respect to the harmonic measure. Therefore, according to Theorem 1.5, we obtain

$$\ln |\mathfrak{F}(z)| = \int_\Gamma K(z, \xi) [\ln |\mathfrak{F}(\xi)| d\omega(E, z_0, G) + d\nu],$$

where $d\nu \leq 0$ and $d\nu$ is singular. But

$$\ln |\mathfrak{F}(\xi)| = \ln |f(\xi)| - \ln |B_0(\xi)| = \ln |f(\xi)| - C_k$$

on γ_k . According to Theorem 2.1(2) we obtain

$$\int_{\Gamma} \mathcal{P}(z, \zeta) \ln |f(\zeta)| d\omega = \int_{\Gamma} \mathcal{P}(z, \zeta) \ln |\mathcal{F}(\zeta)| d\omega.$$

Therefore, we can replace the measure $d\mu$ in (4.1) by the measure $d\mu' = \ln |f(\zeta)| d\omega + d\nu$. The proof of the converse statement is the same as in Theorem 4.1.

THEOREM 4.3. *$f(z) \in H_p(G)$ if and only if (4.1) holds with $d\mu = \ln |f(\zeta)| d\omega + d\nu$, where $d\nu \leq 0$ and $d\nu$ is singular with respect to $d\omega$. Moreover,*

$$\int_{\Gamma} |f(\zeta)|^p d\omega(E, z_0, G) \leq \text{const} < +\infty.$$

We omit the proof of this theorem since it can be obtained in the same way as the proofs of Theorems 4.1 and 4.2. The following theorem characterizes the Blaschke products (cf. [8, 9, 15, 19]).

THEOREM 4.4. *$f(z)$ is representable in the form $f(z) = Q(z)B_0(z)$, where $B_0(z)$ is the generalized Blaschke product, $Q(z)$ is the same as in Theorem 4.1, if and only if there exist real numbers C_1, \dots, C_n , $C_n = 0$, such that*

$$(4.2) \quad \lim_{i \rightarrow \infty} \int_{\gamma_k^i} |\ln |f| - C_k| d\omega^i(E, z_0, G^i) = 0, \quad k = 1, \dots, n.$$

Proof. If $f(z) = Q(z)B_0(z)$, then (4.2) follows from Theorem 3.2.

On the other hand, let us assume (4.2) holds. Then $f(z) \in N_+(G)$. Moreover, the integrals $\{\int_{\Gamma^i} |\ln |f|| d\omega\}$ are uniformly absolute continuous with respect to the harmonic measure. Hence, according to Theorem 1.2, $\ln |f(z)|$ is representable by the Green-Lebesgue integral. Therefore, the measure $d\mu$ in (4.1) is absolutely continuous with respect to $d\omega$. Moreover, on each γ_k we have $\ln |f(\zeta)| = C_k$ a.e. From this and from Theorem 2.1(2) we obtain $f(z) = Q(z) \cdot B_0(z)$.

Let us now study the factorization problem for the classes E_p .

Let G be a n -connected domain with the *rectifiable* Jordan boundary Γ . Hence, we can talk about ordinary angular boundary values. Moreover, in that case the singularity with respect to the harmonic measure and the singularity with respect to the Lebesgue measure ds on T are equivalent.

We recall that G is called a *Smirnov domain* ($G \in S$) if $\ln|\psi'(z)|$ is representable by the integral (1.3) (cf. [10]). (As above $\psi(z)$ is a conformal mapping of G onto the circular domain \mathcal{K} .)

THEOREM 4.5. *Let $G \in S$. Then $f(z) \in E_p(G)$, $p > 0$, if and only if (4.1) holds with $d\mu = \ln|f(\zeta)|d\omega + d\nu$, where $d\nu \leq 0$, $d\nu$ is singular and*

$$\int_{\Gamma} |f(\zeta)|^p dS \leq \text{const} < +\infty.$$

Proof. Since $G \in S$, $E_p(G) \subset N_+(G)$ for all $p > 0$ (see [10]). So our theorem follows immediately from the Polubarinova-Kochina theorem in [10].

