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THE MAXIMAL ERGODIC HILBERT TRANSFORM WITH WEIGHTS

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This work is concerned with the characterization of those positive functions, w, such that the ergodic maximal Hilbert transform associated to an invertible, measure preserving, ergodic transformation on a probability space, is a bounded operator in $L_p(wd\mu)$.

1. Introduction. Let $(X, \mathfrak{F}, \mu)$ be a non-atomic probability space, and let $T: X \to X$ be an ergodic, invertible, measure preserving transformation. We consider the ergodic maximal Hilbert transform associated to T defined by

$$(1.1)\qquad Hf(x) = \sup_{s,t\geq 0}\left|\sum_{s<|i|<t}\frac{f(T^i x)}{i}\right| \qquad (s, t \in \mathbf{Z})$$

and acting on measurable functions. Our main result is given by the following theorem.

(1.2) THEOREM. *Let w be a positive integrable function. Then $f \to Hf$ is bounded on $L_p(wd\mu)$ if and only if w satisfies condition A'_p, i.e., there exists a constant M such that for a.e. $x \in X$ and for all positive integers k*

$$(1.3)\qquad k^{-1}\sum_{i=0}^{k-1} w(T^i x)\cdot\left[k^{-1}\sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)}\right]^{p-1} \leq M.$$

2. Main results. In this section we will prove the theorem above stated using the concept of ergodic rectangle and some ideas in (3) adapted to our context.

(2.1) DEFINITION. Let B be a subset of X with positive measure and k a positive integer such that

$$T^iB \cap T^jB = \varnothing, \qquad i \neq j, 0 \leq i, j \leq k-1.$$

Then the set $R = \bigcup_{i=0}^{k-1} T^iB$ will be called an "(ergodic) rectangle" with base B and length k.

Obviously $\mu(R) = k\mu(B)$.

In the proof of the theorem we will need the following two results which have been proved in (1).

(2.2) PROPOSITION. *Let k be a positive integer and let $A \subset X$ be a subset with positive measure. Then there exists $B \subset A$ such that B is base of a rectangle of length k.*

(2.3) LEMMA. *For any positive integer k, X can be written as a countable union of bases of rectangles of length k.*

The boundedness of the operator $f \to Hf$ on $L_p(wd\mu)$, p > 1, implies w satisfies A'_p. Let k be a positive integer and let's fix a rectangle with base B and length $4k$. We consider, for each integer n, the subset of B given by

$$B_n = \left\{x \in B: 2^n \leq (2k)^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} < 2^{n+1}\right\}. \tag{2.4}$$

Its obvious that $B = \bigcup_n B_n$.

Now fix n and let $A \subset B_n$ be an arbitrary measurable subset with positive measure. Consider

$$Q_1 = A \cup TA \cup \cdots \cup T^{k-1}A,$$

$$Q_2 = T^k A \cup T^{k+1}A \cup \cdots \cup T^{2k-1}A.$$

If f is a non-negative function we have

$$Hf(T^j x) \geq (2k)^{-1} \sum_{l=0}^{k-1} f(T^l x) \tag{2.5}$$
$$(x \in A, \sup f \subset Q_1, k \leq j \leq 2k-1),$$

$$Hf(T^j x) \geq (2k)^{-1} \sum_{l=k}^{2k-1} f(T^l x) \tag{2.6}$$
$$(x \in A, \sup f \subset Q_2, 0 \leq j \leq k-1).$$

Applying (2.6) to χ_{Q_2} we obtain

$$Hf(T^j x) \geq \tfrac{1}{2} \qquad (x \in A, 0 \leq j \leq k-1). \tag{2.7}$$

It follows immediately that

$$\left(\frac{1}{2}\right)^p \int_A w(T^j x)\, d\mu \leq \int_A (Hf(T^j x))^p w(T^j x)\, d\mu. \tag{2.8}$$

Summing over j, $j = 0, \ldots, k-1$, and using the boundedness of our operator we have

$$\int_{Q_1} w\, d\mu \leq 2^p C \int_{Q_2} w\, d\mu. \tag{2.9}$$

Throughout this paper C will denote an universal constant not necessarily the same at each occurrence. Applying now (2.5) to $f = w^{-1/(p-1)}\chi_{Q_1}$ we find that

$$Hf(T^jx) \geq (2k)^{-1}\sum_{l=0}^{k-1} w(T^lx)^{-1/(p-1)} \geq 2^n, \tag{2.10}$$

since $k \leq j \leq 2k-1$ and $x \in A \subset B_n$. Thus, for $f = w^{-1/(p-1)}\chi_{Q_1}$ it follows that

$$2^{np}\int_A w(T^jx)\,d\mu \leq \int_A Hf(T^jx)^p w(T^jx)\,d\mu. \tag{2.11}$$

Adding up in j for $j = k,\ldots,2k-1$ and applying again our assumption of boundedness we can write

$$2^{np}\int_{Q_2} w\,d\mu \leq C\int_{Q_1} w^{-1/(p-1)}\,d\mu$$

which, because of (2.9) yields

$$2^{np}\int_{Q_1} w\,d\mu \cdot \left(\int_{Q_1} w^{-1/(p-1)}\,d\mu\right)^{-1} \leq 2^pC^2. \tag{2.12}$$

