COHOMOLOGY OF COMODULES

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The purpose of the present paper is to develop some aspects of a cohomology theory in the category of C comodules, where C is a bialgebra. This cohomology theory generalizes the rational cohomology of affine algebraic groups and also the cohomology of Lie Algebras. We develop the theory up to the point where we can guarantee the convergence of the analogue of the Hochschild Serre spectral sequence and the convergence of another spectral sequence, which is a natural generalization of the result known as Shapiro's lemma in the cohomology theory of finite groups.

1. Introduction. The attempts to generalize the Hochschild-Serre (or Lyndon-Hochschild-Serre) spectral sequence for the rational cohomology have a long history. One of the main difficulties was the fact that it is no longer evident (as it is for finite groups) that if K is a closed normal subgroup of G then the algebra $P(G)$ of polynomial functions on G is injective as a K-module. A first remark about the need to prove that result was made by Hochschild in [4]. Recently, Cline, Parshall and Scott in [1] and Habousch in [3] proved the above injectivity result and established the validity of the spectral sequence. Here we prove a generalization of the above result, namely that if C and D are commutative Hopf Algebras over a field and $\pi\colon C \to D$ is a normal surjective Hopf Algebra map, then C is injective as a D-comodule (Theorem 4.13).

We begin with a brief description of each section:

2. Cohomology of comodules. Here we introduce the cohomology theory with which we shall be concerned throughout the paper. We present it axiomatically and also construct an explicit resolution functor whose homology gives us the cohomology of comodules. The definition of this comodule cohomology seems to have been known to some specialists and was communicated to the author by G. Hochschild.

3. Normal maps and actions on the cohomology. This section is of a technical nature and has the main purpose of establishing the generalization to the category of comodules of the conjugation action of a group G on the cohomology $H^i(K, M)$ where M is a G-module and K a normal subgroup of G.

4. Induced comodules and injective comodules. Some of the constructions of this section generalize methods of [1] and [4]. We construct a functor π^* that generalizes the induced representation functor, and we prove that C is injective as D-comodule iff the functor π^* is exact (Corollary 4.9). We use this criterion for showing that under certain normality assumptions C is injective as a D-comodule (Theorem 4.13). Finally we prove Theorem 4.15, which is a generalization of the condition for injectivity given in Proposition 2.2 of [5].

5. Two spectral sequences. We establish the validity of the Hochschild Serre spectral sequence for the cohomology of comodules as well as the spectral sequence of Theorem 5.3 that generalizes Shapiro's lemma on the cohomology of induced representations.

We shall follow the terminological conventions of [6]. Coalgebras, bialgebras and Hopf Algebras will be considered over a commutative ring R with identity, or over an arbitrary field k. In the first case, we assume that they are flat when viewed as R-modules (we want the comodules to form an abelian category). If C is a Hopf Algebra then Δ, ε will denote the comultiplication and counit, μ and u will denote the multiplication and the unit, η will stand for the antipode map, and 1_C will stand for the unit of C. If M is a C-comodule then χ or χ_M will indicate the comodule structure on M, and we define $M^C = \{m \in | \chi_M(m) = m \otimes 1_C\}$. If M and N are R-modules then $M \otimes_R N$ will be written as $M \otimes N$ unless there is a danger of confusion. Given an R-coalgebra C (that is flat as an R-module) the C-comodules, together with the C-comodule homomorphisms, form an abelian category that will be denoted by $\mathcal{C}\mathfrak{M}(C)$. The category of R-modules will be denoted by $\mathfrak{M}(R)$. If M and N are R modules then $s\colon M \otimes N \to N \otimes M$ stands for the R-linear switching map sending each $m \otimes n$ onto $n \otimes m$. We use similar notation for any switching map, for example $s_{12,3}$ stands for the R-linear map sending each $x_1 \otimes x_2 \otimes x_3$ onto $x_3 \otimes x_1 \otimes x_2$.

Some of the results of this paper were part of the PhD dissertation of the author at U.C. Berkeley. That dissertation was written under the supervision of Professor G. Hochschild, and the author would like to thank him for his advice and help.

2. Cohomology of comodules. Let C be a bialgebra over a ring R. An object I in $\mathcal{C}\mathfrak{M}(C)$ is said to be relatively injective if it is injective with respect to the injective C-comodule maps that split when considered as R-module maps. The following two results are well known, we include them due to the lack of an adequate reference.

LEMMA 2.1. *Let X be an arbitrary R-module. Then the map* id $\otimes \Delta$ *is a C-comodule structure on $X \otimes C$. Moreover with this structure $X \otimes C$ is a relatively injective object in* $\mathcal{C}\mathfrak{M}(C)$.

Proof. Take a pair of objects U and V in $\mathcal{C}\mathfrak{M}(C)$ and an injective C-comodule map α: $U \to V$, for which there is an R-map β: $V \to U$ satisfying $\beta\alpha = \mathrm{id}_U$. Given any C-comodule map t: $U \to X \otimes C$ consider the map $\tilde{t}$: $V \to X \otimes C$, given by $\tilde{t} = j(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(t \otimes \mathrm{id})(\beta \otimes \mathrm{id})\chi_V$, where j is the map from $X \otimes R \otimes C$ to $X \otimes C$ defined by $j(x \otimes r \otimes c) = rx \otimes c$. It is easy to check that $\tilde{t}\alpha = t$, and that $\tilde{t}$ is a C-comodule map. $\square$

LEMMA 2.2. *Every object in* $\mathcal{C}\mathfrak{M}(C)$ *can be relatively embedded in a relatively injective one.*

Proof. Let M be a C-comodule. Then χ_M: $M \to M \otimes C$ is a C-comodule map when we endow M with the structure χ_M and $M \otimes C$ with structure id $\otimes \Delta$. As χ_M is a relative embedding (follow it by id $\otimes \varepsilon$) our result follows from Lemma 2.1. $\square$

DEFINITION 2.3. We say that a C-comodule Y is coinduced if it is of the form $X \otimes C$ with structure id$_X \otimes \Delta$ for some R-module X.

If we consider the functor $M \to M^C$ from $\mathcal{C}\mathfrak{M}$ to $\mathfrak{M}(R)$ we see easily from the general theory of derived functors in abelian categories that the following result holds.

THEOREM 2.4. *There is one, and (up to equivalence) only one cohomological functor $H^n(C, -)$, $n \geq 0$, from $\mathcal{C}\mathfrak{M}(C)$ to $\mathfrak{M}(R)$ such that*
  (a) $H^0(C, M) = M^0$ *for all C-comodules M.*
  (b) $H^p(C, Y) = 0$ *for every $p \geq 1$ and every coinduced comodule Y.*

Now, we shall describe an explicit resolution functor and express the cohomology functors in terms of that resolution. Then we will describe them in terms of another more manageable resolution ("non homogeneous cochains").

Given any R-module V consider $V \otimes C^p$, $p > 0$, with structure id$_{V \otimes C^{p-1}} \otimes \Delta$ where C^p stands for the p-fold tensor power of C, and $C^0 = R$. If V is a C-comodule we endow $V \otimes C^0 = V$ with the structure χ_V and define the maps d_p as follows:
  (a) $d_0 = \chi_V$,
  (b) Given d_{p-1}: $V \otimes C^{p-1} \to V \otimes C^p$ define d_p as

$$d_p = d_{p-1} \otimes \mathrm{id}_C + (-1)\mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta.$$

It is not hard to prove inductively that

(1) d_p is a C-comodule map for every $p \geq 0$.

(2) The sequence $(V \otimes C^p, d_p)$ is relatively exact, with splitting maps $\zeta_p\colon V \otimes C^p \to V \otimes C^{p-1}$ defined by $\zeta_p(x \otimes c) = (-1)^{p+1}\varepsilon(c)x$ with $x \in V \otimes C^{p-1}$ and $c \in C$. In order to verify (2) one has to check that $d_{p+1}d_p = 0$ and that $\zeta_1 d_0 = \mathrm{id}$,

$$\zeta_{p+1}d_p + d_{p-1}\zeta_p = \mathrm{id}_{V \otimes C^p}.$$

We have

$$
\begin{aligned}
d_{p+1}d_p = &\left[(d_p \otimes \mathrm{id}) + (-1)^{p+1}\mathrm{id}_{V \otimes C^p} \otimes \Delta\right] \\
&\times \left[(d_{p-1} \otimes \mathrm{id}) + (-1)^p \mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta\right] \\
= &(-1)^{p+1}\left(\mathrm{id}_{V \otimes C^p} \otimes \Delta\right)\left(d_{p-1} \otimes \mathrm{id}_C\right) \\
&+ (-1)^p\left(d_p \otimes \mathrm{id}_C\right)\left(\mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta\right) \\
&+ (-1)^{2p+1}\left(\mathrm{id}_{V \otimes C^p} \otimes \Delta\right)\left(\mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta\right).
\end{aligned}
$$

Applying the inductive definition of d_p to the second summand above, we obtain

$$
\begin{aligned}
d_{p+1}d_p = &(-1)^{p+1}\left(\mathrm{id}_{V \otimes C^p} \otimes \Delta\right)\left(d_{p-1} \otimes \mathrm{id}_C\right) \\
&+ (-1)^p\left(d_{p-1} \otimes \mathrm{id}_C \otimes \mathrm{id}_C + (-1)^p \mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta \otimes \mathrm{id}_C\right) \\
&\times \left(\mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta\right) \\
&+ (-1)^{2p+1}\left(\mathrm{id}_{V \otimes C^p} \otimes \Delta\right)\left(\mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta\right),
\end{aligned}
$$

and the terms cancel in pairs. The other statements in (2) can be proven in a similar fashion.

Let us call $\mathcal{C}(V)$ the complex

$$\mathcal{C}(V)\colon V \xrightarrow{d_0} V \otimes C \xrightarrow{d_1} \cdots \to V \otimes C^{p-1} \xrightarrow{d_{p-1}} V \otimes C^p \xrightarrow{d_p} \cdots .$$

This is a relatively injective resolution of V. Consequently, $H^n(C, V)$ may be identified with the homology of the complex $\mathcal{D}(V)$ obtained applying the first part functor to $\mathcal{C}(V)$ and deleting the first R-module.

$$\mathcal{D}(V)\colon 0 \to (V \otimes C)^C \xrightarrow{d_1} (V \otimes C^2)^C \to$$

$$\to \cdots \to (V \otimes C^{p-1})^C \xrightarrow{d_{p-1}} (V \otimes C^p)^C \to \cdots .$$

If we define $\gamma_{p-1}: V \otimes C^{p-1} \to V \otimes C^p$ by $\gamma_{p-1}(x) = x \otimes 1_C$, then γ_{p-1} is a bijective R-map between $V \otimes C^{p-1}$ and $(V \otimes C^p)^C$. A direct computation shows that the differential d_p when restricted to $(V \otimes C^p)^C$ induces a map $\delta_p = \gamma_p^{-1} d_p \gamma_{p-1}: V \otimes C^{p-1} \to V \otimes C^p$, given by

$$(3) \qquad \delta_1 = d_0 - \gamma_0 \quad \text{or} \quad \delta_1(x) = \chi_V(x) - x \otimes 1_C, \quad \text{and}$$

$$(4) \qquad \delta_{p+1} = \delta_p \otimes \mathrm{id}_C + (-1)^{p+1}\left(\mathrm{id}_{V \otimes C^p} \otimes 1_C - \mathrm{id}_{V \otimes C^{p-1}} \otimes \Delta \right.$$
$$\left. + \mathrm{id}_{V \otimes C^{p-1}} \otimes 1_C \otimes \mathrm{id}_C\right).$$

This shows that the complex $\mathfrak{D}(V)$ is isomorphic to the complex

$$\mathfrak{E}(V): 0 \to V \xrightarrow{\delta_1} V \otimes C \xrightarrow{\delta_2} \cdots \to V \otimes C^{p-1} \xrightarrow{\delta_p} V \otimes C^p \xrightarrow{\delta_{p+1}} \cdots.$$

Thus, we can compute $H^n(C, V)$ explicitly as the homology of the complex above.

Let us write δ_1 and δ_2 explicitly

$$\delta_1(x) = \chi_V(x) - x \otimes 1_C.$$
$$\delta_2(x) = \delta_1 \otimes \mathrm{id}_C + \left(\mathrm{id}_{V \otimes C} \otimes 1_C - \mathrm{id} \otimes \Delta + \mathrm{id}_V \otimes 1_C \otimes \mathrm{id}_C\right)$$
$$= \chi_V \otimes \mathrm{id}_C + \mathrm{id}_{V \otimes C} \otimes 1_C - \mathrm{id} \otimes \Delta.$$

In other words

$$\delta_2(v \otimes c) = \chi(v) \otimes c + v \otimes c \otimes 1_C - v \otimes \Delta(c).$$

Hence, we have

$$H^1(C, V)$$
$$= \left\{ \sum v_i \otimes c_i \,\middle|\, \sum_i \chi_V(v_i) \otimes c_i = -\sum_i v_i \otimes c_i \otimes 1_C + \sum_i v_i \otimes \Delta(c_i)\right\}/U,$$

where $U = \{\chi_V(x) - x \otimes 1_C \,|\, x \in V\}$. In the case where the comodule structure of V is trivial we have

$$H^1(C, V) = V \otimes C_0 \quad \text{where } C_0 = \{x \in C \,|\, \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Recall that the trivial comodule structure on V is the one that maps v into $v \otimes 1$, for every v in V.