To investigate the case $G \notin S$ we first recall the following generalization of the Keldysh-Lavrentjev theorem for multiply-connected domains proved in [10].

$$(4.3) \quad f(z) \in E_p(G) \Leftrightarrow f(z)/\sqrt[p]{\psi'(z)} \in H_p(G).$$

The following lemma is known (see [10]).

LEMMA 4.2. *Let G_j , $\gamma_j \subset \Gamma$, \mathcal{K}_j , $j = 1, \dots, n$, as above. Let $\psi_j(z)$ be a conformal mapping of G_j onto \mathcal{K}_j . Then there exist constants C_1, C_2 such that the inequality*

$$0 < C_1 \leq |\psi_j'(z)/\psi'(z)| \leq C_2 < +\infty$$

holds near γ_j for all $j = 1, \dots, n$.

From (4.3) it follows that $\psi'(z) \in E_1(G)$. Moreover, since $\psi'(z) \neq 0$ in G we can write the representation (4.1) for $\psi'(z)$ as

$$(4.4) \quad \psi'(z) = \exp \left\{ \int_{\Gamma} \mathcal{P}(z, \zeta) d\mu_0 \right\},$$

where $d\mu_0 = \ln|\psi'(\zeta)|d\omega + d\nu_0$; $d\nu_0$ is a singular measure. Since $1/\psi'(z) = [\psi^{-1}(w)]' \in H_1(\mathcal{K})$ (Γ is rectifiable!), from (4.3) and Theorem 4.5 it follows that $d\nu_0 \geq 0$.

REMARK. We assume $\psi(z)$ maps the outer curve of Γ onto the outer curve of ∂K . So $\ln \psi'(z)$ is a single-valued function in G .

THEOREM 4.6. $f(z) \in E_p(G)$, $p > 0$, if and only if (4.1) holds with $d\mu = \ln |f(\zeta)| d\omega + d\nu$, where $d\nu$ is singular, $d\nu \leq (1/p)d\nu_0$ and $\int_{\Gamma} |f(\zeta)|^p ds < +\infty$.

Proof. Let $f(z) \in E_p(G)$. Then, according to (4.1), we have

$$f(z) = Q(z)B_0(z)\exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta)\ln|f(\zeta)|d\omega\right\}\exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta) d\nu\right\},$$

where $d\nu$ is a singular measure. From (4.4) we obtain

$$\begin{aligned} \frac{f(z)}{\sqrt[p]{\psi'(z)}} &= Q(z)B_0(z)\exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta)\ln\left|\frac{f(\zeta)}{\sqrt[p]{\psi'(\zeta)}}\right|d\omega\right\} \\ &\times \exp\left\{\int_{\Gamma} \mathfrak{P}(z, \zeta)\left[d\nu - \frac{1}{p}d\nu_0\right]\right\}. \end{aligned}$$

Therefore, according to (4.3) and Theorem 4.3, we have $d\nu - (1/p)d\nu_0 \leq 0$ and

$$(4.5) \quad \int_{\Gamma} \frac{|f(\zeta)|^p}{|\psi'(\zeta)|} d\omega(E, z_0, G) \leq \text{const} < +\infty.$$

So $d\nu \leq (1/p)d\nu_0$. Since

$$|\psi'_j(\zeta)| = \partial g_j(\zeta, z_0)/\partial n_{\zeta}$$

(g_j is the Green function of G_j), then according to Lemmas 1.1 and 4.2 we obtain that (4.5) is equivalent to $\int_{\Gamma} |f(\zeta)|^p ds < +\infty$.

We omit a proof of the sufficiency since it is the same as in Theorems 4.1–4.3.

COROLLARY 4.2. $\bigcap_{0 < p < \infty} E_p(G) \subset N_+(G)$.

Proof. Let $f(z) \in \bigcap_{0 < p < \infty} E_p(G)$. Then according to Theorem 4.6 the singular part $d\nu$ of the measure $d\mu$ in the representation (4.1) for f satisfies the inequality $d\nu \leq (1/p)d\nu_0$ for all $p > 0$. Letting p tend to ∞ and using Theorem 4.2 we complete the proof.

REMARK. From the results in [13] it is seen that the modulus of the boundary values and the singular measure are really the parameters of the corresponding classes N , N_+ , H_p and E_p .

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Received May 30, 1980 and in revised form November 22, 1982.

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