On the other hand we also have:

$$\mu(A)^{-1}\int_A (2k)^{-1}\sum_{i=0}^{k-1} w(T^ix)^{-1/(p-1)}\,d\mu \leq 2^{n+1},$$

raising to the power p and applying (2.12) it follows that

$$\left((k\mu(A))^{-1}\int_A \sum_{i=0}^{k-1} w(T^ix)^{-1/(p-1)}\,d\mu\right)^p$$
$$\circ \int_{Q_1} w\,d\mu\left(\int_{Q_1} w^{-1/(p-1)}\,d\mu\right)^{-1} \leq 2^{3p}C^2$$

or equivalently

$$\left(\mu(A)^{-1}\int_A k^{-1}\sum_{i=0}^{k-1} w(T^ix)^{-1/(p-1)}\,d\mu\right)^{p-1}$$
$$\cdot\left(\mu(A)^{-1}\int_A k^{-1}\sum_{i=0}^{k-1} w(T^ix)\,d\mu\right) \leq 2^{3p}C.$$

This, immediately, gives

$$k^{-1}\sum_{i=0}^{k-1} w(T^i x) \circ \left(k^{-1}\sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)}\right)^{p-1} \leq 2^{3p}C^2 \qquad \text{(a.e. in } B_n).$$

Now a straightforward application of Lemma (2.3) gives us that w satisfies condition A'_p.

In order to prove the converse we first assume that w satisfies condition A'_∞ and for that we mean that there are positive constants C, $\delta > 0$ so that given any finite set I consisting of consecutive integers and any subset $E \subset I$

$$\frac{\sum_{i\in E} w(T^i x)}{\sum_{i\in I} w(T^i x)} \leq C\left(\frac{\#E}{\#I}\right)^{\delta} \qquad \text{(a.e. in } X)$$

where $\#E$ is the number of elements of E.

In the following the subsets I above described will be called intervals in the integers. Theorem (1.2) will, then, be a consequence of the following results:

(2.13). THEOREM. *If w satisfies A'_∞ then*

$$\int_X (Hf)^p w\, d\mu \leq C\int_X (f^*)^p w\, d\mu \tag{2.14}$$

where f^* is the ergodic no centered maximal function associated to the transformation T..

(2.15). LEMMA. *Condition A'_p implies condition A'_∞.*

(2.16). THEOREM.

$$\int_X (f^*)^p w\, d\mu \leq C\int_X |f|^p w\, d\mu, \quad \textit{if } w \textit{ satisfies } A'_p.$$

Theorem (2.16) has been proved in (1).

The proof of Lemma (2.15) runs as follows:

Let's call I to the interval $\{0, 1, \ldots, k-1\}$ and let E be an arbitrary subset of I.

It was shown in (1) that if w satisfies A'_p then the following "reverse Hölder" inequality holds:

$$k^{-1}\sum_{j=0}^{k-1} w(T^j x)^v \leq Ck^{-v}\left(\sum_{j=0}^{k-1} w(T^j x)\right)^{v}, \tag{2.17}$$

with constants C, $v > 1$ independent of k.

Applying Hölder's inequality we obtain

$$\sum_{j\in E} w(T^j k) \le \left(\sum_{j\in E} w(T^j x)^v\right)^{1/v} (\#E)^{1-1/v}$$
$$\le \left(\sum_{j=0}^{k-1} w(T^j x)^v\right)^{1/v} (\#E)^{1-1/v}.$$

The result now holds using inequality (2.17).

In the proof of Theorem (2.13) we will use the fact (4) that there exists a constant C such that for any sequence$\{b_k\}_{k=-\infty}^{\infty}$ and any $\lambda > 0$ holds

$$\sum_{k:\, Hb_k>\lambda} \le \frac{C}{\lambda}\cdot \sum_{k=-\infty}^{+\infty} |b_k| \tag{2.18}$$

where

$$Hb_k = \sup_{s,t\ge 0}\left|\sum_{s<|k-j|<t} \frac{b_j}{k-j}\right| \qquad (s, t \in \mathbf{Z}).$$

Combining this result with condition A'_∞ we will prove, for any $f \in L^1(d\mu)$, the following fundamental inequality

$$\int_{\{x:\, Hf(x)>\beta\lambda,\, f^*(x)\le\gamma\lambda\}} \le C\left(\frac{\gamma}{\beta'}\right)^{\delta} \int_{\{x:\, Hf(x)>\lambda\}} w\, d\mu. \tag{2.19}$$

where β' depends on β and γ.

