PACIFIC JOURNAL OF MATHEMATICS
Vol. 109, No. 1, 1983

WITT KERNELS OF FUNCTION FIELD EXTENSIONS

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Let F be a field of characteristic not 2. For a non-hyperbolic quadratic form q of dimension at least 2, let $F(q)$ denote the function field of the projective variety $q = 0$. We consider the problem, explicitly raised as problem D by Lam, of determining the kernel of induced map of Witt rings $WF \to WF(q)$. This kernel is the Witt kernel of the field extension and is denoted by $W(F(q)/F)$. The basic tool is a comparison of $W(F(q \perp \langle x \rangle)/F)$ and $W(F(q)/F)$. The Witt kernels $W(F(q)/F)$ where q has small dimension or F has small Hasse number are determined. Applications are made to the question of when a conservative form is embeddable.

In the case q is a Pfister form, the function fields $F(q)$ have been widely used (e.g. the Arason-Pfister Hauptsatz). Central to the applications is that the Witt kernel $W(F(q)/F)$ is qWF for Pfister forms q. Elman, Lam and Wadsworth have considered function fields of several Pfister forms ρ_i, (cf. [**8**]). Again the basic problem is computing the Witt kernel $W(F(\rho_1, \rho_2, \ldots, \rho_r)/F)$ and showing it is a Pfister ideal.

Here also the emphasis is on finding conditions to insure Witt kernels are generated by Pfister forms. In the first section the comparison of $W(F(\varphi \perp \langle x \rangle)/F)$ and $W(F(\varphi)/F)$ is made and this is applied in the second section to forms of small dimension. For example, we show the Witt kernel $W(F(\varphi)/F)$ is a strong Pfister ideal if φ has dimension ≤ 5 and a Pfister ideal if dimension 6. This is used to improve several results of Gentile and Shaprio (in [**12**]) on their question of when $W(F(\varphi)/F)$ contains a non-zero Pfister form.

The last section treats fields F of finite Hasse number. It is shown that all Witt kernels of function fields are strong Pfister ideals if $\tilde{u}(F) \leq 8$. And the Witt kernels $W(F(\varphi)/F)$ are essentially computed for any form φ over F with $\tilde{u}(F) \leq 32$. Examples of fields with Hasse number ≤ 8 are C_3 fields, global and local fields, and finite fields.

The notation and terminology used are basically those of [**15**]. Isometry of forms α and β are denoted by $\alpha \simeq \beta$, while equality in the Witt ring is written $\alpha = \beta$. The uniquely determined maximal anisotropic subform α of a form β is termed the kernel of β and written as $\alpha = \ker(\beta)$. If $x\alpha \simeq \beta$ for some $x \in \dot{F}$, we say α and β are similar. The u-invariant used in the last two sections is the generalized u-invariant of Elman and Lam (see e.g. [**4**]) and not the one discussed in [**15**].

The set of all F-Pfister forms is denoted by $P(F)$ and $P_n(F)$ denotes the set of n-fold F-Pfister forms. The set of forms over F similar to F-Pfister forms [n-fold F-Pfister forms] is denoted by $GP(F)$ [resp. $GP_n(F)$]. If $\rho \in GP(F)$ is anisotropic and $\varphi < \rho$ then φ is a Pfister neighbor if $2\dim\varphi > \dim\rho$ and a conjugate neighbor if $2\dim\varphi = \dim\rho$.

We use the terms conservative and embeddable forms as defined by Gentile and Shapiro. Namely, a form q is conservative if $W(F(q)/F) \neq 0$, or equivalently, if $q \otimes L$ is anisotropic for every field extension L/F with $W(L/F) = 0$. A form q is embeddable if it is similar to a subform of an anisotropic Pfister form.

Following Elman, Lam and Wadsworth, for a subset $N \subset \mathbf{N}$ and $\mathfrak{A}$ an ideal of WF we say $\mathfrak{A}$ is an **N**-*Pfister ideal* of $\mathfrak{A}$ is generated by r-fold Pfister forms, $r \in N$. $\mathfrak{A}$ is a *strong* **N**-*Pfister ideal* if each $q \in \mathfrak{A}$ is isometric to a sum of scalar multiples of r-fold Pfister forms in $\mathfrak{A}$, $r \in N$. We write n-Pfister for $\{n\}$-Pfister.

Let X_F denote the set of orderings on the field F and topologize X_F by taking as an open subbasis the Harrison sets:

$$H_F(a) = \{\alpha \in X \mid a >_\alpha 0\},$$

where a ranges over $\dot{F}$. A form q is indefinite at $\alpha \in X_F$ if $|\operatorname{sgn}_\alpha q| < \dim q$ and indefinite if q is indefinite at all $\alpha \in X_F$. The Hasse number of F is:

$$\tilde{u}(F) = \max\{\dim q \mid q \text{ anisotropic and indefinite over } F\}$$

if the maximum exists, otherwise $\tilde{u}(F) = \infty$.

Knebusch's important paper [**13**] will be used extensively and notation and terminology not found in [**15**] or mentioned above will be taken from it. In particular, we use the degree of a form q. As shown in [**13**], for $q \neq 0$ the $\min\{\dim(\ker(q \otimes K)) \mid K/F$ such that $q \otimes K \neq 0\}$ is a 2-power 2^d. The degree of q is d (if $q = 0$, the degree of q is ∞). We also use the ideal $J_nF = \{q \in WF \mid \deg q \geq n\}$.

1. Witt kernels and strong Pfister ideals. The following basic results will be used frequently:

(a) If φ is a neighbor to the n-fold Pfister form ρ, then $W(F(\varphi)/F)$ is a strong n-Pfister ideal ([**5**, 1.4]).

(b) (Cassels-Pfister theorem.) Let q and φ be anisotropic forms such that $q \otimes F(\varphi) = 0$. Then for each $x \in D(q) \cdot D(\varphi)$, there exists a form η_x over F such that $xq \simeq \varphi \perp \eta_x$.

LEMMA 1.1. *Suppose ψ is a subform of a form φ. Then $W(F(\varphi)/F) \subset W(F(\psi)/F)$.*

Proof. Since $\varphi \otimes F(\psi)$ is isotropic, there is an F-place $F(\varphi) \to F(\psi) \cup \infty$ and so $W(F(\varphi)/F) \subset W(F(\psi)/F)$ (cf. [**13**]).

We begin the computations:

PROPOSITION 1.2. *Let φ and ψ be anisotropic forms over F with $1 \in D(\psi)$ and $\varphi \simeq \psi \perp \langle x \rangle$, for some $x \in \dot{F}$. If $W(F(\psi)/F)$ is a strong n-Pfister ideal then*:

(i) *$W(F(\varphi)/F)$ is a $\{n, n+1\}$-Pfister ideal.*

(ii) *If $\sigma \in W(F(\varphi)/F) \cap P_k(F)$, with $k \geq n+1$, then there is a $\rho \in W(F(\varphi)/F) \cap P_{n+1}(F)$ such that $\rho \mid \sigma$.*

Proof. (i) We have $W(F(\varphi)/F) \subset W(F(\psi)/F)$ by (1.1). Let $q \in W(F(\varphi)/F)$ be anisotropic; we may assume $1 \in D(q)$. Now $q \in W(F(\psi)/F)$, a strong n-Pfister ideal, so we may write:

$$(*) \qquad q \simeq c_1\rho_1 \perp \cdots \perp c_k\rho_k$$

where $c_i \in \dot{F}$ and $\rho_i \in W(F(\psi)/F) \cap P_n(F)$. We use induction on k to show q equals a sum of multiples of n-fold and $(n+1)$-fold Pfister forms in $W(F(\varphi)/F)$. The case $k = 1$ is trivial, so suppose $k > 1$.