3. **Normal maps and actions on the cohomology.** Let G be a finite group and K a normal subgroup. For every G-module M there is an action of G on $H^1(K, M)$ having the following properties:

(a) For every exact sequence $0 \to M \to N \to S \to 0$ of G-modules, all the morphisms of the long exact K-cohomology sequence

$$\cdots \to H^i(K, S) \to H^{i+1}(K, M) \to H^{i+1}(K, N) \to H^{i+1}(K, S) \to \cdots$$

are G-maps.

(b) The action of G on $H^0(K, M)$ coincides with the one induced on M^K by the action of G on M.

(c) K acts trivially on $H^i(K, M)$ for every i.

The natural generalization of this situation would be to take a pair of Hopf Algebras C and D and a normal surjective bialgebra map $\pi: C \to D$ and construct a C^D-comodule structure on $H^i(D, M)$ for every C-comodule M. In order to shorten certain long computations and have a simpler notation we shall consider the following (nominally) more general situation.

DEFINITION 3.1. Let C and D be Hopf Algebras over a commutative ring R (as usual these are supposed to be flat as R modules).An adjoint C-structure on D is a C-comodule structure θ on D such that the following diagrams are commutative.

$$
\begin{array}{ccc}
(1) \quad
\begin{array}{ccc}
D & \overset{\theta}{\to} & D \otimes C \\
{\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u \otimes u} \\
R & \to & R \otimes R
\end{array}
& \quad (2) \quad
\begin{array}{ccc}
D & \overset{\theta}{\to} & D \otimes C \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\
D \otimes D & \overset{\theta \boxtimes \theta}{\to} & D \otimes D \otimes C
\end{array}
& \quad (3) \quad
\begin{array}{ccc}
D & \overset{\theta}{\to} & D \otimes C \\
{\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon_D \otimes \varepsilon_C}, \\
R & \to & R \otimes R
\end{array}
\end{array}
$$

where $\theta \boxtimes \theta$ is the tensor product comodule structure whose properties we recall below.

For any bialgebra C and for any pair of C-comodules M and N with structures χ_M and χ_N, the map

$$\chi_M \boxtimes \chi_N = (\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\chi_M \otimes \chi_N)$$

is a C comodule structure on $M \otimes N$. Recall that s stands for the switching map. When we consider the R-module $M \otimes N$ with the structure $\chi_M \boxtimes \chi_N$ we shall denote it as $M \boxtimes N$. It is easy to prove that if $f: M \to M'$, $g: N \to N'$ are C-comodule maps, then

$$f \otimes g: M \boxtimes N \to M' \boxtimes N'$$

is a C-comodule map.

DEFINITION 3.2. Let α be a covariant functor from $\mathcal{CM}(C)$ to $\mathcal{CM}(D)$ having the following properties:

(a) For every object (M, χ_M) in $\mathcal{CM}(C)$, $\alpha(M, \chi_M) = (M, \chi_{M\alpha})$ for some D-comodule structure $\chi_{M\alpha}$ on M.

(b) For any $f \in \operatorname{Hom}_C(M, N)$ $\alpha(f) = f$.

(c) $\alpha(M \boxtimes N) = \alpha(M) \boxtimes \alpha(N)$ (i.e., $(\chi_M \boxtimes \chi_N)_\alpha = \chi_{M\alpha} \boxtimes \chi_{N\alpha}$).

(d) If $u_C \colon R \to R \otimes C$ is the unit of C, viewed as a C-comodule structure on R, then $u_{C\alpha} \colon R \to R \otimes D$ coincides with the unit of D.

We say the α is compatible with the adjoint structure θ if, moreover, α satisfies the following condition:

(e) For every object (M, χ_M) of $\mathcal{C}\mathfrak{M}(C)$, $\chi_{M\alpha} \colon M \to M \otimes D$ is a C-comodule map from (M, χ_M) to $(M \boxtimes D, \chi_M \boxtimes \theta)$.

LEMMA 3.3. *In the situation above, the following four conditions are satisfied*:

(a) *If M is a C-comodule and $f \in \operatorname{Hom}_R(M, R)$, the maps Δ and $(\mathrm{id} \otimes \mu)(s \otimes \mathrm{id})(\mathrm{id} \otimes \theta_\alpha)\Delta$ coincide on every R-submodule of D of the form $\tilde{f}(M)$ where $\tilde{f} \colon M \to D$ is given by $\tilde{f}(x) = \sum_i f(x_i)d_i$ where $\chi_{M\alpha}(x) = \sum x_i \otimes d_i$. (More briefly: $\tilde{f} = (f \otimes \mathrm{id}_D)\chi_{M\alpha}$.)*

(b) *If M is a C-comodule with structure χ_M, then the following diagram is commutative*:

$$
\begin{array}{ccc}
M & \stackrel{\chi_M}{\to} & M \otimes C \\[1em]
\downarrow{\scriptstyle \chi_{M\alpha}} & & \downarrow{\scriptstyle \mathrm{id}_M \otimes \Delta_\alpha} \\[1em]
M \otimes D & \stackrel{\chi_M \otimes \mathrm{id}}{\to} & M \otimes C \otimes D
\end{array}
$$

where Δ_α stands for $\Delta_{C\alpha} \colon C \to C \otimes D$.

(c) $\Delta_\alpha(1_C) = 1_C \otimes 1_D$, $\theta_\alpha(1_D) = 1_D \otimes 1_D$.

(d) *θ_α is an adjoint D-structure on D.*

Proof. (a) Condition (e) of Definition 3.2 means that $(\chi_M \boxtimes \theta)\chi_{M\alpha} = (\chi_{M\alpha} \otimes \mathrm{id})\chi_M$. Conditions (b) of Definition 3.2, applied to $f = \chi_{M\alpha}$, allows us to deduce that $(\chi_M \boxtimes \theta)_\alpha \chi_{M\alpha} = (\chi_{M\alpha} \otimes \mathrm{id})\chi_{M\alpha}$. After applying condition (c) of Definition 3.2 and the definition of $\boxtimes$, we have

$$
(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \theta_\alpha)(\chi_{M\alpha} \otimes \mathrm{id})\chi_{M\alpha} =
$$
$$
= (\chi_{M\alpha} \otimes \mathrm{id})\chi_{M\alpha},
$$

from which we deduce

$$
(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \theta_\alpha)(\mathrm{id} \otimes \Delta)\chi_{M\alpha} = (\mathrm{id} \otimes \Delta)\chi_{M\alpha}.
$$

If $\chi_{M\alpha}(m) = \sum m_i \otimes d_i$, we have

$$
\sum m_i \otimes (\mathrm{id} \otimes \mu)(s \otimes \mathrm{id})(\mathrm{id} \otimes \theta_\alpha)\Delta(d_i) = \sum m_i \otimes \Delta(d_i).
$$

Applying f we deduce our result.

(b) Next we see immediately from part (b) of Definition 3.2, applied to the map $\chi_M \colon M \to M \otimes C$, that the diagram

$$
\begin{array}{ccc}
M & \overset{\chi_M}{\to} & M \otimes C \\[4pt]
\downarrow \chi_{M\alpha} & & \downarrow (\mathrm{id}_M \otimes \Delta)_\alpha \\[4pt]
M \otimes D & \overset{\chi_M \otimes \mathrm{id}}{\to} & M \otimes C \otimes D
\end{array}
$$

is commutative. Now, if we endow M with the trivial D-comodule structure $\chi_0 \colon M \to M \otimes D$ given by $\chi_0(m) = m \otimes 1_D$, we have $\chi_0 \boxtimes \Delta = \mathrm{id}_M \otimes \Delta$, because

$$
(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\chi_0 \otimes \mathrm{id})(m \otimes d)
$$
$$
= (\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(m \otimes 1 \otimes \Delta(d)) = m \otimes \Delta(d).
$$

Therefore, $(\mathrm{id}_M \otimes \Delta)_\alpha = (\chi_0 \boxtimes \Delta)_\alpha = \chi_{0\alpha} \boxtimes \Delta_\alpha = \mathrm{id}_M \otimes \Delta_\alpha$, so that condition (b) is established. Finally, (c) and (d) follow immediately from the definitions.   □

Let M be an arbitrary C-comodule and consider the complex of R-modules, $\mathcal{E}(M)$ defined as in §2 as follows:

$$
\mathcal{E}(M) \colon M \overset{\delta_1}{\to} M \otimes D \overset{\delta_2}{\to} M \otimes D^2 \to \cdots \to M \otimes D^{p-1} \overset{\delta_p}{\to} M \otimes D^p \to \cdots,
$$

$$
\delta_1 = \chi_{M\alpha} - \mathrm{id}_M \otimes 1_D,
$$

$$
\delta_{p+1} = \delta_p \otimes \mathrm{id}_D +
$$
$$
+ (-1)^p (\mathrm{id}_{M \otimes D^{p-1}} \otimes \Delta - \mathrm{id}_{M \otimes D^p} \otimes 1_D - \mathrm{id}_{M \otimes D^{p-1}} \otimes 1_D \otimes \mathrm{id}).
$$

We proved in §2 that the homology of this complex is $H^{\cdot}(D, \alpha(M))$. Now consider the maps $\chi_k \colon M \otimes D^k \to M \otimes D^k \otimes C$ defined inductively as follows, $\chi_0 = \chi_M$, $\chi_k = \chi_{k-1} \boxtimes \theta$. Using the properties of $\boxtimes$ mentioned after Definition 3.1 and induction we see that χ_k is a C-comodule structure on $M \otimes D^k$.

LEMMA 3.4. *The maps δ_k are C-comodule maps when $M \otimes D^k$ is endowed with the structure χ_k.*

Proof. The proof will proceed by induction and will use repeatedly the properties of $\boxtimes$ stated after Definition 3.1. We want to verify that

$$
\chi_{p+1}\delta_{p+1} = (\delta_{p+1} \otimes \mathrm{id}_C)\chi_p.
$$

For $p = 0$, this says that

$$(\chi_M \boxtimes \theta)(\chi_{M\alpha} - \mathrm{id}_M \otimes 1_D) = (\chi_{M\alpha} \otimes \mathrm{id}_C - \mathrm{id}_M \otimes 1_D \otimes \mathrm{id}_C)\chi_M$$

and it suffices to prove that

$$(\chi_M \boxtimes \theta)\chi_{M\alpha} = (\chi_{M\alpha} \otimes \mathrm{id}_C)\chi_M$$

and

$$(\chi_M \boxtimes \theta)(\mathrm{id}_M \otimes 1_D) = (\mathrm{id}_M \otimes 1_D \otimes \mathrm{id}_C)\chi_M.$$

The first equality is nothing but condition (e) of Definition 3.2, the second follows easily from the fact that $\theta(1_D) = 1_D \otimes 1_C$. As to the general case, δ_{p+1} is defined as the sum of four maps (see definition above), and we shall verify that each of the four summands in a C-comodule map. For the first summand, we have to verify that

$$\left(\chi_p \boxtimes \theta\right)\left(\delta_p \otimes \mathrm{id}_C\right) = \left(\delta_p \otimes \mathrm{id} \otimes \mathrm{id}_C\right)\left(\chi_{p-1} \boxtimes \theta\right).$$

By induction we have

$$\chi_p \delta_p = \left(\delta_p \otimes \mathrm{id}_C\right)\chi_{p-1}.$$

Using that $\theta\,\mathrm{id}_C = (\mathrm{id}_C \otimes \mathrm{id}_D)\theta$, we deduce after taking tensor products (see comments after Def. 3.1), that

$$\left(\chi_p \boxtimes \theta\right)\left(\delta_p \otimes \mathrm{id}_C\right) = \left(\delta_p \otimes \mathrm{id}_C \otimes \mathrm{id}_D\right)\left(\chi_{p-1} \boxtimes \theta\right).$$

For the other summands of δ_{p+1}, the proof is based on the same principles. $\square$

Using a similar inductive argument we prove

LEMMA 3.5. *Let* $f\colon M \to N$ *be a* C-*comodule map. Then the maps* $f \otimes \mathrm{id}_{D^p}\colon M \otimes D^p \to N \otimes D^p$ *are* C-*comodule maps when we endow* $M \otimes D^p$ *and* $N \otimes D^p$ *with the structures* χ_p. $\square$

THEOREM 3.6. *Let* C, D, θ, α *be as before. Then, for every* C-*module* M, *there is a* C-*comodule structure* $\tilde{\chi}_k$ *on* $H^k(D, \alpha(M))$ *such that*

(a) *$\tilde{\chi}_0$ coincides with* χ_M.

(b) *For every exact sequence of* C-*comodules* $0 \to M \to N \to S \to 0$, *all the maps in the long exact sequence*

$$\cdots \to H^i(D, \alpha(M)) \to H^i(D, \alpha(N))$$

$$\to H^i(D, \alpha(S)) \to H^{i+1}(D, \alpha(M)) \to \cdots$$

are C-*comodule maps.*

Proof. The existence of $\tilde{\chi}_k$ follows from Lemma 3.4. The fact that $\tilde{\chi}_0$ coincides with χ_M follows from the very definition of χ_0. The fact that the maps $H^i(D, \alpha(M)) \to H^i(D, \alpha(N))$ and $H^i(D, \alpha(N)) \to H^i(D, \alpha(S))$ are C-comodule maps follow from Lemma 3.5. The only part that still requires proof is the assertion that the connecting homomorphisms are C-comodule maps. This will follow from the considerations below.

