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THE REGULAR REPRESENTATION OF LOCAL AFFINE MOTION GROUPS

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Let F be a nondiscrete locally compact topological field. Then the regular representation of the group of invertible affine motions of F^n, the semidirect product of F^n by $GL_n(F)$, is a type I_∞ factor. An explicit transformation formula is obtained.

1. Introduction. It is of some interest [**4**] to examine the regular representation of the group of affine motions of F^n for a nondiscrete locally compact field F. We show that the regular representation of such a group is a type I_∞ factor, i.e. is a multiple of an irreducible representation on an infinite-dimensional Hilbert space.

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2. Preliminaries. Let F be a nondiscrete locally compact field. It is known (see, for example, [**3**, Theorem 9.21]) that F is either $\mathbf{R}$, $\mathbf{C}$, a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, or the field of formal Laurent series in one variable over a finite field. In particular, if F is not $\mathbf{R}$ or $\mathbf{C}$ it has the following properties:

(i) F is the quotient field of a compact open subring R.

(ii) R has a unique maximal ideal M, which is principal; let $M = (\pi)$.

(iii) R/M is a finite field with (say) q elements.

(iv) There is a character χ on the additive group of F with $R \subseteq \ker \chi$, $\pi^{-1} \notin \ker \chi$; any other character on F is of the form $\chi_u(x) = \chi(ux)$ for some $u \in F$.

(v) R has a nonarchimedean absolute value $|\cdot|$ with $|\pi| = 1/q$.

(vi) If μ (usually denoted dx) is additive Haar measure on F, normalized so that $\mu(R) = 1$, then $\mu(M) = 1/q$ and $dx/|x|$ is multiplicative Haar measure μ^* on F^*, with the measure of R^* equal to $1 - 1/q$.

If F is $\mathbf{R}$ or $\mathbf{C}$, let dx denote Lebesgue measure normalized to make the Fourier inversion formula valid, $|\cdot|$ the ordinary absolute value (squared if $F = \mathbf{C}$), and $\chi(x) = e^{2\pi i \operatorname{Re} x}$.

We now let G_n be the group of invertible affine motions of F^n (the n-dimensional "$ax + b$" group), i.e. $G_n = F^n \cdot GL_n$, the semidirect product of F^n by $GL_n = GL_n(F)$. It will frequently be useful to consider G_n as a subgroup of GL_{n+1} by the identification

$$(b, A) \leftrightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b & & A & \end{bmatrix}.$$

Using this identification, we will think of $G_1 \subseteq GL_2 \subseteq \cdots \subseteq GL_n \subseteq G_n \subseteq GL_{n+1}$.

3. The results.

THEOREM 3.1. *The right regular representation ρ_{G_n} of G_n is a type I_∞ factor.*

Proof. By induction on n. The case $n = 1$ was done in [**2**, §3]; we briefly outline the argument for completeness. $G_1 \cong F \times F^*$ topologically, and $\mu \times \mu^*$ is right Haar measure. If $f \in L^2(G_1)$, set $\hat{f}_u(y, x) = \chi(uy)\int_F f(z, x)\chi(-uz)\,dz$; then $[\rho_{G_1}(b, a)f]\hat{}_u(y, x) = \chi(ubx)\hat{f}_u(y, ax)$. If $\rho_u = \mathrm{ind}_{F\uparrow G_1}\chi_{-u}$, then $\rho_u \cong \rho_v$ for $u, v \neq 0$; since $f(y, x) = \int_F \hat{f}_u(y, x)\,du$, we have $\rho = \int_F \rho_u\,du$.

Now assume $\rho_{G_{n-1}}$ is a factor. Regard F^n as a subgroup of G_n by identifying b with $(b, \mathbf{1})$. $\rho_{G_n} = \mathrm{ind}_{F^n\uparrow G_n}\rho_{F^n}$. $\rho_{F^n} = \int_{F^n}\chi_u\,du$, where χ_u ($u \in F^n$) is the character given by $\chi_u(v) = \chi(u \cdot v)$. By moving the direct integral past the induction, we get $\rho_{G_n} = \int_{F^n}(\mathrm{ind}_{F^n\uparrow G_n}\chi_u)\,du$. If u and v are nonzero vectors in F^n, $\mathrm{ind}\,\chi_u \simeq \mathrm{ind}\,\chi_v$, since u and v are conjugate under the action of GL_n on F^n. Set $e_1 = (1, 0, \ldots, 0)$. We then have $\rho_{G_n} \simeq \int_{F^n}(\mathrm{ind}_{F^n\uparrow G_n}\chi_{e_1})\,du$. $G_n = F^n \cdot GL_n$, so, regarding $G_{n-1} \subseteq GL_n$, let $H_n = F^n \cdot G_{n-1}$. Since the action of G_{n-1} on F^n leaves the first coordinate fixed, we have $H_n = F \times (F^{n-1} \cdot G_{n-1})$.