Since $1 \in D(q)$, we may assume $c_1 = 1$, by [**10**, 3.1]. By the Cassels-Pfister theorem, as $1 \in D(q) \cap D(\varphi)$, $1 \in D(\psi) \cap D(\rho_1)$, we have:

$$q \simeq \varphi \perp q_1 \simeq \psi \perp \langle x \rangle \perp q_1$$
$$\rho_1 \simeq \psi \perp \gamma$$

for some forms q_1 and γ over F. Cancelling ψ from the isometry $(*)$ yields

$$x \in D\left(\gamma \perp \mathop{\perp}_{i=2}^{k} c_i\rho_i\right).$$

Thus $x = a + b$, with

$$a \in D(\gamma) \cup \{0\}, b \in D\left(\mathop{\perp}_{i=2}^{k} c_i\rho_i\right) \cup \{0\}.$$

Case 1. $b = 0$.

Here $x \in D(\gamma)$ and so $\varphi \simeq \psi \perp \langle x \rangle < \rho_1$. Hence $\rho_1 \in W(F(\varphi)/F)$ and $\perp_{i \neq 2}^{k} c_i\rho_i \in W(F(\varphi)/F)$. By induction $\perp_{i=2}^{k} c_i\rho_i$ equals a sum of multiples of n-fold and $(n+1)$-fold Pfister forms in $W(F(\varphi)/F)$. Thus so is $q \simeq \rho_1 \perp \perp_{i=2}^{k} c_i\rho_i$.

Case 2. $b \neq 0$.

Here we may assume $c_2 = b$ by [**10**, 3.10]. Since $x \in D(\gamma \perp \langle b \rangle)$, $\varphi \simeq \psi \perp \langle x \rangle < \rho_1 \perp \langle b \rangle < \rho_1 \otimes \langle 1, b \rangle$. Now, since $W(F(\psi)/F)$ is a strong n-Pfister ideal, ρ_1 and ρ_2 are linked ([**10**, 3.1]). Say $\rho_i = \mu \otimes \langle 1, y_i \rangle$ $(i = 1, 2)$, where $\mu \in P_{n-1}(F)$ and $y_1, y_2 \in \dot{F}$. Then:

$$\begin{aligned}\rho_1 \perp b\rho_2 &= \mu \otimes \langle 1, y_1, b, by_2 \rangle \\ &= \mu \otimes \langle\langle y_1, b \rangle\rangle \perp by_2\mu \otimes \langle 1, -y_1y_2 \rangle.\end{aligned}$$

Note $\mu \otimes \langle\langle y_1, b \rangle\rangle \simeq \rho_1 \otimes \langle 1, b \rangle \in W(F(\varphi)/F)$, since it contains φ as a subform. So:

$$q \perp -\mu \otimes \langle\langle y_1, b \rangle\rangle = by_2\mu \otimes \langle 1, -y_1y_2 \rangle \perp \mathop{\perp}_{i=3}^{k} c_i\rho_i \in W(F(\varphi)/F).$$

The left hand side is also in $W(F(\psi)/F)$, a strong n-Pfister ideal, and thus its kernel is isometric to a sum of multiples of at most $k - 1$ n-fold Pfister forms in $W(F(\psi)/F)$. Thus by induction, $q \perp -\mu\langle\langle y_1, b \rangle\rangle$, and hence q is a sum of multiples of n-fold and $(n + 1)$-fold Pfister forms in $W(F(\varphi)/F)$.

(ii) Repeat the argument in (i), with σ replacing q. In Case 1, $\rho_1 \in W(F(\varphi)/F) \cap P_n(F)$ and ρ_1 is a subform of σ. Hence $\rho_1 \mid \sigma$, by [**5**, 2.7]. So take any form $\rho \in W(F(\varphi)/F) \cap P_{n+1}(F)$ such that $\rho_1 \mid \rho$ and $\rho \mid \sigma$.

In Case 2, let $\rho = \rho_1 \otimes \langle 1, b \rangle$. We know $\rho_1 \perp \langle b \rangle$ is a neighbor of ρ and a subform of σ. Thus $F(\rho) \sim_F F(\rho_1 \perp \langle b \rangle)$ and $\sigma \otimes F(\rho_1 \perp \langle b \rangle) = 0$, by [**13**, 4.1]. So $\sigma \otimes F(\rho) = 0$ and $\rho \mid \sigma$ by [**5**, 1.4]. Also, the argument in (i) showed $\varphi < \rho_1 \perp \langle b \rangle$, so $\varphi < \rho$ and thus $\rho \in W(F(\varphi)/F) \cap P_{n+1}(F)$.

COROLLARY 1.3. *Suppose $\varphi \simeq \psi \perp \langle x \rangle$ with $x \in \dot{F}$ and $W(F(\psi)/F)$ a strong $(n - 1)$-Pfister ideal. If $W(F(\varphi)/F) \cap P_{n-1}(F) = 0$ and $W(F(\varphi)/F)$ is n-linked, then $W(F(\varphi)/F)$ is a strong n-Pfister ideal.*

Proof. $W(F(\varphi)/F) \cap P_{n-1}(F) = 0$ and (1.2)(i) imply $W(F(\varphi)/F)$ is an n-Pfister ideal. Then (1.2)(ii) and the hypothesis on linkage imply the result by [**10**, 3.1].

PROPOSITION 1.4. *Let ψ be a neighbor of $\rho \in P_n(F)$ and let $\varphi \simeq \psi \perp \langle x \rangle$, with $x \in \dot{F}$, be anisotropic. Then either:*

(a) *φ is a neighbor of ρ and $W(F(\varphi)/F) = \rho WF$, a strong n-Pfister ideal, or*

(b) *φ is not a neighbor of ρ and $W(F(\varphi)/F)$ is a strong $(n + 1)$-Pfister ideal.*

Proof. We need only show (b) so assume φ is not a neighbor of ρ. By scaling if necessary, we may assume ψ, and hence φ, represent 1.

$W(F(\psi)/F) = \rho WF$ is a strong n-Pfister ideal. We wish to apply (1.3). Suppose $0 \neq \sigma \in W(F(\varphi)/F) \cap P_n(F)$. Since $1 \in D(\varphi) \cap D(\sigma)$, the Cassels-Pfister theorem implies that φ is a subform of σ and hence so is ψ. Since ψ is a neighbor of the n-fold Pfister form ρ, $\dim \psi > 2^{n-1}$. Thus ψ is a neighbor of σ and $\rho \simeq \sigma$ ([**13**, 7.4]). Thus φ is a neighbor of ρ. Contradiction.