Let X be an arbitrary bialgebra that is flat when considered as an R-module, and let $\mathbf{A}$ be a complex of X-comodules and X-comodule maps. We define an X-comodule structure on $H(\mathbf{A})$ as follows. Let $r + d_{i-1}(A_{i-2}) \in H^{i-1}(\mathbf{A})$ then $(d \otimes \mathrm{id})\chi(r) = \chi dr = \chi 0 = 0$, since X is R-flat this shows that $\chi(r) \in (\mathrm{Ker}\, d) \otimes X$, i.e., there are $r_j \in \mathrm{Ker}\, d$ and $x_j \in X$ such that $\chi(r) = \sum r_j \otimes x_j$. If $r \in d_{i-1}(A_{i-2})$ we have $r = dr_1$ and then

$$\chi(r) = \chi(dr_1) = (d \otimes \mathrm{id})\chi(r_1) \in d_{i-1}(A_{i-2}) \otimes X.$$

Thus χ induces a map $\tilde{\chi}$ from $H(\mathbf{A})$ to $H(\mathbf{A}) \otimes X$ where

$$\tilde{\chi}(r + d_{i-1}(A_{i-2})) = \sum_j (r_j + d_{i-1}(A_{i-2})) \otimes x_j.$$

Let $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$ be a short exact sequence in the category of complexes of X-comodules. We have the diagram

$$
\begin{array}{ccccccccc}
0 & \to & A_{i-2} & \xrightarrow{\alpha_{i-2}} & B_{i-2} & \xrightarrow{\beta_{i-2}} & C_{i-2} & \to & 0 \\
 & & \downarrow{\scriptstyle d_{i-1}} & & \downarrow{\scriptstyle d_{i-1}} & & \downarrow{\scriptstyle d_{i-1}} & & \\
0 & \to & A_{i-1} & \xrightarrow{\alpha_{i-1}} & B_{i-1} & \xrightarrow{\beta_{i-1}} & C_{i-1} & \to & 0 \\
 & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} & & \\
0 & \to & A_i & \xrightarrow{\alpha_i} & B_i & \xrightarrow{\beta_i} & C_i & \to & 0 \\
 & & \downarrow{\scriptstyle d_{i+1}} & & \downarrow{\scriptstyle d_{i+1}} & & \downarrow{\scriptstyle d_{i+1}} & & \\
0 & \to & A_{i+1} & \xrightarrow{\alpha_{i+1}} & B_{i+1} & \xrightarrow{\beta_{i+1}} & C_{i+1} & \to & 0
\end{array}
$$

where all the squares are commutative, all the rows are exact, all the objects are X-comodules and all the maps are X-comodule maps. Now we recall the definition of $\delta: H^{i-1}(\mathbf{C}) \to H^i(\mathbf{A})$. Take an element $c \in C_{i-1}$ such that $d_i c = 0$ and consider $a \in A_i$ such that $\alpha_i a = d_i b$ with $b \in B_{i-1}$ such that $\beta_{i-1} b = c$. Then, as $\alpha_{i+1} d_{i+1} a = d_{i+1} \alpha_i a = d_{i+1} d_i b = 0$, we have $a \in \mathrm{Ker}\, d_{i+1}$. In these terms we have $\delta(c + d_{i-1}(C_{i-2})) = a + d_{i-1}(A_{i-1})$. Let $\chi(a) = \sum a_k \otimes x_k$ with $a_k \in \mathrm{Ker}\, d_{i+1}$. Then $\chi \alpha a = (\alpha \otimes \mathrm{id})\chi a = \sum a_k \otimes x_k$. Thus $\chi db = \sum \alpha a_k \otimes x_k$. On the other hand, if

$\chi b = \Sigma\, b_j \otimes y_j$, then $\chi db = \Sigma\, db_j \otimes y_j$. Thus, we have the equality

(1)
$$\sum db_j \otimes y_j = \sum \alpha a_k \otimes x_k.$$

Now $\chi(c) = \chi\beta b = (\beta \otimes \mathrm{id})\chi b = \Sigma\, \beta b_j \otimes y_j$. Now take $t_r \in \mathrm{Ker}\, d$ and $z_r \in X$ such that $\chi(c) = \Sigma\, t_r \otimes z_r$. Then

$$\tilde{\chi}\big(c + d_{i-1}(C_{i-2})\big) = \sum_r \big(t_r + d_{i-1}(C_{i-2})\big) \otimes z_r.$$

Consequently, we have

$$(\delta \otimes \mathrm{id})\tilde{\chi}\big(c + d_{i-1}(C_{i-2})\big) = \sum \delta\big(t_r + d_{i-1}(C_{i-2})\big) \otimes z_r.$$

Take $m_r \in B_{i-1}$ such that $\beta m_r = t_r$ and then choose $s_r \in A_i$ such that $dm_r = \alpha s_r$. Now, we have

$$(\delta \otimes \mathrm{id})\tilde{\chi}\big(c + d_{i-1}(C_{i-2})\big) = \sum \big(s_r + d_i(A_{i-1})\big) \otimes z_r.$$

On the other hand,

$$\tilde{\chi}\big(\delta(c + d_{i-1}(C_{i-2}))\big) = \tilde{\chi}\big(a + d_i(A_{i-1})\big) = \sum \big(a_k + d_i(A_{i-1})\big) \otimes x_k.$$

From $\Sigma\, \beta b_j \otimes y_j = \chi(c) = \Sigma\, t_r \otimes z_r$, we deduce that

$$\sum \beta b_j \otimes y_j = \sum \beta m_r \otimes z_r.$$

Consequently,

$$\sum b_j \otimes y_j - \sum m_r \otimes z_r \in \mathrm{Ker}(\beta \otimes \mathrm{id}) = \mathrm{Ker}(\beta) \otimes X = \mathrm{Im}\,\alpha \otimes X.$$

Thus, there are elements a_l' in A_{i-1} and w_l in X such that

$$\sum b_j \otimes y_j - \sum m_r \otimes z_r = \sum \alpha(a_l') \otimes w_l.$$

Hence, $\Sigma\, db_j \otimes y_j - \Sigma\, dm_r \otimes z_r = \Sigma\, d\alpha a_l' \otimes w_l$. Using $dm_r = \alpha s_r$ and (1) we obtain

$$\sum \alpha a_k \otimes x_k - \sum \alpha s_r \otimes z_r = \sum \alpha d a_l' \otimes w_l.$$

As α is injective and X is flat as an R-module, we conclude that

$$\sum a_k \otimes x_k - \sum s_r \otimes z_r = \sum d a_l' \otimes w_l.$$

Then

$$\sum \big(s_r + d_i(A_{i-1})\big) \otimes z_r = \sum \big(a_k + d_i(A_{i-1})\big) \otimes x_k,$$

i.e., $\tilde{\chi}\delta = (\delta \otimes \mathrm{id})\tilde{\chi}$. $\qquad \square$

The following results generalize the fact that for finite groups not only G acts on the cohomology $H^i(K, M)$ by conjugation, but that this action factors through K, giving an action of G/K on $H^i(K, M)$.

THEOREM 3.7. *Let X be a Hopf Algebra over a commutative ring R and let θ_X be an adjoint X-structure on X. Then if* $(\mathrm{id} \otimes \mu)(s \otimes \mathrm{id})(\mathrm{id} \otimes \theta_X)\Delta = \Delta$, *the identity functor* id: $\mathcal{CM}(X) \to \mathcal{CM}(X)$ *is compatible with the structure* θ_X. *Moreover, the structure $\tilde{\chi}_k$ defined as in Theorem 3.6 is trivial* (*i.e.* $\tilde{\chi}_k(x) = x \otimes 1$ *for every* $x \in H^k(X, M)$).

Proof. Referring to Definition 3.2, conditions (a), (b), (c) and (d) are trivially verified. As to (e), we have to prove that if χ_M is an X-comodule structure on M, then

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \theta_X)(\chi \otimes \mathrm{id})\chi = (\chi \otimes \mathrm{id})\chi.$$

It is clear that $(\chi_M \otimes \mathrm{id})\chi_M = (\mathrm{id} \otimes \Delta)\chi_M$ and our hypothesis, imply the equality above. As to the triviality of $\tilde{\chi}_k$ we proceed by induction (dimension shifting). The axioms of the cohomology theory we are looking at, guarantee, for every M, the existence of an exact sequence $0 \to M \to X_M \to S_M \to 0$ in the category $\mathcal{CM}(X)$ such that the maps

$$\delta: H^i(X, S_M) \to H^{i+1}(X, M)$$

are surjective for every $i \geq 0$. By Theorem 3.6, $\tilde{\chi}_{i+1}\delta = (\delta \otimes \mathrm{id})\tilde{\chi}_i$. Given $x \in H^{i+1}(X, M)$ there is a $y \in H^i(X, S_M)$ such that $\delta y = x$. Then, $\tilde{\chi}_{i+1}(x) = \tilde{\chi}_{i+1}\delta y = (\delta \otimes \mathrm{id})\tilde{\chi}_i(y)$. If $\tilde{\chi}_i(y) = y \otimes 1$ we deduce that $\tilde{\chi}_{i+1}(x) = x \otimes 1$. As the result is obvious for $k = 0$, our proof is finished.   □

Now, let us return to the situation of Theorem 3.6. Let Δ be the comultiplication on C, and let $\Delta_\alpha: C \to C \otimes D$ be the D-structure associated with Δ via α. Assume that the map $U: C \to D$ given by $U(x) = \sum \varepsilon(c_i)d_i$ with $\Delta_\alpha(x) = \sum x_i \otimes d_i$, is surjective. Lemma 3.3 part (a) guarantees that θ_α verifies $(\mathrm{id} \otimes \mu)(s \otimes \mathrm{id})(\mathrm{id} \otimes \theta_\alpha)\Delta_D = \Delta_D$. Consequently, by Theorem 3.7, θ_α will be an adjoint D-structure on D compatible with the identity functor on $\mathcal{CM}(D)$. Now consider the map χ_k-$\chi_M \boxtimes \theta \boxtimes \cdots \boxtimes \theta$ from $M \otimes D^k$ to $M \otimes D^k \otimes C$. We have $\chi_{k\alpha} = \chi_{M\alpha} \boxtimes \theta_\alpha \boxtimes \cdots \boxtimes \theta_\alpha$. If $(\chi_{k\alpha})^{\sim}$ denotes the D-comodule structure induced by $\chi_{k\alpha}$ on $H^k(D, \alpha(M))$, Theorem 3.7 guarantees that $(\chi_{k\alpha})^{\sim}(t) = t \otimes 1_D$, for $t \in H^k(D, \alpha(M))$. If $\tilde{\chi}_k$ denotes the C-structure defined by χ_k on $H^k(D, M)$, it is clear from our definitions that $(\tilde{\chi}_k)_\alpha = (\chi_{k\alpha})^{\sim}$. The above considerations amount to the fact that, for every $t \in H^k(D, \alpha(M))$, we have $(\tilde{\chi}_k)_\alpha(t) = t \otimes 1_D$.

THEOREM 3.8. *Let C and D be Hopf Algebras over a field k, let θ be an adjoint C-structure on D and let α be a functor compatible with the structure θ. Assume also that the map $U: C \to D$ defined as above is surjective. Then $\alpha(C)^D$ is a subcoalgebra of C and for every C-comodule M there is a natural $\alpha(C)^D$ comodule structure $\tilde{\chi}_k$ on $H^k(D, \alpha(M))$ such that*

(a) $\tilde{\chi}_0$ *coincides with* χ_M.

(b) *For every exact sequence of C-comodules $0 \to M \to N \to S \to 0$ all the maps of the sequence*

$$\cdots \to H^i(D, \alpha(M)) \to H^i(D, \alpha(N)) \to H^i(D, \alpha(S))$$
$$\to H^{i+1}(D, \alpha(M)) \to \cdots$$

are $\alpha(C)^D$ comodule maps.

Proof. The only assertions requiring a proof (see Theorem 3.6) are

(i) $\alpha(C)^D$ is a subcoalgebra of C.

(ii) $\tilde{\chi}_i(H^i(D, \alpha(M))) \subset H^i(D, \alpha(M)) \otimes \alpha(C)^D$.

Clearly (i) is a particular case of (ii) with $M = C$ and $i = 0$. Now consider $x \in H^i(D, \alpha(M))$. By Lemma 3.3(b), we have $(\tilde{\chi}_k \otimes \text{id})(\tilde{\chi}_k)_\alpha = (\text{id} \otimes \Delta_\alpha)\tilde{\chi}_k$. If $\tilde{\chi}_k(x) = \sum x_i \otimes c_i$, then, since $(\tilde{\chi}_k)_\alpha(x) = x \otimes 1$ (see comments after Theorem 3.7), we have $\sum x_i \otimes c_i \otimes 1 = \sum x_i \otimes \Delta_\alpha(c_i)$. Since we are working over a field, we may conclude that $\tilde{\chi}_k(x) \in H^k(D, \alpha(M)) \otimes \alpha(C)^D$. $\square$

LEMMA 3.9. *In the situation above, $(\alpha(M)^D)^{\alpha(C)^D} = M^C$.*

Proof. It is clear that $(\alpha(M)^D)^{\alpha(C)^D} \subset M^C$. Our result will be proved as soon as we have shown that $M^C \subset \alpha(M)^D$. Using Lemma 3.3, part (b), we deduce that if $m \in M^C$, then $m \otimes 1 \otimes 1 = (\chi_M \otimes \text{id})\chi_{M\alpha}(m)$. Applying $(\text{id} \otimes \varepsilon \otimes \text{id})$ we obtain $m \otimes 1 = \chi_{M\alpha}(m)$. $\square$

DEFINITION 3.10. Let C be a Hopf Algebra defined over a ring R and assume that the antipode of C is involutory. The map $\theta: C \to C \otimes C$, $\theta = (\eta \otimes \mu)(\Delta\eta \otimes \text{id})\Delta$ is called the conjugate comodule structure on C.