We split the induction into two steps,

$$\rho_{G_n} \simeq \int_{F^n}\mathrm{ind}_{H_n\uparrow G_n}(\mathrm{ind}_{F^n\uparrow H_n}\chi_{e_1})\,du.$$

Let us examine $\pi = \mathrm{ind}_{F^n\uparrow H_n}\chi_{e_1}$. $\chi_{e_1} = \chi \otimes \mathbf{1}$ on $F^n = F \times F^{n-1}$, and $H_n = F \times (F^{n-1} \cdot G_{n-1})$, so $\pi \simeq \chi \otimes (\mathrm{ind}_{F^{n-1}\uparrow(F^{n-1}\cdot G_{n-1})}\mathbf{1}) \simeq \chi \otimes \rho_{G_{n-1}}$ (where $\rho_{G_{n-1}}$ is considered as a representation of $F^{n-1} \cdot G_{n-1}$ with kernel F^{n-1}). By the induction hypothesis, $\rho_{G_{n-1}}$ is a I_∞ factor representation of G_{n-1}, so π is a I_∞ factor representation of H_n. We now use Mackey's theorem ([**1**], Theorem 6, p. 58) to show that $\mathrm{ind}_{H_n\uparrow G_n}\pi$ is a I_∞ factor representation of G_n, since H_n is precisely the stability group of χ_{e_1} under the action of G_n on F^n. □

We now get an explicit formula for this transformation. Throughout, we will always consider $GL_k \subseteq G_k \subseteq GL_{k+1} \subseteq G_{k+1}$, so that all groups will be thought of as being embedded in GL_{n+1}. Let $f \in L^2(G_n)$. We first take the Fourier transform along F^n: define

$$\hat{f}_u(y, X) = \chi(u \cdot y) \int_{F^n} f(z, X) \chi(-u \cdot z)\, dz.$$

Then

$$\hat{f}_u \in \mathcal{H}_u^n = \left\{ f\colon G_n \to \mathbf{C}\colon f(y, X) = \chi(u \cdot y) f(0, X), \right.$$
$$\left. \int_{GL_n} |f(0, X)|^2\, dX < \infty \right\}$$

where dX is Haar measure on GL_n.

By the Fourier inversion formula, $f(y, X) = \int_{F^n} \hat{f}_u(y, X)\, du$.

$$[\rho(b, A) f]\hat{}_u (y, X) = \chi(u \cdot y) \int_{F^n} [\rho(b, A) f](z, X) \chi(-u \cdot z)\, dz$$
$$= \chi(u \cdot y) \int_{F^n} f(z + Xb, XA) \chi(-u \cdot z)\, dz$$

Set $t = z - Xb$.

$$= \chi(u \cdot y) \int_{F^n} f(t, XA) \chi(-u \cdot t) \chi(u \cdot Xb)\, dt$$
$$= \chi(u \cdot Xb) \hat{f}_u(y, XA).$$

This is precisely the representation $\operatorname{ind}_{F^n \uparrow G_n} \chi_u$ on $\mathcal{H}_u^n[\chi_u(v) = \chi(u \cdot v)]$. So we have written

$$L^2(G_n) \simeq \int_{F^n} \mathcal{H}_u^n\, du, \qquad \rho_{G_n} \simeq \int_{F^n} \Big(\operatorname*{ind}_{F^n \uparrow G_n} \chi_u \Big)\, du.$$

Let $e_1^n = (1, 0, \ldots, 0) \in F^n$. We now take an equivalence in each piece, $\mathcal{H}_u^n \to \mathcal{H}_{e_1^n}^n$, $\operatorname{ind} \chi_u \to \operatorname{ind} \chi_{e_1^n}$ by setting $\tilde{f}_u(y, X) = \hat{f}_u(B_u(y, X))$ where

$$B_u = \begin{bmatrix} 1/u_1 & -u_2/u_1 & \cdots & -u_n/u_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{for } u = (u_1, \ldots, u_n),\ u_1 \neq 0.$$

We interchangeably think of B_u as an element of GL_n, G_n, and GL_{n+1} to simplify notation. The reason for choosing this B_u is that $u \cdot B_u v = B_u^t u \cdot v = e_1^n \cdot v$ for all v.