Thus $W(F(\varphi)/F) \cap P_n(F) = 0$. Since $W(F(\varphi)/F) \subset W(F(\psi)/F) = \rho WF$, any two $(n+1)$-fold Pfister forms in $W(F(\varphi)/F)$ are linked by ρ. So the result follows from (1.3).

COROLLARY 1.5. *Let φ be an anisotropic form such that* $\dim \varphi = 4$ *and* $\varphi \notin GP(F)$. *If* $W(F(\varphi)/F) \neq 0$, *then* $W(F(\varphi)/F)$ *is a strong* 3*-Pfister ideal. In particular, φ is conservative if and only if φ is a conjugate neighbor.*

Proof. By scaling we may assume $\varphi \simeq \langle 1, a, b, x \rangle$, for some a, b, $x \in \dot{F}$. The first statement then follows from (1.4) and the second from the Cassels-Pfister theorem.

2. $W(F(\varphi)/F)$ for small dimensional φ.

REMARK. Let ρ be an n-fold Pfister form over F. Suppose $\rho \simeq \psi \perp \gamma$, with $\dim \psi > \dim \gamma$, and $\varphi \simeq \psi \perp \langle x \rangle$ is anistropic. Further suppose φ is not a neighbor of ρ. Then, $W(F(\varphi)/F)$ is a strong $(n+1)$-Pfister ideal by (1.4). By examining the proof of (1.2) we see that:

$$P_{n+1}(F) \cap W(F(\varphi)/F) = \{\rho \otimes \langle 1, a \rangle \mid a \in D(\langle x \rangle \perp -\gamma)\}.$$

Now $W(F(\psi)/F) = \rho WF$ and:

$$\rho WF \cap \langle 1, x \rangle WF \cap P_{n+1}(F) = \{\rho \otimes \langle 1, a \rangle \mid a \in D(\langle x \rangle \perp -\rho')\},$$

where ρ' is the pure part of ρ. Thus:

$$W(F(\varphi)/F) \subset W(F(\psi)/F) \cap \langle 1, x \rangle WF,$$

but the inclusion may be strict.

We wish to examine in detail the structure of $W(F(\varphi)/F)$ for four dimensional forms:

EXAMPLE. Let φ be conservative and $\dim \varphi = 4$; we may assume $\varphi \simeq \langle 1, a, b, x\rangle$. Suppose $x \neq ab$. Then:

$$\begin{aligned} W(F(\varphi)/F) \cap P_3(F) &= \{\langle\langle a, b, \alpha\rangle\rangle \mid \alpha \in D(\langle x, -ab\rangle)\} \\ &= \{\langle\langle a, b, xt^2 - abs^2\rangle\rangle \mid s, t \in F\}. \end{aligned}$$

If $t = 0$, then $\langle\langle a, b, xt^2 - abs^2\rangle\rangle \simeq \langle\langle a, b, -ab\rangle\rangle = 0$. So we may assume $t \neq 0$. Hence:

$$W(F(\varphi)/F) \cap P_3(F) = \{\langle\langle a, b, x - abs^2\rangle\rangle \mid s \in F\} \cup \{0\}.$$

In particular:

$$W(F(\varphi)/F) = \langle\langle a, b\rangle\rangle \sum_{s\in F} \langle\langle x - abs^2\rangle\rangle WF.$$

by (1.5).

Comparing with (1.5), we also have:

$$\begin{aligned} &\varphi \text{ is conservative iff } \varphi \text{ is a conjugate neighbor} \\ &\qquad \text{iff } D(\langle -x, ab\rangle) \not\subset D(\langle\langle a, b\rangle\rangle). \end{aligned}$$

To treat 5 and 6 dimensional forms, we need:

THEOREM 2.1. *Let ψ be a codimension 1 neighbor of $\rho \in P_n(F)$, $\varphi \simeq \psi \perp \langle x, y\rangle$ anisotropic and suppose φ is not a Pfister neighbor. Then $W(F(\varphi)/F)$ is a strong $(n + 2)$-Pfister ideal.*

Proof. By (1.4) $W(F(\psi \perp \langle x\rangle)/F)$ is a strong $(n + 1)$-Pfister ideal, and so $W(F(\varphi)/F)$ is a $\{n + 1, n + 2\}$-Pfister ideal, by (1.2). Since $\dim \varphi = 2^n + 1$ and φ is not a Pfister neighbor $W(F(\varphi)/F) \cap P_{n+1}(F) = 0$. Thus, by (1.3), we need only show $W(F(\varphi)/F)$ is $(n + 2)$-linked.

Let $\rho_1, \rho_2 \in W(F(\varphi)/F) \cap P_{n+2}(F)$. By the Cassels-Pfister theorem, φ is similar to a subform of each ρ_i and so the Witt index $i(\rho_1 \perp -\rho_2) \geq 2^n + 1$. But $i(\rho_1 \perp -\rho_2)$ must be a power of 2, by [**5**, 4.5]. Thus $i(\rho_1 \perp -\rho_2) \geq 2^{n+1}$ and hence ρ_1 and ρ_2 are linked.

COROLLARY 2.2. *Let φ be a conservative form of dimension 5. Then either:*

(a) *φ is a neighbor to a Pfister form ρ and $W(F(\varphi)/F) = \rho WF$ is a strong 3-Pfister ideal, or*

(b) *φ is not a Pfister neighbor and $W(F(\varphi)/F)$ is a strong 4-Pfister ideal.*

It is quite possible that a 5 dimensional form φ is not a Pfister neighbor. Indeed φ is a Pfister neighbor if and only if $d(\varphi) \in D(\varphi)$, by [**13**, p. 10].

COROLLARY 2.3. *Let φ be a conservative form of dimension 6. If φ is not a Pfister neighbor then $W(F(\varphi)/F)$ is a $\{4,5\}$-Pfister ideal.*

EXAMPLE. If φ has dimension 6, $W(F(\varphi)/F)$ need not be a strong Pfister ideal. Since no such example is in the literature, I will work out one in some (but not complete) detail.

Let $F = \mathbf{R}(x, y, z)$, $\varphi = \langle 1, 1, 1, x, y, z\rangle$, $\rho_1 = \langle\langle 1, 1, x, y, z\rangle\rangle$ and $\rho_2 = \langle\langle 1, 1, x, y-1, z-x\rangle\rangle$. By considering an ordering for which $z > y > x > 1$, one sees that ρ_1 and ρ_2 are anisotropic. A simple computation shows $\varphi < \rho_1$ and $\varphi < \rho_2$, while a more tedious one shows ρ_1 and ρ_2 are not linked. Let $\psi = \ker(\rho_1 \perp -\rho_2) \in W(F(\varphi)/F)$.

Fix an ordering α on F with x infinitely large positive, y infinitely small positive and z infinitely larger than x.

Claim. There does not exist $\sigma \in W(F(\varphi)/F) \cap P_4(F)$ such that $\operatorname{sgn}_\alpha \sigma = 16$.