As $\Delta\eta = s(\eta \otimes \eta)\Delta$ and $\eta^2 = \text{id}$, we have

$$\theta = (\text{id} \otimes \mu)(\text{id} \otimes \eta \otimes \text{id})(s \otimes \text{id})(\Delta \otimes \text{id})\Delta.$$

DEFINITION 3.11. Let C and D be Hopf Algebras defined over a ring R, and assume that the antipode of C is involutory. Let $\pi: C \to D$ be a surjective bialgebra map. We say that π is normal if $\theta(\text{Ker } \pi) \subset (\text{ker } \pi \otimes C)$ where θ is the conjugate structure defined above.

In the situation of Definition 3.11 there is a unique map $\tilde{\theta}$ that makes the diagram that follows commutative.

$$
\begin{array}{ccc}
C & \xrightarrow{\theta} & C \otimes C \\
\downarrow \pi & & \downarrow \pi \otimes \mathrm{id} \\
D & \xrightarrow{\tilde{\theta}} & D \otimes C
\end{array}
$$

LEMMA 3.12. *Let* C, D, θ, $\tilde{\theta}$ *and* π *as above. Then* $\tilde{\theta}$ *is an adjoint* C*-structure on* D, *in the sense of Definition* 3.1.

Proof. Using the fact that π is a bialgebra map it is easy to reduce that Lemma to the case $C = D$, $\pi = \mathrm{id}$, $\theta = \tilde{\theta}$. In that case, we have to prove

(a) θ is a C-comodule map.

(b) $\theta u = u \otimes u$.

(c) $(\Delta \otimes \mathrm{id})\theta = (\theta \boxtimes \theta)\Delta$.

(d) $(\varepsilon \otimes \varepsilon)\theta = \varepsilon$.

The verification of (b) and (d) is more or less immediate. As to (a) and (c), we shall verify (a). The verification of (c) is similar but longer. We leave if for the patient reader. We check that $(\mathrm{id} \otimes \Delta)\theta = (\theta \otimes \mathrm{id})\theta$. Using the definition of θ and the equalities $\Delta\eta = s(\eta \otimes \eta)\Delta$,

$$
\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\Delta \otimes \Delta),
$$

we have

$$
(\mathrm{id} \otimes \Delta)\theta = (\mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(s \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta
$$
$$
= (\mathrm{id} \otimes \mu \otimes \mu)(\mathrm{id} \otimes \mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \Delta \otimes \Delta)
$$
$$
(\mathrm{id} \otimes \eta \otimes \mathrm{id})(s \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta
$$
$$
= (\mathrm{id} \otimes \mu \otimes \mu)(\mathrm{id} \otimes \mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes s \otimes \mathrm{id} \otimes \mathrm{id})
$$
$$
(\mathrm{id} \otimes \eta \otimes \eta \otimes \mathrm{id} \otimes \mathrm{id})(s_{12,3} \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id} \otimes \Delta)(\Delta \otimes \mathrm{id})\Delta.
$$

Similarly

$$
(\theta \otimes \mathrm{id})\theta = (\mathrm{id} \otimes \mu \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})
$$
$$
(s \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \eta \otimes \mathrm{id})
$$
$$
(\Delta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(s \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta.
$$

A direct computation with $(\Delta \otimes \text{id} \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id} \otimes \text{id})(s \otimes \text{id})$ gives

$$(\theta \otimes \text{id})\theta = (\text{id} \otimes \mu \otimes \mu)(\text{id} \otimes \eta \otimes \text{id} \otimes \text{id} \otimes \text{id})$$

$$(s \otimes \text{id} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{id} \otimes \text{id} \otimes \eta \otimes \text{id})$$

$$(s_{1,234} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(\Delta \otimes \text{id})\Delta.$$

By coassociativity,

$$(\Delta \otimes \text{id} \otimes \Delta)(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(\Delta \otimes \text{id})\Delta.$$

Therefore, in order to prove that $(\theta \otimes \text{id})\theta = (\text{id} \otimes \Delta)\theta$, all we have to verify is that

$$(\text{id} \otimes \mu \otimes \mu)(\text{id} \otimes \eta \otimes \text{id} \otimes \text{id} \otimes \text{id})(s \otimes \text{id} \otimes \text{id} \otimes \text{id})$$
$$(\text{id} \otimes \text{id} \otimes \text{id} \otimes \eta \otimes \text{id})(s_{1,234} \otimes \text{id})$$
$$= (\text{id} \otimes \mu \otimes \mu)(\text{id} \otimes \text{id} \otimes s \otimes \text{id})(\text{id} \otimes s \otimes \text{id} \otimes \text{id})$$
$$(\text{id} \otimes \eta \otimes \eta \otimes \text{id} \otimes \text{id})(s_{12,3} \otimes \text{id} \otimes \text{id}).$$

This equality can be verified directly. $\qquad\qquad\square$

Now for arbitrary coalgebras, we can make the following definition.

DEFINITION 3.13. Let C and D be coalgebras defined over R, a commutative ring with identity, and assume that C and D are flat as R-modules. Let π be a surjective coalgebra map from C onto D. The map π induces a functor $\pi_*: \mathcal{C}\mathfrak{M}(C) \to \mathcal{C}\mathfrak{M}(D)$ as follows:

(a) If (M, χ_M) is an object in $\mathcal{C}\mathfrak{M}(C)$,

$$\pi_*(M, \chi_M) = (M, (\text{id} \otimes \pi)\chi_M).$$

(b) If $f: M \to N$ is a C-comodule map $\pi_*(f) = f$.

There are a few things to verify in the definition above, all of which are immediate.

LEMMA 3.14. Let C, D, π, θ, and $\tilde{\theta}$ be as in Lemma 3.12. Then, the functor π_* is compatible with the adjoint structure $\tilde{\theta}$.

Proof. Referring to definition 3.2, the only condition not immediately clear is condition (e). Again an easy argument reduces our verification to the case $C = D$, $\pi = \text{id}$, $\theta = \tilde{\theta}$, in which case condition (e) of Definition 3.2 reads: $(\chi_M \boxtimes \theta)\chi_M = (\chi_M \otimes \text{id})\chi_M$. Actually, we have

$$(\chi_M \boxtimes \theta)\chi_M$$
$$= (\text{id} \otimes \text{id} \otimes \mu)(\text{id} \otimes s \otimes \text{id})(\text{id} \otimes \text{id} \otimes \theta)(\chi_M \otimes \text{id})\chi_M$$
$$= (\text{id} \otimes \text{id} \otimes \mu)(\text{id} \otimes s \otimes \text{id})(\text{id} \otimes \text{id} \otimes \theta)(\text{id} \otimes \Delta)\chi_M.$$

As $(\chi_M \otimes \text{id})\chi_M = (\text{id} \otimes \Delta)\chi_M$, it is enough to prove that

$$(\text{id} \otimes \mu)(s \otimes \text{id})(\text{id} \otimes \theta)\Delta = \Delta.$$

If we write down the definition of θ in the formula above we obtain

$$(\text{id} \otimes \mu)(s \otimes \text{id})(\text{id} \otimes \theta)\Delta$$
$$= (\text{id} \otimes \mu)(s \otimes \text{id})(\text{id} \otimes \text{id} \otimes \mu)(\text{id} \otimes s \otimes \text{id})(\text{id} \otimes \eta \otimes \text{id} \otimes \text{id})$$
$$(\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)\Delta$$
$$= (\text{id} \otimes \mu)(s \otimes \text{id})(\mu(\text{id} \otimes \eta)\Delta \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)\Delta.$$

As

$$\mu(\text{id} \otimes \eta)\Delta = u\varepsilon \quad \text{and} \quad (\text{id} \otimes \mu)(s \otimes \text{id})(u\varepsilon \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta) = \text{id},$$

we deduce our result. $\square$

In the case of Lemma 3.14, $\Delta_{\pi_*} = (\text{id} \otimes \pi)\Delta$. Then the map U considered in Theorem 3.8 coincides with π and consequently is surjective. Therefore, Theorem 3.8 yields the following result:

THEOREM 3.15. *Let C and D be Hopf Algebras over a field, that have involutory antipodes, and let π be a surjective normal bialgebra map from C to D. Then, given any C-comodule M there is a natural $\pi_*(C)^D$ comodule structure $\tilde{\chi}_k$ on $H^k(D, \pi_*(M))$ such that:*
   (a) *$\tilde{\chi}_0$ coincides with χ_M.*
   (b) *If $0 \to M \to N \to S \to 0$ is an exact sequence of C-comodules, the maps in the sequence*

$$\cdots \to H^k\big(D, \pi_*(M)\big) \to H^k\big(D, \pi_*(N)\big) \to H^k\big(D, \pi_*(S)\big)$$
$$\to H^{k+1}\big(D, \pi_*(M)\big) \to \cdots$$

are $\pi_*(C)^D$ comodule maps.

4. Induced comodules and injective comodules.

In this section we shall study cetain injective comodules in order to be able to prove that the spectral sequences of §5 converge. We shall generalize results on injective comodules from [4, 5] [1] and [3].

LEMMA 4.1. *Let C be a Hopf Algebra over a field, let M be an arbitrary C-comodule and let V be a finite dimensional C-comodule. If we endow V* (dual space of V) with its dual C-comodule structure then $(M \boxtimes V^*)^C = \text{Hom}_C(V, M)$.*

Proof. Let us choose a basis of V, (v_i) $i = 1,\ldots,n$, and let (f_i) be the dual basis of V^*. If $\chi_V(v_j) = \sum_k v_k \otimes c_{jk}$ the dual C-comodule structure on V^* is given by $\chi_{V^*}(f_i) = \sum_k f_k \otimes \eta(c_{ki})$. Consider the k-linear isomorphism $\Phi: M \otimes V^* \to \mathrm{Hom}_k(V, M)$ given by $\Phi(\sum_i m_i \otimes \gamma_i)(v) = \sum_i \gamma_i(v)m_i$. We shall prove that $\Phi((M \boxtimes V^*)^C) = \mathrm{Hom}_C(V, M)$. The condition for an element $\sum_i m_i \otimes f_i$ to be in $(M \boxtimes V^*)^C$ can be expressed as follows, where $\chi_M(m_i) = \sum_t m_{ti} \otimes c'_{ti}$.

$$\sum_i m_i \otimes f_i \otimes 1_C = \sum_{ikt} m_{ti} \otimes f_k \otimes c'_{ti}\eta(c_{ki}).$$

Thus, $\sum m_i \otimes f_i \in (M \boxtimes V^*)^C$ if and only if the following condition is satisfied:

(*) $$m_k \otimes 1_C = \sum_{it} m_{ti} \otimes c'_{ti}\eta(c_{ki}) \quad \text{for } k = 1,\ldots,n.$$

The condition $\Phi(\sum m_i \otimes f_i) \in \mathrm{Hom}_C(V, M)$ means that the maps $\chi_M(\Phi(\sum m_i \otimes f_i))$ and $(\Phi(\sum m_i \otimes f_i) \otimes \mathrm{id})\chi_V$ from V to $M \otimes C$ coincide. This happens if and only if they coincide at v_j for $j = 1,\ldots,n$. As $\Phi(\sum m_i \otimes f_i)(v_j) = m_j$, the maps coincide if and only if $\chi_M(m_j) = \sum_k m_k \otimes c_{jk}$. Thus $\Phi(\sum m_i \otimes f_i) \in \mathrm{Hom}_C(V, M)$ if and only if the following condition is satisfied:

(**) $$\sum_t m_{tj} \otimes c'_{tj} = \sum_k m_k \otimes c_{jk}.$$

As c_{jk} are the matrix coefficients of a comodule structure, they satisfy

$$\Delta(c_{jt}) = \sum_{kt} c_{kt} \otimes c_{jk} \quad \text{and} \quad \varepsilon(c_{jk}) = \delta_{jk}.$$

Hence,

$$\sum_k c_{kt}\eta(c_{jk}) = \mu(\mathrm{id} \otimes \eta)\Delta(c_{jt}) = u\big(\varepsilon(c_{jt})\big) = \delta_{jt},$$

and

$$\sum_k \eta(c_{kt})c_{jk} = \delta_{jt}.$$

If we tensor (**) with c_{rj}, apply $\mathrm{id} \otimes \mathrm{id} \otimes \eta$ and multiply, we obtain

$$\sum_{jt} m_{jt} \otimes c'_{tj}\eta(c_{rj}) = \sum_{kj} m_i \otimes c_{jk}\eta(c_{rj})$$

$$= \sum_k m_k \otimes \sum_j c_{jk}\eta(c_{rj})$$

$$= \sum_k m_k \otimes \delta_{rk} = m_r \otimes 1_C.$$

This is exactly condition (∗). In a similar way we can go from (∗) to (∗∗).   □

THEOREM 4.2. *Let C be a Hopf Algebra over a field with involutory antipode. Let M be an arbitrary C-comodule, and let F_M denote the functor from $\mathcal{CM}(C)$ to $\mathcal{M}(k)$ given by $F_M(V) = (M \boxtimes V)^C$. Then M is injective as a C-comodule if and only if the functor F_M is exact. (Note that the functor F_M is always left exact.)*

Proof. It is easy to see (using a Zorn's Lemma type of argument) that M is injective as a C-comodule if and only if for every pair (V_1, V_2) of *finite dimensional* C-comodules, for every injective C-comodule map $\alpha: V_1 \to V_2$ and for every C-comodule map $f: V_1 \to M$, there exists a C-comodule map $\tilde{f}: V_2 \to M$ such that $\tilde{f}\alpha = f$.