$\hat{f}_u \to \tilde{f}_u$ is an isometry of $\mathcal{H}_u^n$ onto $\mathcal{H}_{e_1^n}^n$: this can be seen most easily by identifying $\mathcal{H}_u^n$ with $L^2(GL_n)$ by $\hat{f}_n \leftrightarrow \hat{f}_u(0, \cdot)$ and noting that GL_n is unimodular (we have assumed right Haar measure). By associating f with $\int_{F^n} \tilde{f}_u \, du$, we get

$$L^2(G_n) \simeq \int_{F^n} \mathcal{H}_{e_1^n}^n \, du, \qquad \rho_{G_n} \simeq \int_{F^n} \Big(\operatorname*{ind}_{F^n \uparrow G_n} \chi_{e_1^n} \Big) \, du.$$

$\tilde{f}_u(y, X) = \chi(e_1^n \cdot y) \int_{F^n} f(v, B_u X) \chi(-u \cdot v) \, dv$. We now change variables, setting $v = B_u t$, $dv = 1/|u_1| \, dt$.

$$\tilde{f}_u(y, X) = \chi(e_1^n \cdot y) \int_{F^n} f(B_u(t, X)) \chi(-u \cdot B_u t) \frac{1}{|u_1|} \, dt$$

$$= \chi(e_1^n \cdot y) \int_{F^n} f(B_u(t, X)) \chi(e_1^n \cdot t) \, dt.$$

Now we split the induction into two steps,

$$\operatorname*{ind}_{F^n \to G_n} \chi_{e_1^n} = \operatorname*{ind}_{H_n \uparrow G_n} \Big(\operatorname*{ind}_{F^n \uparrow H_n} \chi_{e_1^n} \Big).$$

Set

$$\bar{f}_u(y, X)(Z) = \tilde{f}_u(y, ZX) \quad \text{for } y \in F^n, X \in GL_n, Z \in G_{n-1} \subseteq GL_n.$$

$$\bar{f}_u \in \Big\{ f\colon G_n \to L^2(G_{n-1})\colon f([(b, C)(y, X)])(Z) = \chi(e_1^n \cdot b) f(y, X)(ZC)$$
$$\text{for } X \in GL_n, Z, C \in G_{n-1}, b, y \in F^n; \int_{GL_n} |f(X)(\mathbf{1})|^2 \, dX < \infty \Big\}.$$

If we look at the representation σ^n of H_n on $L^2(G_{n-1})$ given by $[\sigma^n(b, C)g](Z) = \chi(e_1^n \cdot b) g(ZC)$ for $b \in F^n$, $C \in G_{n-1}$, we see that

$$\sigma^n \simeq \operatorname*{ind}_{F^n \uparrow H_n} \chi_{e_1^n}, \quad \text{and} \quad \operatorname*{ind}_{F^n \uparrow G_n} \chi_{e_1^n} \simeq \operatorname*{ind}_{H_n \uparrow G_n} \sigma^n.$$

Also, $\sigma^n \simeq \chi_{e_1^n} \otimes \rho_{G_{n-1}}$ as an inner tensor product.

We now decompose $\rho_{G_{n-1}}$ in the same manner as before. Let

$$\hat{f}_{u,r}(y, X)(t, S) = \chi(r \cdot t) \int_{F^{n-1}} \bar{f}_u(y, X)(w, S) \chi(-r \cdot w) \, dw$$
$$\big(t \in F^{n-1}, S \in GL_{n-1}\big).$$

Then

$$\bar{f}_u(y, X)(t, S) = \int_{F^{n-1}} \hat{f}_{u,r}(y, X)(t, S) \, dr; \qquad \hat{f}_{u,r}(y, X) \in \mathcal{H}_r^{n-1}.$$

Let

$$B_r = \begin{bmatrix} 1/r_1 & -r_2/r_1 & \cdots & -r_{n-1}/r_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in GL_{n-1}$$

$$\left(\text{for } r \in F^{n-1}, r_1 \neq 0\right).$$

Set $\tilde{f}_{u,r}(y, X)(t, S) = \hat{f}_{u,r}(y, X)(B_r(t, S))$.

$$[\sigma^n(b, (d, C))f]^{\hat{}}_{u,r}(y, X)(t, S)$$
$$= \chi(r \cdot t)\int_{F^{n-1}} [\sigma^n(b, (d, C))f]^{\tilde{}}_u(y, X)(w, S)\chi(-r \cdot w)dw$$
$$= \chi(r \cdot t)\int_{F^{n-1}} \chi(e_1 \cdot b)\tilde{f}_u(y, X)(w + Sd, SC)\chi(-r \cdot w)dw.$$