We first note that $\langle 1, 1, 1, x, y\rangle$ is not a Pfister neighbor — otherwise $xy \in D(\langle 1, 1, 1, x, y\rangle)$ and $\langle 1, 1, 1, x\rangle \perp y\langle 1, -x\rangle$ is isotropic, which is impossible. Thus if there is a σ invalidating the claim, the proof of (1.2) shows we may write $\sigma \simeq \langle\langle 1, 1, p^2x - q^2, r^2y - \beta\rangle\rangle$, where $p, q, r, \beta \in \mathbf{R}[x, y, z]$ and $\beta \in (p^2x - q^2)D(\langle 1, 1, 1, x\rangle)$.

We will show $z \notin D(\sigma)$ and hence $\varphi \not< \sigma$. We need some simple calculations. For a polynomial $g(x, y, z) \in \mathbf{R}[x, y, z]$ let $\deg_x g$ denote the degree of g as a polynomial in x over $\mathbf{R}[y, z]$. Define $\deg_y g$ and $\deg_z g$ similarly.

Consider p^2x, q^2 and r^2y as polynomials in z over $\mathbf{R}[x, y]$, with leading coefficients $w_1(x, y)$, $w_2(x, y)$ and $w_3(x, y)$ respectively. Note that p^2x, q^2 and r^2y have even z-degree. It is easy to check the following:

(a) $\deg_x w_1$ is odd and $\deg_x w_2$ is even,

(b) $\deg_y(w_1 - w_2)$ is even,

(c) $\deg_y w_3$ is odd.

We thus obtain:

(i) $\deg_z(p^2x - q^2)$ is even (by (a)),

(ii) $\deg_z \beta$ is even (by (i)),

(iii) $\deg_z(r^2y - \beta)$ is even (by (ii), (b) and (c)).

Suppose finally that $z \in D(\sigma)$. Then:

$$(*) \quad z = s_0 + (p^2x - q^2)s_1 + (r^2y - \beta)s_2 + (p^2x - q^2)(r^2y - \beta)s_3$$

with each s_i a sum of four squares in F. Let $w_4(x, y)$ be the z-leading coefficient of β. Set

$$V = \{(a, b) \in \mathbf{R}^2 \mid w_i(a, b) = 0, \text{ some } i = 1, 2, 3, \text{ or } 4\};$$

V is a closed subvariety of $\mathbf{R}^2$. Since $\operatorname{sgn}_\alpha \sigma = 16$, $p^2x - q^2$ and $r^2y - \beta$ are positive with respect to α and we may find positive $x_0, y_0 \in R - V$ such that:

$$\left.\begin{array}{l} P_1(z) = (p^2x - q^2)(x_0, y_0, z) \\ P_2(z) = (r^2y - \beta)(x_0, y_0, z) \end{array}\right\} \geq 0 \quad \text{for } z \gg 0.$$

By the observations (i) and (iii), we see that P_1 and P_2 have even degree. Thus for sufficiently negative z_0, at (x_0, y_0, z_0) the left hand side of $(*)$ is negative while the right hand side is positive. This proves the claim.

To finish the example, suppose $W(F(\varphi)/F)$ is a strong Pfister ideal. Then we may write:

$$\psi \simeq \perp a_i\mu_i, \quad \text{with } \mu_i \in W(F(\varphi)/F) \cap P(F) \text{ and } a_i \in \dot{F}.$$

Since $\dim \psi = 48$ we have three cases:

(i) Some $\mu_i \in P_3(F)$:
Then φ is a Pfister neighbor and there exists a $\sigma \in W(F(\varphi)/F) \cap P_4(F)$ such that $\sigma \mid \rho_1$. Since $\operatorname{sgn}_\alpha \rho_1 = 32$, $\operatorname{sgn}_\alpha \sigma = 16$. Contradiction.

(ii) Some $\mu_i \in P_5(F)$:
Then $\psi \simeq a_1\mu_1 \perp a_2\mu_2$, with $\mu_1 \in P_5(F)$ and $\mu_2 \in P_4(F)$. But $\deg \psi = 5$ while $\deg(a_1\mu_1 \perp a_2\mu_2) = 4$, which again is a contradiction.

(iii) All $\mu_i \in P_4(F)$:
Then $\psi \simeq a_1\mu_1 \perp a_2\mu_2 \perp a_3\mu_3$, with $\mu_i \in W(F(\varphi)/F) \cap P_4(F)$. Now $\operatorname{sgn}_\alpha \psi = 32$, as $\operatorname{sgn}_\alpha \rho_1 = 32$ and $\operatorname{sgn}_\alpha \rho_2 = 0$. Thus at least one μ_i has α-signature 16, contradicting the claim.

Thus $W(F(\varphi)/F)$ is *not* a strong Pfister ideal.

It is worth noting that $W(F(\varphi)/F)$ does however contain 4-fold Pfister forms. For example, $0 \neq \langle\langle 1, 1, x, 4xy - (xz - xy - 1)^2\rangle\rangle$ is in $W(F(\varphi)/F)$.

We can show $W(F(\varphi)/F)$ is a strong Pfister ideal in some cases.

COROLLARY 2.4. *Let φ be a conservative form of dimension* 6 *which is not a Pfister neighbor. If φ contains a four dimensional subform of determinant* 1, *then $W(F(\varphi)/F)$ is a strong* 4-*Pfister ideal.*

Proof. Write $\varphi \simeq \psi \perp \langle a, b \rangle$, with $\dim \psi = 4$ and $d(\psi) = 1$. If $c \in D(\psi)$, then $c\psi \in P_2(F)$. So we may assume $\varphi \simeq \rho \perp \langle x, y \rangle$, where $\rho \in P_2(F)$ and $x, y \in \dot{F}$. Now $\rho \perp \langle x \rangle$ is a neighbor to $\rho \otimes \langle 1, x \rangle$, so $W(F(\varphi)/F)$ is a strong 4-Pfister ideal by (1.4).

3. Conservative and embeddable forms. In [**12**], Gentile and Shapiro raised the question whether a conservative form φ over F must be embeddable. They showed the answer was yes, if $\dim \varphi \leq 5$ or if $u(F) < 24$ ([**12**, Corollaries 8 and 19]). The results of Section 2 can be used to improve these bounds. As an immediate consequence of (2.3) we have:

COROLLARY 3.1. *Let* $\dim \varphi \leq 6$. *Then* φ *is conservative iff* φ *is embeddable.*

PROPOSITION 3.2. *Let* φ *be a conservative form over F which is not a Pfister neighbor and such that* $\dim \varphi \geq 5$. *Let* $q \in W(F(\varphi)/F)$ *be anisotropic. Then*:

(a) $16 \mid \dim q$

(b) $q \equiv \rho \bmod I^5F$, *where* $\rho \in P_4(F) \cap W(F(\varphi)/F)$.

Proof. We first note that for (b) we need only show the equation holds for some $\sigma \in GP_4(F)$. Namely then $q = \alpha\rho \perp q_1$, where $\alpha \in \dot{F}$, $\rho \in P_4(F)$ and $q_1 \in I^5F$. Now $\alpha\rho \otimes F(\varphi) = -q_1 \otimes F(\varphi) \in I^5F(\varphi)$. By the Arason-Pfister Hauptsatz ([**2**]), $\rho \otimes F(\varphi) = 0$ and so $\rho \in W(F(\varphi)/F)$. Further $q = \rho \perp \langle -1, \alpha \rangle \rho \perp q_1$ and so $q \equiv \rho \bmod I^5F$.