In a similar fashion we can reduce the proof that F_M is right exact to the finite dimensional case. Denote $\mathcal{CM}_f(M)$ the category of finite dimensional C-comodules, and let us call i the functor from $\mathcal{CM}_f(C)$ to itself that sends V to V^*. We have proved that, on $\mathcal{CM}_f(C)$, F_M i is naturally equivalent to $\mathrm{Hom}_C(-, M)$. The C-comodule M is injective if and only if the functor $\mathrm{Hom}_C(-, M)$ from $\mathcal{CM}_f(C)$ to $\mathcal{M}(k)$ is left exact. Thus, it is clear that, on $\mathcal{CM}_f(C)$, F_M right exact implies that $\mathrm{Hom}_C(-, M)$ is left exact. Conversely, as $\eta^2 = \mathrm{id}$, F_m i i is naturally equivalent to F_M on $\mathcal{CM}_f(C)$. Consequently, on $\mathcal{CM}_f(C)$, F_M is equivalent to $\mathrm{Hom}_C(-, M)$ i. Thus our conclusion follows.   □

COROLLARY 4.3. *In the situation of Theorem 4.2, M is an injective C-comodule if and only if $H^i(C, M \boxtimes V) = 0$ for every C-comodule V.*   □

Now consider the following situation. Let C and D be coalgebras over a field k, and let $\pi: C \to D$ be a surjective coalgebra map. In §3, we constructed the functor $\pi_*: \mathcal{CM}(C) \to \mathcal{CM}(D)$. We shall see that π_* has a right adjoint π^* that is closely related to the functor F_M constructed above, for $M = \pi_*(C)$.

Let V be a D-comodule with structure χ_V. We define a map $\Phi_V: V \otimes C \to V \otimes D \otimes C$ by $\Phi_V = (\chi_V \otimes \mathrm{id}_C) - \mathrm{id}_V \otimes (\pi \otimes \mathrm{id})\Delta_C$. It is immediate that, with respect to the C-comodule structure $\mathrm{id} \otimes \Delta$ on $V \otimes C$, the kernel of Φ_V is a subcomodule. Using the notations of [8], $\mathrm{Ker}\,\Phi_V$ is the cotensor product of the right D-comodule V with structure χ_V and the left D-comodule C with structure $(\pi \otimes \mathrm{id})\Delta_C$. Another interpretation of the C-comodule $\mathrm{Ker}\,\Phi_V$ with structure $\mathrm{id} \otimes \Delta$ is the following. The coalgebra structure of C induces an algebra structure on C^* (dual

space of C). If V is a C-comodule, we can associate to it a C^*-module in the usual fashion. (The base space is the same V and if $v \in V$ and $f \in C^*$, $f \cdot v = \Sigma_i f(c_i)v_i$, where $\chi_V(v) = \Sigma c_i \otimes v_i$.) If $\pi: C \to D$ is a surjective coalgebra map, it induces an injective algebra map from D^* into C^*. In that case if V is a D-comodule, $\mathrm{Ker}\, \Phi_V$ is a C-comodule that, with the associated C^*-module structure, is isomorphic with $\mathrm{Hom}_{D^*}(C^*, V)$. Here, we are considering V as a D^*-module as above and C^* as a D^*-module via the injection from D^* into C^*, and $\mathrm{Hom}_{D^*}(C^*, V)$ is endowed with a C^*-module structure in the usual fashion.

If $C = P(G)$ and $D = P(K)$, with K a closed subgroup of the affine algebraic group G, and π is the restriction map, the space $\mathrm{Ker}\, \Phi_V$ with the structure (id $\otimes \Delta$) is the induced representation functor studied in [1].

It is easy to see that if $f: V \to W$ is a morphism of D-comodules, then $(f \otimes \mathrm{id})(\mathrm{Ker}\, \Phi_V) \subset \mathrm{Ker}\, \Phi_W$.

DEFINITION 4.4. In the situation above, we define the functor π^* from $\mathcal{C}\mathfrak{M}(D)$ to $\mathcal{C}\mathfrak{M}(C)$ as follows. If V is a D-comodule then $\pi^*(V) = \mathrm{Ker}\, \Phi_V$, endowed with the structure $\mathrm{id}_V \otimes \Delta$; if f is a D-comodule map from V to W then $\pi^*(f)$ is the restriction of $f \otimes \mathrm{id}$ to $\pi^*(V)$.

LEMMA 4.5. *In the situation above, $\pi^*(D)$ is isomorphic with C, as a C-comodule.*

Proof. Consider the map $\gamma: C \to D \otimes C$, where $\gamma = (\pi \otimes \mathrm{id})\Delta$. We have

$$\Phi_D \gamma = ((\Delta \otimes \mathrm{id}) - \mathrm{id} \otimes (\pi \otimes \mathrm{id})\Delta)(\pi \otimes \mathrm{id})\Delta$$

$$= (\Delta\pi \otimes \mathrm{id})\Delta - (\pi \otimes \pi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta$$

$$= (\Delta\pi \otimes \mathrm{id})\Delta - ((\pi \otimes \pi)\Delta \otimes \mathrm{id})\Delta = 0.$$

Thus, $\gamma(C) \subset \pi^*(D)$. Consider now $\delta: D \otimes C \to C$, where $\delta(d \otimes c) = \varepsilon(d)c$. The restriction of δ to $\pi^*(D)$ is the inverse of γ, because if $x \in C$ and $\Delta(x) = \Sigma x_i \otimes x_i'$ then $(\delta\gamma)(x) = \Sigma_i \varepsilon(\pi(x_i))x_i' = \Sigma \varepsilon(x_i)x_i' = x$. Conversely, if $\Sigma d_i \otimes c_i \in \pi^*(D)$, then

$$\Sigma \Delta(d_i) \otimes c_i = \Sigma d_i \otimes (\pi \otimes \mathrm{id})\Delta(c_i).$$

Applying $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ we get

$$\Sigma d_i \otimes c_i = \Sigma \varepsilon(d_i)(\pi \otimes \mathrm{id})\Delta(c_i) = (\gamma\delta)(\Sigma d_i \otimes c_i).$$

Finally, γ is a C-map because

$$(\mathrm{id} \otimes \Delta)\gamma = (\mathrm{id} \otimes \Delta)(\pi \otimes \mathrm{id})\Delta = (\pi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta$$

$$= (\pi \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta = (\gamma \otimes \mathrm{id})\Delta. \qquad \square$$

If V is a D-comodule, we consider the map $E_V: \pi^*(V) \to V$ given by $E_V(\Sigma\, v_i \otimes c_i) = \Sigma\, v_i \varepsilon(c_i)$.

LEMMA 4.6. (a) *If V is a D-comodule then E_V is a morphism of D-comodules from $\pi_*(\pi^*(V))$ to V.*

(b) *If W is a C-comodule, V a D-comodule and $f: \pi_*(W) \to V$ a D-comodule map, there is one and only one morphism of C-comodules $\tilde{f}: W \to \pi^*(V)$ such that $E_V \pi_*(\tilde{f}) = f$.*

Proof. (a) If $\Sigma\, v_i \otimes c_i$ verifies $\Sigma\, \chi_V(v_i) \otimes c_i = \Sigma\, v_i \otimes (\pi \otimes \mathrm{id})\Delta(c_i)$ when we apply $\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon$, we have

$$(*) \qquad\qquad \sum \varepsilon(c_i)\chi_V(v_i) = \sum v_i \otimes \pi(c_i).$$

Now, if $\Delta(c_i) = \Sigma\, c_{ik} \otimes c'_{ik}$, we have

$$(E_V \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi)(\mathrm{id} \otimes \Delta)\left(\sum_i v_i \otimes c_i\right)$$

$$= \sum_{ik}(E_V \otimes \mathrm{id})(v_i \otimes c_{ik} \otimes \pi(c'_{ik}))$$

$$= \sum_{ik} \varepsilon(c_{ik})v_i \otimes \pi(c'_{ik}) = \sum_i v_i \otimes \sum_k \varepsilon(c_{ik})\pi(c'_{ik})$$

$$= \sum_i v_i \otimes \pi(c_i).$$

Using equality $(*)$ we deduce that

$$(E_V \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi)(\mathrm{id} \otimes \Delta)\left(\sum v_i \otimes c_i\right)$$

$$= \sum \varepsilon(c_i)\chi_V(v_i) = (\chi_V E_V)\sum(v_i \otimes c_i).$$

(b) Consider $\tilde{f}: W \to V \otimes C$ defined by $\tilde{f} = (f \otimes \mathrm{id})\chi_W$. We have

$$\Phi_V(f \otimes \mathrm{id})\chi_W$$

$$= (\chi_V f \otimes \mathrm{id})\chi_W - (f \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \pi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\chi_W.$$

As f is a morphism of D-comodules, we have $\chi_V f = (f \otimes \mathrm{id})(\mathrm{id} \otimes \pi)\chi_W$. Hence, $\Phi_V(f \otimes \mathrm{id})\chi_W = 0$ and $\tilde{f}(W) \subset \pi^*(V)$.

It is almost immediate that $E_V(f \otimes \mathrm{id})\chi_W = f$. The verification that $\tilde{f}$ is a C-comodule map goes along similar lines as the proof of Lemma 4.5.

It only remains to be proven that if $h\colon W \to \pi^*(V)$ is a C-comodule map such that $E_V \pi_*(h) = 0$, then $h = 0$. We have $(h \otimes \mathrm{id})\chi_W = (\mathrm{id} \otimes \Delta)h$. Applying $(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})$, we obtain

$$((\mathrm{id} \otimes \varepsilon)h \otimes \mathrm{id})\chi_W = (\mathrm{id} \otimes (\varepsilon \otimes \mathrm{id})\Delta)h.$$

If $h(x) = \Sigma\, v_i \otimes c_i$ we deduce from the last equality that

$$((\mathrm{id} \otimes \varepsilon)h \otimes \mathrm{id})\chi_W(x) = \sum v_i \otimes 1 \otimes c_i.$$

As $E_V \pi_* h = 0$ we deduce that the left hand side is zero when we identify $V \otimes R$ with V and $(\mathrm{id} \otimes \varepsilon)h$ with $E_V \pi_*(h)$. In these terms, the right hand side becomes $\Sigma\, v_i \otimes c_i$. Thus we have $h(x) = 0$. $\square$

COROLLARY 4.7. π^* is a right adjoint of π_*.

In the situation of Definition 4.4, we can consider the functor $F_{\pi_*}(C)$ from $\mathcal{C}\mathfrak{M}(D)$ to $\mathfrak{M}(k)$. Now, $F_{\pi_*(C)}(V) = (\pi_*(C) \boxtimes V)^D$, and $\pi_*(C) \otimes V$ has a natural right C-comodule structure given by

$$\Delta \otimes \mathrm{id}_V \colon \pi_*(C) \otimes V \to C \otimes \pi_*(C) \otimes V.$$

We shall verify that, when $\pi_*(C) \otimes V$ is endowed with this right C-comodule structure, then $(\pi_*(C) \boxtimes V)^D$ is a right C-subcomodule of $\pi_*(C) \otimes V$. Consider $(\Delta \otimes \mathrm{id} \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta \boxtimes \chi_V)$. By direct computation one finds that

$$\begin{aligned}
&(\Delta \otimes \mathrm{id} \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta \boxtimes \chi_V) \\
&= (\mathrm{id} \otimes (\mathrm{id} \otimes \pi)\Delta \boxtimes \chi_V)(\Delta \otimes \mathrm{id}).
\end{aligned}$$

It is clear then that if

$$((\mathrm{id} \otimes \pi)\Delta \boxtimes \chi_V)(x) = x \otimes 1,$$

then

$$(\Delta \otimes \mathrm{id})(x) \otimes 1 = (\mathrm{id} \otimes (\mathrm{id} \otimes \pi)\Delta \boxtimes \chi_V)(\Delta \otimes \mathrm{id})(x).$$

Consequently, $(\pi_*(C) \boxtimes V)^D$ has a natural structure of right C-comodule.

Now assume that C is a Hopf Algebra. We know that, using the antipode of C, we can transform any right C-comodule structure χ_M on M into a left C-comodule structure $\tilde{\chi}_M \colon M \to M \otimes C$, where

$$\tilde{\chi}_M = (\mathrm{id} \otimes \eta)s\chi_M.$$

In that fashion, when C is a Hopf Algebra, $(\pi_*(C) \boxtimes V)^D$ has a natural left C-comodule structure given by $(\mathrm{id} \otimes \mathrm{id} \otimes \eta)s_{1,23}(\Delta \otimes \mathrm{id}_V)$.

LEMMA 4.8. *Let C and D be Hopf Algebras over a field, each having involutory antipode, and let π: $C \to D$ be a surjective bialgebra map. Then the functors π^*: $\mathcal{CM}(D) \to \mathcal{CM}(C)$ and $F_{\pi_*(C)}$: $\mathcal{CM}(D) \to \mathcal{CM}(C)$ are naturally equivalent.*

Proof. Given an arbitrary D-comodule M, consider the map $(\mathrm{id} \otimes \eta)s$: $C \otimes M \to M \otimes C$. Clearly $(\mathrm{id} \otimes \eta)s$ is bijective. We want to verify:

(a) $(\mathrm{id} \otimes \eta)s((\pi_*(C) \boxtimes M)^D) = \mathrm{Ker}\,\Phi_M = \pi^*(M)$.