Set $v = w + Sd$.

$$= \chi(e_1 \cdot b)\chi(r \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(v, SC)\chi(-r \cdot v)\chi(r \cdot Sd)dv$$
$$= \chi(e_1 \cdot b)\chi(r \cdot Sd)\hat{f}_{u,r}(y, X)(t, SC).$$

$$[\sigma^n(b, (d, C))f]^{\tilde{}}_{u,r}(y, X)(t, S)$$
$$= \chi(e_1 \cdot b)\chi(r \cdot B_r Sd)\hat{f}_{u,r}(y, X)(B_r(t, SC))$$
$$= \chi(e_1 \cdot b)\chi(e_1 \cdot Sd)\tilde{f}_{u,r}(y, X)(t, SC).$$

Thus by associating $\tilde{f}_u$ with

$$\int_{F^{n-1}} \tilde{f}_{u,r}\, dr, \qquad \sigma^n \simeq \int_{F^{n-1}} \chi_{e_1^n} \otimes \Big(\operatorname*{ind}_{F^{n-1}\uparrow G_{n-1}} \chi_{e_1^{n-1}} \Big).$$

$$\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(w, B_r S)\chi(-r \cdot w)dw.$$

We want to pull the B_r past the w, so we change variables as before. Set $w = B_r v$, $dw = 1/|r_1|\, dv$. Then

$$\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(B_r(v, S))\chi(-r \cdot B_r v)\frac{1}{|r_1|}dv$$
$$= \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(B_r(v, S))\chi(e_1 \cdot v)\frac{1}{|r_1|}dv$$
$$= \chi(e_1^n \cdot y)\chi(e_1^{n-1} \cdot t)$$
$$\cdot \int_{F^{n-1}} \left[\int_{F^n} f(B_u(w, B_r(v, S)X))\chi(-w_1)\frac{1}{|u_1|}dw \right] \chi(-v_1)\frac{1}{|r_1|}dv.$$

We now pull the B_r past the w, by letting $w = B_r z$, $dw = 1/|r_1|\, dz$. Note that $z_1 = w_1$ since B_r does not affect the first column.

$$\tilde{f}_{u,r}(y, X)(t, S) = \int_{F^{n-1}} \left[\int_{F^n} f(B_u B_r(z, (v, S)X)) \chi(-z_1) \frac{1}{|u_1 r_1|} dz \right] \chi(-v_1) \frac{1}{|r_1|} dv.$$

We now repeat the process until we get down to F^1. We end up with

$$\begin{aligned} &\tilde{f}_{u,r,\ldots,s}(y, X)(t, S) \cdots (q, T) \\ &\qquad\qquad ((y, X) \in G_n, (t, S) \in G_{n-1}, \ldots, (q, T) \in G_1) \\ &= \chi(e_1^n \cdot y) \chi(e_1^{n-1} \cdot t) \cdots \chi(q) \\ &\quad \cdot \int_F \int_{F^2} \cdots \int_{F^n} f(B_u B_r \cdots B_s(w, (v, \ldots (z, T) \ldots, S)X)) \\ &\quad \cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} dw\, dv \cdots dz. \end{aligned}$$

$$\tilde{f}_{u,r,\ldots,s} \in \mathcal{H}^n = \left\{ f \colon G_n \to \mathcal{H}^{n-1} \colon f([(b, C)(y, X)])(Z) = \chi(e_1^n \cdot b) f(y, X)(ZC) \quad \text{for } X \in GL_n,\ Z, C \in G_n,\ b, y \in F^n;\ \int_{G_{n-1}\backslash G_n} |f(y, X)|^2 < \infty \right\}.$$

$[\mathcal{H}^0 = \mathbf{C}]$.

Set $\bar{f}_{u,r,\ldots,s}(y, X) = \tilde{f}_{u,r,\ldots,s}(y, X)(0, \mathbf{1}) \cdots (0, \mathbf{1})$.