Let ψ be a 5-dimensional subform of φ. By (1.1), $q \in W(F(\psi)/F)$.

Case 1. ψ is not a Pfister neighbor:

Here we may write $q \simeq \perp_{i=1}^{m} \alpha_i\sigma_i$, with each $\alpha_i \in \dot{F}$ and $\sigma_i \in W(F(\psi)/F) \cap P_4(F)$, by (2.2). In particular, (a) holds. Now write:

$$q \equiv \mathop{\perp}_{i=1}^{n} a_i\rho_i \mod I^5F$$

with $a_i \in F$, $\rho_i \in W(F(\psi)/F) \cap P_4(F)$ and n minimal. Suppose $n > 1$. Since $W(F(\psi)/F)$ is a strong 4-Pfister ideal, ρ_1 and ρ_2 are linked. Thus there is an $a_{n+1} \in \dot{F}$ and $\rho_{n+1} \in W(F(\psi)/F) \cap P_4(F)$ such that

$a_2(\rho_2 \perp -\rho_1) = a_{n+1}\rho_{n+1}$. We have:

$$\begin{aligned} q &\equiv a_1\rho_1 \perp a_2\rho_1 \perp -a_2\rho_1 \perp a_2\rho_2 \perp \mathop{\perp}_{i=3}^{n} a_i\rho_i && \bmod I^5F \\ &\equiv \langle a_1, a_2\rangle \rho_1 \perp a_{n+1}\rho_{n+1} \perp \mathop{\perp}_{i=3}^{n} a_i\rho_i && \bmod I^5F \\ &\equiv \mathop{\perp}_{i=3}^{n+1} a_i\rho_i && \bmod I^5F. \end{aligned}$$

This contradicts the minimality of n and proves (b) for this case.

Case 2. ψ is a Pfister neighbor:

Let ψ be a neighbor to the (3-fold) Pfister form σ. Then $q \simeq \sigma \otimes \langle b_1,\dots,b_m\rangle$ by [**5**, 1.4]. To prove (a), we need only show m is even. Suppose m is odd. Since $q \otimes F(\varphi) = 0$, $(\sigma \otimes F(\varphi)) \otimes (\langle b_1,\dots,b_m\rangle \otimes F(\varphi)) = 0$. If $\sigma \otimes F(\varphi) \neq 0$, then $\langle b_1,\dots,b_m\rangle \otimes F(\varphi)$ is an odd dimensional zero advisor, which is impossible ([**15**, VIII 6.7]). Thus $\sigma \otimes F(\varphi) = 0$. Since $\deg \sigma = 3$ and $\dim \varphi > 5$, the Cassels-Pfister theorem implies φ is a neighbor to σ, contrary to hypothesis. Thus m is even and (a) holds.

Now write $\langle b_1,\dots,b_m\rangle \equiv \langle 1, x\rangle \bmod I^2F$ for some $x \in \dot{F}$. Then $q \equiv \langle 1, x\rangle\sigma \bmod I^5F$ as desired.

COROLLARY 3.3. *If F is 5-linked then for all conservative φ over F, $W(F(\varphi)/F)$ is a Pfister ideal.*

Proof. Let $q \in W(F(\varphi)/F)$; we may write $q = a_1\rho_1 \perp q_1$ with $a_1 \in \dot{F}$, $\rho_1 \in W(F(\varphi)/F) \cap P_4(F)$ and $q_1 \in W(F(\varphi)/F) \cap I^5F$, by (3.2). By [**10**, 5.1], $W(F(\varphi)/F) \cap I^5F$ is a Pfister ideal, hence q_1, and q, lie in $W(F(\varphi)/F)_{Pf}$.

COROLLARY 3.4. *Suppose φ is a conservative form over F that is not embeddable. Then $W(F(\varphi)/F) \subset I^5F$.*

Proof. Clearly φ is not a Pfister neighbor, and $\dim \varphi \geq 7$ by (3.1). The result then follows from (3.2) since $W(F(\varphi)/F) \cap P_4(F) = 0$.

In [**9**] it was shown that if $q \in W(F(\varphi)/F)$ then $2^n q \in W(F(\varphi)/F)_{Pf}$, where $n = \dim q$. Thus if φ is conservative but not embeddable then $W(F(\varphi)/F) \subset W_tF$ (see also [**12**]). Hence we have:

COROLLARY 3.5. *Suppose I^5F is torsion-free. Then a form φ over F is conservative if and only if it is embeddable.*

In particular, if $\text{tr.d.}_{\mathbf{R}}(F) \leq 4$, then φ is conservative if and only if it is embeddable.

COROLLARY 3.6. *Suppose φ is a conservative form over F that is not embeddable. If $q \in W(F(\varphi)/F)$ is non-zero, then* $\dim q \geq 48$.

In particular, if $u(F) < 48$, then a form over F is conservative if and only if it is embeddable.

Proof. We may assume q is anisotropic. By (3.4), $q \in I^5F$ and so by the Arason-Pfister Hauptsatz ([**2**]), $\dim q \geq 32$. If $\dim q = 32$, then $q \in GP(F)$ and φ is embeddable; thus $\dim q > 32$. By (3.2), $16 \mid \dim q$, so $\dim q \geq 48$.

4. Witt kernels over fields of finite Hasse number. As was done in [**11**], for an anisotropic form q we define $N(q)$ to be $\dim q - q^{\deg q}$.

LEMMA 4.1. *Suppose $\varphi \notin GP(F)$ and q is an anisotropic form with $q \in W(F(\varphi)/F)$. Then,*

(i) $2^{\deg q} > \dim \varphi$;

(ii) *if* $N(q) < 2 \cdot \dim \varphi$ *then* $q \in GP(F)$.

Proof. (i) follows from [**12**, Prop. 13] and (ii) follows from [**11**, 1.6].

REMARK. A stronger inequality than (i) is shown in [**12**], namely that $2^{\deg q} \geq \dim \varphi + 2^{\deg \varphi}$. It would be interesting to know if this can be improved to $2^{\deg q} \geq 2 \cdot \dim \varphi$ for non-Pfister neighbors φ. Note that if there exists a $q \in W(F(\varphi)/F)$ such that $2 \dim \varphi \geq 2^{\deg q}$ and φ is not a Pfister neighbor then $W(F(\varphi)/F)$ is not a Pfister ideal. Namely, suppose $q = \perp_{i=1}^{n} x_i \rho_i$ with $\rho_i \in W(F(\varphi)/F) \cap P(F)$. Then for some i, $\deg \rho_i \leq \deg q$ and the Cassels-Pfister theorem then implies φ is a Pfister neighbor.

We next recall a definition due to Knebusch, Rosenberg and Ware (cf. [**14**, 1.2]) which will be used frequently in this section:

DEFINITION. We say F satisfies the Strong Approximation Property (SAP) if for every clopen $S \subset X_F$ there exists an $e \in \dot{F}$ such that $e > 0$ on S and $e < 0$ outside of S.

The following lemma is well-known.