(b) $(\mathrm{id} \otimes \eta)s$ is a C-comodule map when we endow $(\pi_*(C) \boxtimes M)^D$ with the structure $(\mathrm{id} \otimes \mathrm{id} \otimes \eta)s_{1,23}(\Delta \otimes \mathrm{id}_M)$ and $\pi^*(M)$ with the structure $\mathrm{id}_M \otimes \Delta$. In order to verify (a), let $x = \sum c_i \otimes m_i \in (\pi_*(C) \boxtimes M)^D$, write $\chi(m_i) = \sum_j m_{ij} \otimes d_{ij}$ and $\Delta(c_i) = \sum_k c_{ik} \otimes c'_{ik}$. Then we have $x \in (\pi_*(C) \boxtimes M)^D$ if and only if the following condition is satisfied:

$$(*) \qquad \sum_i c_i \otimes m_i \otimes 1 = \sum_{ijk} c_{ik} \otimes m_{ij} \otimes \pi(c'_{ik})d_{ij}.$$

Analogously, we have $(\mathrm{id} \otimes \eta)(x) \in \pi^*(M)$ if and only if either one of the following two mutually equivalent equalities hold:

$$(**) \qquad \begin{cases} \displaystyle\sum_i \chi m_i \otimes \eta c_i = \sum_{ik} \chi_{ik} m_i \otimes \pi\eta c'_{ik} \otimes \eta c_{ik}, \\ \displaystyle\sum_i \chi m_i \otimes c_i = \sum_{ik} m_i \otimes \pi\eta c'_{ik} \otimes c_{ik}. \end{cases}$$

From $(**)$, applying $\mathrm{id} \otimes \mathrm{id} \otimes s(\mathrm{id} \otimes \pi)\Delta$, we deduce

$$\sum_{ijk} m_{ij} \otimes d_{ij} \otimes \pi(c'_{ik}) \otimes c_{ik}$$
$$= \sum_{ik} m_i \otimes \pi\eta c'_{ik} \otimes s(\mathrm{id} \otimes \pi)\Delta c_{ik}.$$

Changing the order and multiplying on the left hand side, we deduce that

$$\sum_{ijk} m_{ij} \otimes \pi(c'_{ik})d_{ij} \otimes c_{ik}$$
$$= \sum_i m_i \otimes (\mu s \otimes \mathrm{id})(\pi\eta \otimes s(\mathrm{id} \otimes \pi)\Delta)s\Delta(c_i).$$

By direct computation with the switching operators we get

$$\sum_{ijk} m_{ij} \otimes \pi(c'_{ik})d_{ij} \otimes c_{ik}$$

$$= \sum_i m_i \otimes s(\mathrm{id} \otimes \mu)((\mathrm{id} \otimes \pi)\Delta \otimes \pi\eta)\Delta(c_i)$$

$$= \sum_i m_i \otimes s(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \pi \otimes \pi)$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \eta)(\mathrm{id} \otimes \Delta)\Delta(c_i)$$

$$= \sum_i m_i \otimes s(\mathrm{id} \otimes \pi)(\mathrm{id} \otimes \mu(\mathrm{id} \otimes \eta)\Delta)\Delta(c_i)$$

$$= \sum_i m_i \otimes s(\mathrm{id} \otimes \pi)(c_i \otimes 1)$$

$$= \sum_i m_i \otimes 1 \otimes c_i.$$

This is exactly the equality (*). In a similar way we can go from (*) to (**). Establishing (b) amounts to verifying that

$$(\mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \eta)s = (\mathrm{id} \otimes \eta \otimes \mathrm{id})(s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)s_{1,23}(\Delta \otimes \mathrm{id}).$$

Now

$$(\mathrm{id} \otimes \eta \otimes \mathrm{id})(s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)s_{1,23}(\Delta \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \eta \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)(s \otimes \mathrm{id})s_{1,23}(\Delta \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \eta \otimes \eta)(\mathrm{id} \otimes s)(\mathrm{id} \otimes \Delta)s.$$

The last equality follows from an elementary manipulation with switching maps

$$(\mathrm{id} \otimes s)(\mathrm{id} \otimes \Delta)s(x \otimes y) = (\mathrm{id} \otimes s)(\mathrm{id} \otimes \Delta)(y \otimes x)$$

$$= \sum_i (\mathrm{id} \otimes s)(y \otimes x_i \otimes x'_i) = \sum_i y \otimes x'_i \otimes x_i$$

and

$$(s \otimes \mathrm{id})s_{1,23}(\Delta \otimes \mathrm{id})(x \otimes y) = \sum_i (s \otimes \mathrm{id})s_{1,23}(x_i \otimes x'_i \otimes y)$$

$$= \sum_i (s \otimes \mathrm{id})(x'_i \otimes y \otimes x_i) = \sum_i y \otimes x'_i \otimes x_i.$$

Thus,

$$(\mathrm{id} \otimes \eta \otimes \mathrm{id})(s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta)s_{1,23}(\Delta \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \eta \otimes \eta)(\mathrm{id} \otimes s)(\mathrm{id} \otimes \Delta)s = (\mathrm{id} \otimes \Delta\eta)s. \qquad \square$$

COROLLARY 4.9. *In the situation above, $\pi_*(C)$ is injective in $\mathcal{CM}(D)$ if and only if π^* is an exact functor or, equivalently, if and only if $H^1(D, \pi_*(C) \boxtimes V) = 0$ for every D-comodule V.*

We apply this criterion of injectivity in order to prove that, if the map π is normal and C and D are commutative, then $\pi_*(C)$ is injective. We shall use the following two results from [7].

THEOREM 4.A ([7] *page 6*). *Let C be a commutative Hopf Algebra over a field k and let X be a sub Hopf Algebra of C. Then the inclusion $X \hookrightarrow C$ is faithfully flat.*

THEOREM 4.B ([7] *page 17*). *Let π be a surjective normal Hopf Algebra map between a pair of commutative Hopf Algebras C and D defined over a field. Then* $\operatorname{Ker} \pi = (\operatorname{Ker} \varepsilon \cap \pi_*(C)^D)C$.

Suppose now that C and D are commutative Hopf Algebras over a field k (this implies that the antipode is involutory) and let π be a normal surjective Hopf Algebra map between C and D. We shall write $^D\pi_*(C)$ or $^D C$ for the set

$$\{x \in C \mid (\pi \otimes \operatorname{id})\Delta(x) = 1 \otimes x\}.$$

It is easy to see that $C^D = {}^D C$ for symmetry reasons. Using the natural structure of C as a $^D C$ module, we endow $C \otimes C$ and $D \otimes C$, as well as $V \otimes C$ for any k-spaces V, with a $^D C$-module structure by multiplication in the second tensor factor. It is elementary to verify that, with respect to the structures mentioned above, the map $(\pi \otimes \operatorname{id})\Delta$: $C \to D \otimes C$ is a $^D C$-module map. Given the multiplication μ: $C \otimes C \to C$, it is clear that if we indicate by i: $C \otimes C \to C \otimes_{{}^D C} C$ the canonical surjection, there is a map $\bar{\mu}$: $C \otimes_{{}^D C} C \to C$ such that $\bar{\mu} i = \mu$. We shall denote this new map $\bar{\mu}$ again by μ because there will be no danger of confusion. Recall that $\otimes$ always indicates tensor product over k. Consider the maps

$$\Delta_D \otimes \operatorname{id}: D \otimes C \to D \otimes D \otimes C$$

and

$$(\pi \otimes \operatorname{id})\Delta_C \otimes_{{}^D C} \operatorname{id}: C \otimes_{{}^D C} C \to D \otimes C \otimes_{{}^D C} C.$$

It is easy to prove that these are right C-comodule structures. Next, look at the maps

$$\psi = (\mathrm{id} \otimes \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes_{D_C}\mathrm{id}\big):$$

$$C \otimes_{D_C} C \xrightarrow{(\pi \otimes \mathrm{id})\Delta \otimes_{D_C}\mathrm{id}} D \otimes C \otimes_{D_C} C \xrightarrow{\mathrm{id} \otimes \mu} D \otimes C,$$

$$\phi_1 = i(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(\Delta \otimes \mathrm{id}):$$

$$C \otimes C \xrightarrow{\Delta \otimes \mathrm{id}} C \otimes C \otimes C \xrightarrow{\mathrm{id} \otimes \eta \otimes \mathrm{id}} C \otimes C \otimes C \xrightarrow{\mathrm{id} \otimes \mu} C \otimes C \xrightarrow{i} C \otimes_{D_C} C$$

where, as before, i stands for the canonical map $C \otimes C \xrightarrow{i} C \otimes_{D_C} C$.

LEMMA 4.10. *In the situation above, there is a unique k-linear map ϕ_0 that makes the following diagram commutative.*

$$
\begin{array}{ccc}
C \otimes C & \xrightarrow{\phi_1} & C \otimes_{D_C} C \\[2mm]
\downarrow{\scriptstyle \pi \otimes \mathrm{id}} & \nearrow{\scriptstyle \phi_0} & \\[2mm]
D \otimes C & &
\end{array}
$$

Proof. All we have to verify is that if $t \in \mathrm{Ker}\,\pi \otimes C$ then $\phi_1(t) = 0$. Theorem 4.B says that $\mathrm{Ker}\,\pi = (\mathrm{Ker}\,\varepsilon \cap {}^D C)C$. Take an element $\sum x_k \otimes y_k \in C \otimes C$ such that $x_k \in (\mathrm{Ker}\,\varepsilon \cap {}^D C)$. We shall verify that $\phi_1(\sum x_k \otimes y_k) = 0$. Let us write $\Delta(x_k) = \sum_i x_{ki} \otimes x'_{ki}$. Then we have $\phi_1(\sum x_k \otimes y_k) = \sum_{ki} x_{ki} \otimes_{D_C} \eta(x'_{ki})y_k$. Since π is normal $x_{ki} \in {}^D C$ (because $\Delta({}^D C) \subset {}^D C \otimes {}^D C$). Consequently,

$$\phi_1\Big(\sum x_i \otimes y_k\Big) = 1 \otimes \sum_{ki} x_{ki}\eta(x'_{ki})y_k = 1 \otimes \sum_k u\varepsilon(x_k)y_k = 0. \qquad \square$$

THEOREM 4.11. *The map ψ is an isomorphism of right D-comodules from $C \otimes_{D_C} C$ to $D \otimes C$, whose inverse is the map ϕ_0 defined above.*

Proof. First, we show that ψ is a morphism of right D-comodules. This amounts to proving that the diagram below is commutative.

$$
\begin{array}{ccccc}
C \otimes_{D_C} C & \xrightarrow{(\pi \otimes \mathrm{id})\Delta \otimes_{D_C}\mathrm{id}} & D \otimes C \otimes_{D_C} C & \xrightarrow{\mathrm{id} \otimes \mu} & D \otimes C \\[2mm]
\downarrow{\scriptstyle (\pi \otimes \mathrm{id})\Delta \otimes_{D_C}\mathrm{id}} & & & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\[2mm]
D \otimes C \otimes_{D_C} C & \xrightarrow{\mathrm{id} \otimes (\pi \otimes \mathrm{id})\Delta \otimes_{D_C}\mathrm{id}} & D \otimes D \otimes C \otimes_{D_C} C & \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes \mu} & D \otimes D \otimes C
\end{array}
$$

If we add to the above diagram the map

$$\Delta \otimes \mathrm{id} \otimes_{\mathit{D}_C} \mathrm{id}: D \otimes C \otimes^{\mathit{D}}_C C \to D \otimes D \otimes C \otimes_{\mathit{D}_C} C,$$

we need only check the commutativity of the two subdiagrams. The resulting square on the left is commutative because π is a coalgebra map, and the resulting square on the right is obviously commutative. Next, we show that ψ and ϕ_0 are inverses to each other. Look at

$$\psi\phi_0(\pi \otimes \mathrm{id}) = (\mathrm{id} \times \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes_{\mathit{D}_C} \mathrm{id}\big)\phi_0(\pi \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes_{\mathit{D}_C} \mathrm{id}\big)i(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(\Delta \otimes \mathrm{id}).$$

From the very definition of i and $\otimes_{\mathit{D}_C}$ we have that

$$\psi\phi_0(\pi \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes \mathrm{id}\big)(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(\Delta \otimes \mathrm{id})$$
$$= (\pi \otimes \mathrm{id})(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \mu \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \eta \otimes \mathrm{id})$$
$$(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) = \pi \otimes \mathrm{id}.$$

The last equality follows from the fact that $\mu(\mathrm{id} \otimes \eta)\Delta = u\varepsilon$. Since $\pi \otimes \mathrm{id}$ is surjective, we conclude that $\psi\phi_0 = \mathrm{id}$. Now, we observe, that:

$$\phi_0\psi i = \phi_0(\mathrm{id} \otimes \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes_{\mathit{D}_C} \mathrm{id}\big)i = \phi_0(\pi \otimes \mathrm{id})(\mathrm{id} \otimes \mu)(\Delta \otimes \mathrm{id}).$$

If we replace $\phi_0(\pi \otimes \mathrm{id})$ with ϕ_1 and write out ϕ_1 explicitly, we find from an argument similar to one above (relying on the fact that η is an antipode) that:

$$\phi_0(\pi \otimes \mathrm{id})(\mathrm{id} \otimes \mu)(\Delta \otimes \mathrm{id}) = i.$$

This shows that $\phi_0\psi i = i$. Since the map $i: C \otimes C \to \otimes_{\mathit{D}_C} C$ is surjective, we deduce that $\phi_0\psi = \mathrm{id}$. $\square$

Let V be a D-comodule. Then $\pi^*(V)$ is a $^{\mathit{D}}C$ submodule of $V \otimes C$ when we endow $V \otimes C$ with the structure of $^{\mathit{D}}C$ module given by multiplication on C.