If $\mu\{x\colon Hf(x) > \lambda\} = 1$ the weak type $(1 - 1)$ of H with respect to the measure μ tells us

$$1 \le \frac{C}{\lambda}\int_X |f|\, d\mu$$

and choosing $\gamma < C^{-1}$ we have

$$\gamma\lambda < \int_X |f|\, d\mu.$$

By the individual ergodic theorem:

$$\gamma\lambda < f^*(x) \quad \text{a.e. in } X$$

and that implies (2.19)

Therefore we may assume that $\mu\{x\colon Hf(x) > \lambda\} < 1$. In particular, if

$$D = \{x\colon T^i x \in O_\lambda\colon i = 0, -1, -2, \dots\}$$

where $O_\lambda = \{x\colon Hf(x) > \lambda\}$, then $\mu(D) = 0$, since T is ergodic.

From this fact is clear that if we call

$$B_i = \{x\colon x, Tx, \dots, T^{i-1}x \in O_\lambda, T^{-1}x, T^i x \notin O_\lambda\}$$

and $R_i = B_i \cup \cdots \cup T^{i-1}B_i$ then $O_\lambda = \bigcup_{i=1}^{\infty} R_i$ (a.e.).

The former decomposition of O_λ and the study of distribution function inequalities in the integers (2), that we now proceed to develop, will be used in the proof of (2.19).So we consider a function F defined in the integers and the associated maximal Hilbert transform

$$HF(k) = \sup_{s,t\geq 0}\left|\sum_{s<|k-j|<t} \frac{F(j)}{k-j}\right| \qquad (s, t \in \mathbf{Z}) \tag{2.20}$$

and the maximal function

$$F^*(k) = \sup_{n,m\geq 0} \frac{1}{n+m+1} \sum_{j=-n}^{m} |F(k+j)|. \tag{2.21}$$

Let λ be a positive number. The set

$$\{k\colon HF(k) > \lambda\}$$

can be written as a countable union of disjoint intervals I_i in the intergers and of maximum length. In this situation we can state the following lemma.

(2.22). Lemma. *There exists positive constants C and C' such that*

$$\#\{j \in I_i\colon HF(j) > \beta\lambda, F^*(j) \leq \gamma\lambda\} \leq C\frac{\gamma}{\beta - 1 - \gamma C'} \#I_i$$

for any I_i and where β is bigger than 1.

For the proof just look at the proof of inequality (4) in (2) and write it in the integers.

Proof of inequality (2.19). For n fixed we call $E_{n,l}$ the nonempty subsets of $\{0, 1, \dots, n-1\}$ $(l = 1, 2, \dots, 2^n - 1)$.

For each x of B_n we write

$$E_n^x = \{i\colon 0 \leq i \leq n-1\colon HF(T^i x) > \beta\lambda, f^*(T^i x) \leq \gamma\lambda\}$$

and

$$B_{n,l} = \{x \in B_n\colon E_n^x = E_{n,l}\}.$$

By Lemma (2.22) if $x \in B_n$ we have

$$\#E_n^x \leq \frac{C\gamma}{\beta'} \#\{0, 1, \dots, n-1\}$$

which implies

$$\sum_{j \in E_n^x} w(T^j x) \leq C \cdot \left(\frac{\gamma}{\beta'}\right)^{\delta} \sum_{j=0}^{n-1} w(T^j x) \qquad (x \in B_n)$$

since w satisfies A'_∞. Integrating over B_n we obtain

$$\int_{\bigcup_{j \in E_{n,l}} T^j B_{n,l}} w d\mu \leq C \cdot \left(\frac{\gamma}{\beta'}\right)^{\delta} \int_{\bigcup_{j=0}^{n-1} T^j B_{n,l}} w \, d\mu .$$

Summing first over l and then over n and keeping in mind that $O_\lambda = \bigcup_{n=1}^{\infty} R_n$ (a.e.) we get inequality (2.19).

As is well known a standard argument shows that the "good-λ inequality" (2.19) implies (2.14) (see for example (2)). Therefore we have Theorem (2.13) for f in $L^1(d\mu)$.

Theorem (1.2) now follows combining Theorem (2.16) with standard density arguments.

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