LEMMA 4.2. *If $\tilde{u}(F) \leq 2^n$, then F is n-linked. In particular, F is* SAP.

Proof. Let $\rho_1, \rho_2 \in P_n(F)$. Then for any ordering α on F,

$$|\operatorname{sgn}_\alpha(\rho_1 \perp -\rho_2)| = \dim \rho_1 \text{ or } 0.$$

In particular, $\rho_1 \perp -\rho_2$ is indefinite. Hence $\dim(\ker(\rho_1 \perp -\rho_2)) \leq 2^n$ and the Witt index $i(\rho_1 \perp -\rho_2) \geq 2^{n-1}$. Then, ρ_1 and ρ_2 are linked, by [**5**, 4.4].

For the second statement, F is n-linked, so stably linked (cf. [**6**]) and hence F is SAP by [**6**, 3.5].

LEMMA 4.3. *Let $q \in W(F(\varphi)/F)$. If φ is indefinite at $\alpha \in X_F$, then* $\operatorname{sgn}_\alpha q = 0$.

Proof. Since φ is indefinite at α, α extends to $F(\varphi)$ ([**9**, 3.5]). Since $q \otimes F(\varphi) = 0$, $\operatorname{sgn}_\alpha q = 0$.

PROPOSITION 4.4. *Suppose $u(F) \leq 2^n$, and φ is a conservative indefinite form over F. Then*:

(i) *If $2^{n-1} < \dim \varphi \leq 2^n$, then φ is a Pfister neighbor. In particular, $W(F(\varphi)/F)$ is a strong n-Pfister ideal.*

(ii) *If $2^{n-2} < \dim \varphi \leq 2^{n-1}$, then either*:

(a) *φ is a Pfister neighbor and $W(F(\varphi)/F)$ is a strong $(n-1)$-Pfister ideal, or*

(b) *φ is not a Pfister neighbor and every non-zero anistropic $q \in W(F(\varphi)/F)$ is in $GP_n(F)$. In particular, $W(F(\varphi)/F)$ is a strong n-Pfister ideal.*

Proof. Let $0 \neq q \in W(F(\varphi)/F)$ be anistropic. By (4.3) $\operatorname{sgn}_\alpha q = 0$ for all $\alpha \in X_F$, so by Pfister's Local-Global Principle q is torsion. Thus $\dim q \leq 2^n$.

(i) Here $\dim q < 2 \dim \varphi$ and so $q \in GP_n(F)$ by (4.1). In particular, φ is a Pfister neighbor.

(ii) Part (a) is known so suppose φ is not a Pfister neighbor. By (4.1), $2^{\deg q} > \dim \varphi > 2^{n-2}$ thus $\deg q \geq n-1$ and $N(q) \leq 2^n - 2^{n-1} < 2 \dim \varphi$. (4.1) then implies $q \in GP(F)$. If $\deg q = n-1$, then φ is a Pfister neighbor, contrary to the assumption of (b). Hence $q \in GP_n(F)$ and $W(F(\varphi)/F)$ is a strong n-Pfister ideal.

Both the statement and the proof of the following lemma are similar to the Pfister neighbor criterion of Elman, Lam and Wadsworth [**8**, 4.6]:

LEMMA 4.5. *Let F be formally real with $\tilde{u}(F) \leq 2^n$. Let φ be a form over F, definite at some $\alpha \in X_F$, with $1 \in D(\varphi)$ and* $\dim \varphi > 2^{n-2}$.

(i) *Let m be the least integer such that $n \leq m$ and* $\dim \varphi \leq 2^m$. *Let S be* *a non-empty clopen subset of* X_F *such that* $S \subset \{\alpha \mid \varphi$ *is (positive) definite at* $\alpha\}$. *Then there exists* $\rho \in W(F(\varphi)/F) \cap P_{m+1}(F)$ *such that* ρ *is definite at* α *iff* $\alpha \in S$.

(ii) *If* $\dim \varphi = 2^m + 1$, $m \geq n$, *then* φ *is a Pfister neighbor.*

Proof. In part (ii) let $S = \{\alpha \mid \varphi \text{ is definite at } \alpha\}$. S is clopen since $S = \hat{\varphi}^{-1}(\{\dim \varphi\})$, where $\hat{\varphi}\colon X_F \to Z$ is the continuous function $\alpha \mapsto \operatorname{sgn}_\alpha(\varphi)$.

For both parts (i) and (ii) there is an $e \in \dot{F}$ such that $e >_\alpha 0$ iff $\alpha \in S$, since F is SAP. Set $\rho = 2^m\langle 1, e\rangle$. For $\alpha \in X_F$ then:

$$\operatorname{sgn}_\alpha(\rho \perp -\varphi) = \begin{cases} -\operatorname{sgn}_\alpha \varphi, & \text{if } e <_\alpha 0 \\ \dim \rho - \dim \varphi & \text{if } e >_\alpha 0. \end{cases}$$

In (i), $|\operatorname{sgn}_\alpha \varphi| \leq \dim \varphi \leq 2^m \leq \dim \rho - \dim \varphi$. In (ii), if $e <_\alpha 0$, $|\operatorname{sgn}_\alpha \varphi| \leq \dim \varphi - 2 = 2^m - 1 = \dim \rho - \dim \varphi$. Thus in both cases

$$|\operatorname{sgn}_\alpha(\rho \perp -\varphi)| \leq \dim \rho - \dim \varphi, \quad \text{for all } \alpha \in X_F.$$

Set $\psi = \ker(\rho \perp -\varphi)$.

Suppose $\dim \psi > \dim \rho - \dim \varphi$. Then ψ is indefinite. In (i), this forces $\dim \psi \leq 2^n \leq 2^m \leq \dim \rho - \dim \varphi$, and in (ii), since $\dim \psi$ is odd, $\dim \psi \leq 2^n - 1 \leq \dim \rho - \dim \varphi$. In both cases we get a contradiction.

So $\dim \psi \leq \dim \rho - \dim \varphi$. In particular, the Witt index $i(\rho \perp -\varphi) \geq \dim \varphi$. Thus φ is a subform of ρ.

THEOREM 4.6. *Suppose* $\tilde{u}(F) \leq 2^n$ *and* φ *is a conservative form over* F. *If* $2^{m-1} < \dim \varphi \leq 2^n$, *with* $m \geq n$, *then either*:

(i) φ *is a Pfister neighbor and* $W(F(\varphi)/F)$ *is a strong m-Pfister ideal*, *or*

(ii) φ *is not a Pfister neighbor and* $W(F(\varphi)/F)$ *is a strong* $(m + 1)$-*Pfister ideal.*

Proof. We may assume $1 \in D(\varphi)$. We may also assume φ is not indefinite and, in particular, that F is formally real, by (4.4). Case (i) is known so assume φ is not a Pfister neighbor.

Let $0 \neq q \in W(F(\varphi)/F)$ be anisotropic. We will show q is isometric to a sum of multiples of $(m + 1)$-fold Pfister forms in $W(F(\varphi)/F)$ by induction on $\dim q$.