THEOREM 4.12. *Let C and D be as before, and let V be a D-comodule, then the k-linear maps*

$$\mathrm{id} \otimes \mu: V \otimes C \otimes_{\mathit{D}_C} C \to V \otimes C \quad \textit{and} \quad \alpha = (\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id}),$$

the composite

$$V \otimes C \overset{\chi \otimes \mathrm{id}}{\to} V \otimes D \otimes C \overset{\mathrm{id} \otimes \phi_0}{\to} V \otimes C \otimes_{\mathit{D}_C} C,$$

have the following properties:

   (a) $\alpha(V \otimes C) \subset \pi^*(V) \otimes_{D_C} C$.

   (b) α *is the inverse of the restriction of* $\mathrm{id} \otimes \mu$ *to* $\pi^*(V) \otimes_{D_C} C$.

Proof. Let us verify first that if $t \in \pi^*(V) \otimes_{D_C} C$, then

$$\alpha(\mathrm{id}_V \otimes \mu)(t) = t.$$

If $t = \Sigma_i \, v_i \otimes c_i \otimes_{D_C} \bar{c}_i$, then

$$\sum_i \chi_V(v_i) \otimes c_i \otimes_{D_C} \bar{c}_i = \sum_i v_i \otimes (\pi \otimes \mathrm{id})\Delta(c_i) \otimes_{D_C} \bar{c}_i.$$

Hence,

$$\alpha(\mathrm{id} \otimes \mu)(t) = (\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})(\mathrm{id} \otimes \mu)(t)$$

$$= (\mathrm{id} \otimes \phi_0)\Big(\sum_i \chi(v_i) \otimes c_i \bar{c}_i \Big)$$

$$= (\mathrm{id}_V \otimes \phi_0)\Big(\sum_i v_i \otimes (\mathrm{id} \otimes \mu) \Big)\big((\pi \otimes \mathrm{id})\Delta \otimes_{D_C} \mathrm{id}\big)\big(c_i \otimes_{D_C} \bar{c}_i\big)$$

$$= \sum_i v_i \otimes \phi_0(\mathrm{id} \otimes \mu)\big((\pi \otimes \mathrm{id})\Delta \otimes_{D_C} \mathrm{id}\big)\big(c_i \otimes_{D_C} \bar{c}_i\big)$$

$$= \sum_i v_i \otimes \phi_0 \psi\big(c_i \otimes_{D_C} \bar{c}_i\big) = \sum_i v_i \otimes c_i \otimes_{D_C} \bar{c}_i = t.$$

The equality before the last comes from the fact that ϕ_0 and ψ are mutually inverse maps.

   Next, we check that

$$(\mathrm{id} \otimes \mu)\alpha = \mathrm{id}, \qquad (\mathrm{id} \otimes \mu)\alpha = (\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id}).$$

   Now observe that

$$\mu \phi_0(\pi \otimes \mathrm{id}) = \mu i(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes \eta \otimes \mathrm{id})(\Delta \otimes \mathrm{id})$$

$$= \mu(\mu \otimes \mathrm{id})(\mathrm{id} \otimes \eta \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) = \mu(u\varepsilon \otimes \mathrm{id}).$$

We have $\mu \phi_0(\pi \otimes \mathrm{id})(c_1 \otimes c_2) = \varepsilon(c_1)c_2 = j(\varepsilon_D \otimes \mathrm{id})(\pi \otimes \mathrm{id})(c_1 \otimes c_2)$, where j stands for the map $j: k \otimes C \to C$, given by $j(\lambda \otimes c) = \lambda c$. Consequently we have $\mu \phi_0 = j(\varepsilon_D \otimes \mathrm{id})$. Hence,

$$(\mathrm{id} \otimes \mu)\alpha = (\mathrm{id} \otimes j) \quad (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\chi \otimes \mathrm{id}) = \mathrm{id}_{V \otimes C}.$$

It remains only to verify (a). We have (recall Definition 4.4)

$$\left(\phi_V \otimes_{{}^D C} \mathrm{id}\right)(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})$$
$$= \left((\chi \otimes \mathrm{id} \otimes_{{}^D C} \mathrm{id}) - (\mathrm{id} \otimes (\pi \otimes \mathrm{id})\Delta \otimes_{{}^D C} \mathrm{id})\right)(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})$$
$$= (\chi \otimes \mathrm{id} \otimes_{{}^D C} \mathrm{id})(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})$$
$$\quad - (\mathrm{id} \otimes (\pi \otimes \mathrm{id})\Delta \otimes_{{}^D C} \mathrm{id})(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id}).$$

By Theorem 4.11 ϕ_0 is a right D-comodule map. Therefore,

$$\left((\pi \otimes \mathrm{id}) \otimes_{{}^D C} \mathrm{id}\right)\phi_0 = (\mathrm{id} \otimes \phi_0)(\Delta \otimes \mathrm{id})$$

Hence,

$$\left(\phi_V \otimes_{{}^D C} \mathrm{id}\right)(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})$$
$$= (\chi \otimes \mathrm{id} \otimes_{{}^D C} \mathrm{id})(\mathrm{id} \otimes \phi_0)(\chi \otimes \mathrm{id})$$
$$\quad - (\mathrm{id} \otimes \mathrm{id} \otimes \phi_0)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\chi \otimes \mathrm{id}) = 0,$$

because $(\mathrm{id} \otimes \Delta)\chi = (\chi \otimes \mathrm{id})\chi$.

Thus, $(\phi_V \otimes_{{}^D C} \mathrm{id})\alpha(V \otimes C) = 0$, whence

$$\alpha(V \otimes C) \subset \mathrm{Ker}(\phi_V \otimes_{{}^D C} \mathrm{id}) = \mathrm{Ker}(\phi_V) \otimes_{{}^D C} C,$$

because C is flat as a ${}^D C$-comodule. $\square$

THEOREM 4.13. *Let C and D be commutative Hopf Algebras over a field k, and let π be a surjective normal Hopf Algebra map from C to D. Then $\pi_*(C)$ is injective as a D-comodule.*

Proof. By Corollary 4.9, it is enough to prove that if V_1 and V_2 are D-comodules and α a surjective D-comodule map from V_1 to V_2, then the map induced by α from $\pi^*(V_1)$ to $\pi^*(V_2)$ is surjective. As the inclusion ${}^D C \hookrightarrow C$ is faithfully flat (Theorem 4.A), the map $\alpha \otimes \mathrm{id}$ is surjective if and only if $\alpha \otimes \mathrm{id} \otimes_{{}^D C} \otimes \mathrm{id}: \pi^*(V_1) \otimes_{{}^D C} C \to \pi^*(V_2) \otimes_{{}^D C} C$, is surjective. The maps $\mathrm{id}_{V_i} \otimes \mu: \pi^*(V_i) \otimes_{{}^D C} C \to V_i \otimes C$ are isomorphisms and commute with α in the sense that

$$(\alpha \otimes id)(\mathrm{id}_{V_1} \otimes \mu) = (\mathrm{id}_{V_2} \otimes \mu)(\alpha \otimes \mathrm{id} \otimes_{{}^D C} \mathrm{id}).$$

Consequently $(\alpha \otimes \mathrm{id})(\pi^*(V_1)) = \pi^*(V_2)$ if and only if

$$\alpha \otimes \mathrm{id}: V_1 \otimes C \to V_2 \otimes C$$

is surjective. Since α is surjective our conclusion follows. $\square$

Now we have the injectivity results for comodules that we need in order to establish the spectral sequences of §5. Before proceeding to do that, we establish one other result that is sometimes useful as a criterion for injectivity, and is a generalization of [5], Proposition 2.2.

Lemma 4.14 is contained in [6], our proof is not different from Sweedler's but is more compatible with our notation. For the next two results our coalgebras need not be flat as R-modules.

LEMMA 4.14 *Let C and D be Hopf Algebras defined over a ring R, and let $\pi\colon C \to D$ be a bialgebra map. Suppose moreover that there exists a D-comodule map $\sigma\colon D \to \pi_*(C)$. Then the map $Q = \mu(\mathrm{id} \otimes \sigma\pi\eta)\Delta\colon C \to C$ sends C into $\pi_*(C)^D$.*

Proof. In convolution notation, we have $Q = \mathrm{id} * \sigma\pi\eta$.

$$(\mathrm{id} \otimes \pi)\Delta Q = (\mathrm{id} \otimes \pi)\Delta\mu(\mathrm{id} \otimes \sigma\pi\eta)\Delta$$
$$= (\mu \otimes \mu)(\mathrm{id} \otimes \mathrm{id} \otimes \pi \otimes \pi)(\mathrm{id} \otimes s \otimes \mathrm{id})(\Delta \otimes \Delta)(\mathrm{id} \otimes \sigma\pi\eta)\Delta.$$

For the last equality we used that $\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\Delta \otimes \Delta)$ and the fact that π is an algebra map.

Thus,

$$(\mathrm{id} \otimes \pi)\Delta Q = (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes id)((\mathrm{id} \otimes \pi)\Delta \otimes (\mathrm{id} \otimes \pi)\Delta\sigma\pi\eta)\Delta.$$

As σ is a D-comodule map, we have

$$(\mathrm{id} \otimes \pi)\Delta\sigma = (\sigma \otimes \mathrm{id})\Delta$$

and

$$(\mathrm{id} \otimes \pi)\Delta Q = (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta \otimes (\sigma \otimes \mathrm{id})\Delta\pi\eta)\Delta.$$

Using that $\Delta\pi = (\pi \otimes \pi)\Delta$ and $\Delta\eta = s(\eta \otimes \eta)\Delta$, we get

$$(\mathrm{id} \otimes \pi)\Delta Q = (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes s)$$
$$(\mathrm{id} \otimes \pi \otimes \pi \otimes \sigma\pi)(\mathrm{id} \otimes \mathrm{id} \otimes \eta \otimes \eta)(\Delta \otimes \Delta)\Delta.$$

Using associativity of μ, coassociativity of Δ and the fact that π is an algebra map, we get

$$(\mathrm{id} \otimes \pi)\Delta Q$$
$$= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes s)(\mathrm{id} \otimes \pi \otimes \sigma\pi)(\mathrm{id} \otimes \mu(\mathrm{id} \otimes \eta)\Delta \otimes \eta)(\mathrm{id} \otimes \Delta)\Delta$$
$$= (\mu \otimes \mathrm{id})(\mathrm{id} \otimes s)(\mathrm{id} \otimes \pi \otimes \sigma\pi)(\mathrm{id} \otimes u\varepsilon \otimes \eta)(\mathrm{id} \otimes \Delta)\Delta.$$

Now if $\Delta(x) = \Sigma\, x_i \otimes x_i'$, we have

$$(\mathrm{id} \otimes \pi)\Delta Q(x) = \sum (\mu \otimes \mathrm{id})(\mathrm{id} \otimes s)\big(x_i \otimes 1 \otimes \sigma\pi\eta(x_i')\big)$$

$$= \sum x_i\sigma\pi\eta x_i' \otimes 1.$$

But, by the very definition of Q, we have $Qx = \Sigma\, x_i\sigma\pi\eta x_i'$. Thus, $Q(x) \in \pi_*(C^D)$. $\square$

THEOREM 4.15. *Let C and D be Hopf Algebras over a ring R and let π be a bialgebra map from C to D. Let $\sigma\colon D \to \pi_*(C)$ be an algebra map that is at the same time a D-comodule homomorphism. Then $\pi_*(C)$ is a relatively injective D-comodule.*

Proof. Let us consider $C \otimes D$ with structure $\mathrm{id} \otimes \Delta$. We shall prove that $\pi_*(C)$ is a direct D-comodule summand of $C \otimes D$. Consider the maps $\alpha = (\mathrm{id} \otimes \pi)\Delta\colon C \to C \otimes D$ and $\beta = \mu(Q \otimes \sigma)\colon C \otimes D \to C$. We have

$$\beta\alpha = \mu(Q \otimes \sigma)(\mathrm{id} \otimes \pi)\Delta = Q * \sigma\pi = (\mathrm{id} * \sigma\pi\eta) * \sigma\pi$$

$$= \mathrm{id} * (\sigma\pi\eta * \sigma\pi).$$

As $\sigma\pi$ is an algebra homomorphism, $\sigma\pi\eta * \sigma\pi = \mu(\sigma\pi \otimes \sigma\pi)(\eta \otimes \mathrm{id})\Delta = \sigma\pi\mu(\eta \otimes \mathrm{id})\Delta = \sigma\pi u\varepsilon = u\varepsilon$. Consequently, $\beta\alpha = \mathrm{id} * u\varepsilon = \mathrm{id}$. Now, α is obviously a D-map, therefore, the only thing that still requires a proof is that β is a D-comodule map, i.e. that

$$(\mathrm{id} \otimes \pi)\Delta\mu(Q \otimes \sigma) = (\mu(Q \otimes \sigma) \otimes \mathrm{id})(\mathrm{id} \otimes \Delta).$$

We have,

$$(\mathrm{id} \otimes \pi)\Delta\mu(Q \otimes \sigma)$$
$$= (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes \pi \otimes \mathrm{id} \otimes \pi)(\Delta \otimes \Delta)(Q \otimes \sigma)$$
$$= (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta Q \otimes (\mathrm{id} \otimes \pi)\Delta\sigma).$$

Using that σ is a D-comodule map, we obtain

$$(\mathrm{id} \otimes \pi)\Delta\mu(Q \otimes \sigma)$$
$$= (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta Q \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \Delta).$$

To finish the proof that β is a D-comodule map, all that needs to be verified is that

$$(\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta Q \otimes \sigma \otimes \mathrm{id}) = \mu(Q \otimes \sigma) \otimes \mathrm{id}.$$

We have

$$(\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})((\mathrm{id} \otimes \pi)\Delta Q \otimes \sigma \otimes \mathrm{id})(x \otimes y \otimes z)$$
$$= (\mu \otimes \mu)(\mathrm{id} \otimes s \otimes \mathrm{id})(Q(x) \otimes 1 \otimes \sigma y \otimes z)$$
$$= Q(x)\sigma y \otimes z.$$

As $(\mu(Q \otimes \sigma) \otimes \mathrm{id})(x \otimes y \otimes z) = Q(x)\sigma(y) \otimes z$, our proof is finished. $\square$

5. Two spectral sequences. First, we establish the validity of the Hochschild-Serre spectral sequence for the cohomology of comodules.