$$\bar{f}_{u,r,\ldots,s} \in \mathcal{H} = \left\{ f \colon G_n \to \mathbf{C} \colon f(C(y, X)) = \phi(C) f(y, X) \quad \text{for } C \in \Gamma_n,\ \int_{\Gamma_n \backslash G_n} |f(y, X)|^2 < \infty \right\}$$

where

$$\Gamma_n = \left\{ \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ * & & & 1 \end{bmatrix} \right\}, \quad \phi\left(\begin{bmatrix} 1 & & & \\ a_{11} & 1 & & 0 \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} & 1 \end{bmatrix} \right) = \Sigma a_{ii}.$$

$$\bar{f}_{u,r,\ldots,s}(y, X) = \int_F \cdots \int_{F^n} f\left(B_u B_r \cdots B_s \begin{bmatrix} 1 & & & \\ w_1 & 1 & & 0 \\ \vdots & & \ddots & \\ w_n & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & v_1 & \ddots & \\ 0 & v_{n-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & \cdots & z_1 & 1 \end{bmatrix} (y, X) \right)$$

$$\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} dw\, dv \cdots dz.$$

$$\int_F \cdots \int_{F^n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & & & & \\ w_1 & 1 & & & \\ w_2 & v_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ w_n & v_{n-1} & \cdots & z_1 & 1 \end{bmatrix} (y, X) \right)$$

$$\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} dw\, dv \cdots dz$$

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(y, X) \right) \phi(-\gamma) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} d\gamma$$

since Haar measure on Γ_n is $dw\, dv \cdots dz$.

$$[\rho(b, A)f]^{-}_{u,r,\ldots,s}(y, X)$$

$$= \int_{\Gamma_n} [\rho(b, A)f]\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(y, X)\right)$$

$$\phi(-\gamma)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\, d\gamma$$

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(Xb, \mathbf{1})(y, XA)\right)$$

$$\cdot \phi(-\gamma)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\, d\gamma$$

[Set $\beta = \gamma(Xb, \mathbf{1})$.]

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \beta(y, XA)\right)$$

$$\cdot \chi(e_1 \cdot Xb)\phi(-\beta)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\, d\beta$$

$$= \chi(e_1 \cdot Xb)\bar{f}_{u,r,\ldots,s}(y, XA).$$

This is precisely $\operatorname{ind}_{\Gamma_n \uparrow G_n} \phi$ on $\mathcal{H}$. So we have

$$L^2(G_n) \simeq \int_F \cdots \int_{F^n} \mathcal{H}\, du\, dr \cdots ds,$$

$$\rho_{C_n} \simeq \int_F \cdots \int_{F^n} \Big(\operatorname*{ind}_{\Gamma_n \uparrow G_n} \phi\Big)\, du\, dr \cdots ds.$$

Let

$$\Delta_n = \left\{ \begin{bmatrix} u_1 & & \cdots & u_n \\ 0 & r_1 & \cdots & r_{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & s_1 \end{bmatrix} : u_1 \neq 0, \ldots, s_1 \neq 0 \right\}$$

$$= \text{group of upper triangular invertible } n \times n \text{ matrices.}$$

Right Haar measure on Δ_n is

$$\frac{du_1 \cdots du_n dr_1 \cdots dr_{n-1} \cdots ds_1}{|u_1 r_1^2 \cdots s_1^n|}.$$

We may identify Δ_n with $\Gamma_n \backslash G_n$ as a measure space, and hence we may regard $\mathrm{ind}_{\Gamma_n \uparrow G_n} \phi$ as a representation σ on $L^2(\Delta_n)$.

We now renormalize $\bar{f}_{u,r,\ldots,s}$ so that we can recapture f as an integral over Δ_n.

We have

$$f = \int_F \cdots \int_{F^n} \bar{f}_{u,r,\ldots,s} \, du \, dr \cdots ds.$$

Set $f_{u,r,\ldots,s} = \sqrt{|u_1 r_1^2 \cdots s_1^n|} \, \bar{f}_{u,r,\ldots,s}$; then

$$f = \int_F \cdots \int_{F^n} f_{u,r,\ldots,s} \frac{du \, dr \cdots ds}{|u_1 r_1^2 \cdots s_1^n|} = \int_{\Delta_n} f_\alpha \, d\alpha;$$

$$f_\alpha(y, X) = \left(|u_1 r_1^2 \cdots s_1^n|\right)^{-1/2} \int_{\Gamma_n} f(\alpha^{-1}\gamma(y, X))\phi(-\gamma) \, d\gamma,$$

where

$$\alpha = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}.$$

We thus have $L^2(G_n) \simeq \int_{\Delta_n} L^2(\Delta_n) \, d\alpha$, $\rho_{G_n} \simeq \int_{\Delta_n} \sigma \, d\alpha$. We may identify $\int_{\Delta_n} L^2(\Delta_n) \, d\alpha$ with $L^2(\Delta_n) \otimes L^2(\Delta_n)$, $\rho_{G_n} \simeq \sigma \otimes \mathbf{1}$.

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