Case 1. $\dim q \leq 2^{m+1}$:

By (4.1), $2^{\deg q} > \dim \varphi > 2^{m-1}$. So $\deg q \geq m$ and $N(q) \leq 2^{m+1} - 2^m = 2^m < 2 \dim \varphi$. This implies $q \in GP(F)$ by (4.1). If $\deg q = m$, then φ is a Pfister neighbor, contrary to our assumption. Thus $\deg q \geq m+1$. Since $\dim q \leq 2^{m+1}$ we obtain $q \in GP_{m+1}(F)$.

Case 2. $\dim q > 2^{m+1}$:

Set $S_1 = \{\alpha \in X_F \mid \operatorname{sgn}_\alpha q \neq 0\}$ and $S_2 = \{\alpha \in S_1 \mid \operatorname{sgn}_\alpha q > 0\}$. Both S_1 and S_2 are clopen, S_1 is non-empty (as $\dim q > \tilde{u}(F)$) and $S_1 \subset \{\alpha \mid \varphi$ is (positive) definite at $\alpha\}$ by (4.3). Thus there is an e_2 such that $e_2 >_\alpha 0$ iff $\alpha \in S_2$, since F is SAP (set $e_2 = -1$ if $S_2 \neq \varnothing$), and a $\rho \in W(F(\varphi)/F) \cap P_{m+1}(F)$ such that ρ is definite at α iff $\alpha \in S_1$, by (4.5).

Set $q_1 = \ker(e_2 q \perp -\rho)$. Let $\alpha \in X_F$. Then:

$$\operatorname{sgn}_\alpha q_1 = \begin{cases} 0, & \text{if } \alpha \notin S_1, \\ -\operatorname{sgn}_\alpha q - \dim \rho,\ \operatorname{sgn}_\alpha q < 0, & \text{if } \alpha \in S_1 - S_2, \\ \operatorname{sgn}_\alpha q - \dim \rho,\ \operatorname{sgn}_\alpha q > 0, & \text{if } \alpha \in S_2. \end{cases}$$

Thus for each $\alpha \in X_F$, $2 - \dim \rho \leq \operatorname{sgn}_\alpha q_1 \leq \dim q - \dim \rho$ that is:

$$|\operatorname{sgn}_\alpha q_1| \leq \max\{\dim q - 2^{m+1}, 2^{m+1} - 2\}.$$

Thus, since $\tilde{u}(F) \leq 2^n$,

$$(*) \qquad \dim q_1 \leq \max\{\dim q - 2^{m+1}, 2^{m+1} - 2, 2^n\}.$$

Now since q, $\rho \in W(F(\varphi)/F)$, $q_1 \in W(F(\varphi)/F)$. Applying the argument in Case 1 to q_1 (instead of q) we see that $\dim q_1 \geq 2^{m+1} > 2^n$. Hence, the largest term on the right in (*) must be $\dim q - 2^{m+1}$. So $\dim q_1 \leq \dim q - 2^{m+1}$.

Since $q_1 = e_2 q \perp -\rho$, $\dim q_1 \geq \dim q - \dim \rho = \dim q - 2^{m+1}$. So $\dim q_1 = \dim q - 2^{m+1}$, $e_2 q \simeq \rho \perp q_1$ and $q \simeq e_2 \rho \perp e_2 q_1$. Lastly, $e_2 q_1 \in W(F(\varphi)/F)$ and $\dim q_1 < \dim q$, so we are done by induction.

REMARK. Case (ii) of Theorem 4.6 can occur. Consider $\varphi = \langle 1,1,1,1,1,7\rangle$ over $F = \mathbf{Q}$. Since φ is not indefinite, φ is conservative— namely $\langle\langle 1,1,1,1,1,7\rangle\rangle \in W(F(\varphi)/F)$. If φ were a Pfister neighbor of some $\rho \in P(F)$, then since $5\langle 1\rangle < \varphi < \rho$, $\rho \simeq 8 \cdot \langle 1\rangle$ ([**5**, 2.7]). Thus $7 \in D(\langle 1,1,1\rangle)$, a contradiction. Hence φ is a conservative non-Pfister neighbor while $\tilde{u}(F) = 4$ and $\dim \varphi = 6$. However we do have:

PROPOSITION 4.7. *Suppose $\tilde{u}(F) \leq 2^n$ and φ is an anisotropic form over F. Then*:

(i) *If* $\dim \varphi = 2^m + 1$, $m \geq n$, *then φ is a Pfister neighbor.*

(ii) *If* $\dim \varphi = 2^m$, $m \geq n$ *and φ is not indefinite, then φ is a conjugate neighbor.*

Proof. We may assume $1 \in D(\varphi)$. Only (ii) is new and here $\varphi \perp \langle 1 \rangle$ is anisotropic since φ is not indefinite. Part (i) implies $\varphi \perp \langle 1 \rangle$ is a Pfister neighbor, and hence φ is a conjugate neighbor.

We now consider the forms φ over F with $\tilde{u}(F) \leq 2^n$ and $2^{n-2} < \dim \varphi \leq 2^{n-1}$. This requires two lemmas, the first of which is well-known:

LEMMA 4.8. *If $\tilde{u}(F) \leq 2^n$, then $J_k F = I^k F$ for $k \geq n$.*

Proof. We may assume F is real. Let $s = \mathrm{st}(F)$ be the reduced stability index as defined by Bröcker in [3]. SAP fields have $s = 1$ ([7]) so:

$$J_k F = I^k F + (J_k F)_t$$

for each k by [1, Lemma 2], where $(J_k F)_t$ denotes the torsion part of $J_k F$. Since $k \geq n$, $(J_k F)_t \subset I^k F$ and $J_k F = I^k F$.

LEMMA 4.9. *Suppose $\tilde{u}(F) \leq 2^n$ and φ is an anisotropic form over F with $2^{n-2} < \dim \varphi \leq 2^{n-1}$. Suppose also that there exists a $q \in W(F(\varphi)/F)$ of degree $n - 1$. Then φ is a Pfister neighbor.*

Proof. We may assume q is anisotropic, $1 \in D(q)$ and, by (4.4), that φ is not indefinite. We induct on $\dim q$. If $\dim q \leq 2^n$, then $N(q) \leq 2^n - 2^{n-1} < 2 \dim \varphi$. (4.1) then implies $q \in GP_{n-1}(F)$ and so φ is a Pfister neighbor by the Cassels-Pfister theorem.

Now suppose $\dim q > 2^n$; q is thus not indefinite. Set $S = \{\alpha \in X_F \mid q$ is (positive) definite at $\alpha\}$. S is non-empty and clopen in X_F. Using (4.3) and (4.5) we obtain a $\rho \in W(F(\varphi)/F) \cap P_{n+1}(F)$ such that ρ is definite at ρ iff $\alpha \in S$.

For $\alpha \in X_F$:

$$\mathrm{sgn}_\alpha(q \perp -\rho) = \begin{cases} \dim q - 2^{n+1}, & \text{if } \alpha \in S \\ \mathrm{sgn}_\alpha q, & \text{if } \alpha \notin S. \end{cases}$$

If $\dim q \leq 2^{n+1}$, then $|\dim q - 2^{n+1}| \leq 2^n < \dim q$. If $\dim q > 2^{n+1}$, then $|\dim q - 2^{n+1}| < \dim q$. And if $\alpha \notin S$, then $|\mathrm{sgn}_\alpha q_1| < \dim q$ for all $\alpha \in X_F$.