THEOREM 5.1. *Let C and D be commutative Hopf Algebras over a field k, and let $\pi\colon C \to D$ be a surjective normal bialgebra map from C to D. For every object M in $\mathcal{C}\mathfrak{M}(C)$, there is a third quadrant spectral sequence $\{E_r^{p,q}\}$, natural in M, such that*

$$E_2^{p,q} = H^p\Big(\pi_*(C)^D, H^q\big(D, \pi_*(M)\big)\Big), \quad E_2^{p,q} \Rightarrow_p H^{p+q}(C, M).$$

Here we regard $H^q(D, \pi_*(M))$ as a $\pi_*(C)^D$-comodule, with the structure established in §3.

Proof. Consider the functors

$$F\colon \mathcal{C}\mathfrak{M}(C) \to \mathcal{C}\mathfrak{M}\Big(\pi_*(C)^D\Big), \qquad G\colon \mathcal{C}\mathfrak{M}\Big(\pi_*(C)^D\Big) \to \mathfrak{M}(k)$$

defined (in the object part) as follows:
  (a) If $M \in \mathcal{C}\mathfrak{M}(C)$ then $F(M) = \pi_*(M)^D$.
  (b) If $X \in \mathcal{C}\mathfrak{M}(\pi_*(C)^D)$ then $G(X) = X^{\pi_*(C)^D}$.
We proved before (Theorem 3.15 and Lemma 3.9) that $F(M) \in \mathcal{C}\mathfrak{M}(\pi_*(C)^D)$ and that $(G \circ F)(M) = M^C$. The general theory of Grothendieck spectral sequences (see [2]) will give the theorem as soon as we prove
  (i) I injective in $\mathcal{C}\mathfrak{M}(C)$ implies $F(I)$ G-acyclic.
  (ii) The derived functors of F are π_* composed with the D-cohomology functors.
  It is enough to check (i) for $I = C$. In that case $F(I) = \pi_*(C)^D$ and our conclusion follows from Lemma 2.1.
  As to (ii), since π is normal, $\pi_*(C)$ is injective in $\mathcal{C}\mathfrak{M}(D)$. Consequently, the canonical resolution for M as a C-comodule that we constructed in §2, gives us (after applying π_*) an injective resolution $\pi_*(M)$ by injective D-comodules. Consequently the ith derived functors of F are functors $H^i(D, \pi_*())$. $\square$

In particular we can establish as corollaries the exactness of the Inflation Restriction sequence, as well as the exactness of the seven term sequence:

$$0 \to H^1\left(\pi_*(C)^D, \pi_*(M)^D\right) \to H^1(C, M)$$

$$\to H^1\left(D, \pi_*(M)\right)^{\pi_*(C)^D}$$

$$\to H^2\left(\pi_*(C)^D, \pi_*(M)^D\right) \to \mathrm{Ker}\,\rho_M$$

$$\to H^1\left(\pi_*(C)^D, H^1\left(D, \pi_*(M)\right)\right)$$

$$\to H^3\left(\pi_*(C)^D, \pi_*(M)^D\right),$$

$$\text{where } \rho_M \colon H^2(C, M) \to H^2\left(D, \pi_*(M)\right)^{\pi_*(C)^D}$$

is the so called restriction map.

It may be useful for some calculations to have a direct proof of the exactness of the sequence above. We shall sketch here such a proof.

Let M be a C-comodule and consider an exact sequence in $\mathcal{CM}(C)$

$$(1) \qquad\qquad 0 \to M \to X \to L \to 0$$

with X coinduced. Taking the D cohomology sequence of (1) we obtain an exact sequence and an isomorphism

$$(2) \qquad 0 \to M^D \to X^D \to L^D \to H^1(D, M) \to 0,$$

$$(2') \qquad\qquad H^1(D, L) \cong H^2(D, M).$$

We have used here that X, when considered as a D-comodule, is injective. We split the sequence (2) into a pair of short exact sequences in the category $\mathcal{CM}(C^D)$ as follows:

$$(3) \qquad\qquad 0 \to M^D \to X^D \to Y \to 0,$$

$$(4) \qquad\qquad 0 \to Y \to L^D \to H^1(D, M) \to 0.$$

The long exact C^D cohomology sequence that we obtain from (4) will give, after we identify all the terms, the seven term sequence.

$$(5) \qquad 0 \to Y^{C^D} \to \left(L^D\right)^{C^D} \to H^1(D, M)^{C^D} \to H^1(C^D, Y)$$

$$\to H^1(C^D, L^D) \to H^1\left(C^D, H^1(D, M)\right) \to H^2(C^D, Y).$$

Taking the C^D cohomology sequence from (3) and the C cohomology sequence from (1) we obtain:

(6) $$0 \to M^C \to X^C \to Y^{C^D} \to H^1(C^D, M^D) \to 0,$$

(6') $$H^1(C^D, Y) \cong H^2(C^D, M^D), H^2(C^D, Y) \cong H^3(C^D, M^D),$$

(7) $$0 \to M^C \to X^C \to L^C \to H^1(C, M) \to 0,$$

(7') $$H^1(C, L) \cong H^2(C, M).$$

It is easy to see that the map from Y^{C^D} to L^C in sequence (5) defines a map from $H^1(C^D, M^D)$ to $H^1(C, M)$ that is injective (see sequences (6) and (7)) and when substituted into (5) gives the seven term sequence

(5') $$0 \to H^1(C^D, M^D) \to H^1(C, M) \to H^1(D, M)^{C^D} \to H^2(C^D, M^D)$$
$$\to H^1(C^D, L^D) \to H^1(C^D, H^1(D, M)) \to H^3(C^D, M^D).$$

Note that we also substituted the 4th and 7th term of (5) using the isomorphisms of (6'). The only term in (5') that still needs identification is the 5th. When we apply the first three terms of the sequence (5') to L we obtain the sequence:

$$0 \to H^1(C^D, L^D) \to H^1(C, L) \to H^1(D, L).$$

Using (2') and (7') we have that $H^1(C^D, L^D) \cong \operatorname{Ker} \rho_M$, with $\rho_M \colon H^2(C, M) \to H^2(D, M)$.

Now consider Hopf Algebras C and D defined over a ring R, and let $\pi \colon C \to D$ be a surjective bialgebra map. Given any D-comodule M we consider the D-comodule $\pi_*(C) \boxtimes M$ and the complex

$$E(\pi_*(C) \boxtimes M) \colon 0 \to \pi_*(C) \boxtimes M \overset{\delta_1}{\to} \pi_*(C) \boxtimes M \otimes D \to \cdots$$

$$\to \pi_*(C) \boxtimes M \otimes D^{p-1} \overset{\delta_p}{\to} \pi_*(C) \boxtimes M \otimes D^p \to \cdots$$

where the maps δ_i are defined inductively by formulas (3) and (4) of §2. We endow every object on the complex with a right C-comodule structure $\eta_k = \Delta \otimes \operatorname{id}_M \otimes \operatorname{id}_{D^k}$.

It is obvious that, with respect to these structures, the differentials of the complex are C-comodule maps. Let $\tilde{\eta}_k$ denote the left C-comodule structure constructed from η_k in the same fashion as in the considerations before Lemma 4.8. The maps δ_k are now left C-comodule maps, and from the general considerations at the end of the proof of Theorem 3.6

we deduce the existence of left C-comodule structures $\bar{\eta}_k$ on $H^k(D, \pi_*(C) \boxtimes M)$ that are compatible with the connecting homomorphisms and, at zero, coincide with the structure

$$(\text{id} \otimes \text{id} \otimes \eta)s_{1,23}(\Delta \otimes \text{id}_M).$$

Now let us look at the functor

$$F_{\pi_*(C)}: \mathcal{C}\mathfrak{M}(D) \to \mathcal{C}\mathfrak{M}(C), \quad \text{where } F_{\pi_*(C)}(M) = \left(\pi_*(C) \boxtimes M\right)^D.$$

(We endow $(\pi_*(C) \boxtimes M)^D$ with the structure $(\text{id} \otimes \text{id} \otimes \eta)s_{1,23}(\Delta \otimes \text{id}_M)$ as we did in Lemma 4.8.) Consider $L: \mathcal{C}\mathfrak{M}(C) \to \mathfrak{M}(R)$, where $L(M) = M^C$.

Recall that in Lemma 4.8 we proved that $F_{\pi_*(C)}$ is naturally equivalent with the induced representation functor π^* (see Definition 4.4).

LEMMA 5.2. *The functor* $LF_{\pi_*(C)}$ *is naturally equivalent with the functor from* $\mathcal{C}\mathfrak{M}(D)$ *to* $\mathfrak{M}(R)$ *that sends* M *to* M^D.

Proof. We have noticed before that $F_{\pi_*(C)}$ is nothing else but π^*. Thus, our statement can be expressed in the following terms:

Let M be a D-comodule and let $\phi_M: M \otimes C \to M \otimes D \otimes C$ be the map $\chi_M \otimes \text{id}_C - \text{id}_M \otimes (\pi \otimes \text{id})\Delta$. If we endow $M \otimes C$ with the structure $\text{id} \otimes \Delta$, then $(\text{Ker } \phi_M)^C \cong M^D$. Consider the maps $E: M \otimes C \to M$, $E(m \otimes c) = \varepsilon(c)m$ and $\gamma: M \to M \otimes C$, $\gamma(m) = m \otimes 1$. Then we have

(a) $\gamma(M^D) \subset (\text{Ker } \phi_M)^C$. This is because $\gamma(M) \subset (M \otimes C)^C$ (this is nothing but the assertion that $\Delta(1) = 1 \otimes 1$) and if $m \in M^D$ then $\chi m = m \otimes 1$ and $\phi_M m = m \otimes 1 \otimes 1 - m \otimes (\pi \otimes \text{id})\Delta(1) = m \otimes 1 \otimes 1 - m \otimes 1 \otimes 1 = 0$.

(b) If $x \in (M \otimes C)^C$ then $x = E(x) \otimes 1$. This is because if $x = \sum m_i \otimes c_i$ and $\sum m_i \otimes \Delta c_i = \sum m_i \otimes c_i \otimes 1$, then applying $\text{id} \otimes \varepsilon \otimes \text{id}$ yields $\sum m_i \otimes c_i = \sum \varepsilon(c_i)m_i \otimes 1$.

(c) If $x \in \text{Ker } \phi_M$ then $\chi(Ex) = (\text{id} \otimes \pi)(x)$. This is because if $x = \sum m_i \otimes c_i$ and $\sum \chi m_i \otimes c_i = \sum m_i \otimes (\pi \otimes \text{id})\Delta c_i$, then applying $\text{id} \otimes \text{id} \otimes \varepsilon$ yields $\sum \chi(m_i \varepsilon(c_i)) = \sum m_i \otimes \pi c_i$.

(d) $E\gamma = \text{id}_M$.

Condition (b) tells us that if $x \in (\text{Ker } \phi_M)^C$ then $x = \gamma Ex$. This, together with (d), guarantees that E restricted to $(\text{Ker } \phi_M)^C$ and γ restricted to M^D are inverses to each other. Now from (b) and (c) we deduce that if $x \in (\text{Ker } \phi_M)^C = \text{Ker } \phi_M \cap (M \otimes C)^C$ then $\chi(Ex) = (\text{id} \otimes \pi)x = (\text{id} \otimes \pi)(Ex \otimes 1) = Ex \otimes 1$. Thus $E((\text{Ker } \phi_M)^C) \subset M^D$. $\square$

THEOREM 5.3. *Let C and D be arbitrary Hopf Algebras defined over a ring R that are flat as R-modules. Let π be a surjective bialgebra map $\pi\colon C \to D$. Then, for every object M in $\mathcal{C}\mathfrak{M}(D)$, there is a third quadrant spectral sequence, natural in M, $\{E_r^{p,q}\}$ such that*

$$E_2^{p,q} = H^p\Big(C, H^q\big(D, \pi_*(C) \boxtimes M\big)\Big) \underset{p}{\Rightarrow} H^{p+q}(D, M).$$

We regard $H^q(D, \pi_*(C) \boxtimes M)$ as a C-comodule with the structure $\bar{\eta}_q$ defined above.

Proof. The only part that still needs to be verified is that if I is a relatively injective D-comodule, then $\pi^*(I)$ is L-acylic. It is enough to look at coinduced D-comodules of the form $X \otimes D$ with structure $\mathrm{id}_X \otimes \Delta$. In that case $\pi^*(X \otimes D) \cong X \otimes C$, as is seen from Lemma 4.5. By the very definition of our cohomology theory, $X \otimes C$ is L-acylic. $\square$

In the case where C is injective as a D-comodule, the spectral sequence collapses and gives us isomorphisms between $H^p(C, \pi^*(M))$ and $H^p(D, M)$. The spectral sequence of Theorem 5.3 is the form that Shapiro's lemma takes in our cohomology theory.

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