If $\dim q_1 \geq \dim q$, then q_1 is indefinite and of dimension greater than 2^n, which is impossible. So $\dim q_1 < \dim q$. By [13, 6.4], $q = \rho \perp q_1$ implies $\deg q_1 = n - 1$. Thus by induction φ is a Pfister neighbor.

REMARK. Lemma 4.9 says the first inequality of (4.1) *can* be strengthed to $2 \dim \varphi \leq 2^{\deg q}$ for non-Pfister neighbors φ provided $\dim \varphi > 2^{n-2}$ and $\tilde{u}(F) \leq 2^n$.

THEOREM 4.10. *Suppose* $\tilde{u}(F) \leq 2^n$ *and* φ *is a conservative form over* F. *If* $2^{n-2} < \dim \varphi \leq 2^{n-1}$, *then either*:

(i) φ *is a Pfister neighbor and* $W(F(\varphi)/F)$ *is a strong* $(n-1)$*-Pfister ideal*,

(ii) φ *is not a Pfister neighbor and* $W(F(\varphi)/F)$ *is a* $\{n, n+1\}$*-Pfister ideal.*

Proof. (i) is known so we may assume φ is not a Pfister neighbor. Let $0 \neq q \in W(F(\varphi)/F)$ be anisotropic. Then by (4.1), $2^{\deg q} > \dim \varphi > 2^{n-2}$ and so $\deg q \geq n-1$. By (4.9) $\deg q \geq n$, and so $q \in I^nF$ by (4.8). Thus $W(F(\varphi)/F) \subset I^nF$. Since F is n-linked [**10**, 5.1] implies $W(F(\varphi)/F)$ is a N-Pfister ideal, where $N = \{n, n+1, \ldots\}$.

To finish then, we need only show any form in $W(F(\varphi)/F) \cap P_i(F)$, with $i \geq n+2$, is divisible by a form in $W(F(\varphi)/F) \cap P_{n+1}(F)$. Let $\sigma \in W(F(\varphi)/F) \cap P_i(F)$ with $i \geq n+2$. We may assume φ is not indefinite and, in particular, that F is real, by (4.3). We may also assume $1 \in D(\varphi)$. Let $S = \{\alpha \in X_F \mid \varphi \text{ is (positive) definite at } \alpha\}$. S is non-empty and clopen in X_F. There is then a $(n+1) =$ fold Pfister form $\rho \in W(F(\varphi)/F)$ such that ρ is definite at α iff $\alpha \in S$. Using (4.3) we see that for all $\alpha \in X_F$:

$$\operatorname{sgn}_\alpha(\sigma \perp -\rho) = \begin{cases} \operatorname{sgn}_\alpha \sigma - 2^{n+1}, & \text{if } \alpha \in S \\ 0, & \text{if } \alpha \notin S. \end{cases}$$

So $|\operatorname{sgn}_\alpha(\sigma \perp -\rho)| \leq \dim \sigma - \dim \rho$. For all $\alpha \in X_F$. Since $\dim \sigma - \dim \rho > 2^n$, $\dim(\ker(\sigma \perp -\rho)) \leq \dim \sigma - \dim \rho$. Thus $\rho < \sigma$ and $\rho \mid \sigma$ by [**5**, 2.7].

REMARK. The result of (4.4)(ii) for non-real fields is stronger than the corresponding result (4.10) for real fields, namely for real fields we no longer have that $W(F(\varphi)/F)$ is a strong Pfister ideal. To see why this occurs we observe that $W(F(\varphi)/F)$ is a strong n-Pfister ideal iff there exists a $\rho \in W(F(\varphi)/F) \cap P_n(F)$ such that $\operatorname{sgn}_\alpha \rho = 0$ precisely when φ is indefinite at α. This condition holds trivially if F is non-real (take $\rho = 2^{n-1}\langle 1, -1\rangle$).

To verify the observation, we first note that by (4.2) and [**10**, 3.1], $W(F(\varphi)/F)$ is a strong n-Pfister ideal iff for each $\sigma \in W(F(\varphi)/F) \cap P_{n+1}(F)$ there exists a $\rho \in W(F(\varphi)/F) \cap P_n(F)$ such that $\rho \mid \sigma$. Suppose $W(F(\varphi)/F)$ is a strong n-Pfister ideal. Then, since F is SAP, we may find a $\sigma \in W(F(\varphi)/F) \cap P_{n+1}(F)$ such that σ is definite at α iff φ is. Let $\rho \in W(F(\varphi)/F) \cap P_n(F)$ be such that $\rho \mid \sigma$. Then $\operatorname{sgn}_\alpha \rho = 0$ iff φ is indefinite at α. On the other hand, suppose we have such a $\rho \in W(F(\varphi)/F) \cap P_n(F)$ and let $\sigma \in W(F(\varphi)/F) \cap P_{n+1}(F)$. By (4.3),

$$\{\alpha \in X_F \mid \operatorname{sgn}_\alpha \rho = 0\} \subset \{\alpha \in X_F \mid \operatorname{sgn}_\alpha \sigma = 0\},$$

so $|\operatorname{sgn}_\alpha(\sigma \perp -\rho)| \leq 2^n$ for each $\alpha \in X_F$ and $\rho \mid \sigma$ ([**5**, 2.7]). Thus $W(F(\varphi)/F)$ is a strong n Pfister ideal.

COROLLARY 4.11. *If $\tilde{u}(F) \leq 8$, then $W(F(\varphi)/F)$ is a strong k-Pfister ideal, for some k, for every conservative φ over F. In particular, this holds for C_3 fields, global fields and fields of transcendence degree ≤ 1 over* **R**.

Proof. The first statement follows from (1.5) and (4.6). For the second statement see [**4**].

Lastly we can imporve (3.3).

COROLLARY 4.12. *Let $\tilde{u}(F) \leq 32$ and φ a conservative form over F which is not a Pfister neighbor. Then $W(F(\varphi)/F)$ is a*:

(1)	3-*Pfister ideal*	*if* $\dim \varphi = 4$
(2)	4-*Pfister ideal*	*if* $\dim \varphi = 5$
(3)	$\{4,5\}$-*Pfister ideal*	*if* $\dim \varphi = 6$
(4)	$\{4,5,6\}$-*Pfister ideal*	*if* $\dim \varphi = 7$ *or* 8
(5)	$\{5,6\}$-*Pfister ideal*	*if* $9 \leq \dim \varphi \leq 16$
(6)	$(n+2)$-*Pfister ideal*	*if* $2^n < \dim \varphi \leq 2^{n+1}, n \geq 4$.

Proof. All but (4) have been done previously, so assume $\dim \varphi = 7$ or 8. The proof of (3.3) shows $W(F(\varphi)/F)$ is a $\{4,5,\dots\}$-Pfister ideal, while the second paragraph of the proof of (4.10) shows $W(F(\varphi)/F)$ is a $\{4,5,6\}$-Pfister ideal.

Acknowledgement. The author would like to thank the referee for many suggestions improving the readability of the paper.

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Received January 28, 1982.

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