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LOCALLY CONVEX SPACES OF NON-ARCHIMEDEAN VALUED CONTINUOUS FUNCTIONS

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We study the space $C(X, K, \mathcal{P})$ of all continuous functions from the ultraregular space X into the non-Archimedean valued field K with topology of uniform convergence on a family $\mathcal{P}$ of subsets of the Z-repletion of X. We characterize the bornological space associated to $C(X, K, \mathcal{P})$, semi-bornological spaces $C(X, K, \mathcal{P})$, reflexivity and semi-reflexivity both for spherically complete and non-spherically complete K.

1. Introduction. Throughout this paper, K is a complete non-trivially non-Archimedean valued field and X is an ultraregular (= zero-dimensional Hausdorff) space. Then $X \subseteq \upsilon_K X \subseteq \upsilon_0 X \subseteq \beta_0 X$ where $\upsilon_K X$, $\upsilon_0 X$ and $\beta_0 X$ are the K-repletion, $\mathbf{Z}$-repletion and Banaschewski compactification of X, respectively. If K has nonmeasurable cardinal, then $\upsilon_K X = \upsilon_0 X$ [**1**, Theorem 15].

The set $|K| = \{|\lambda| : \lambda \in K\}$ is provided with a topology in which all points are discrete, except for 0, whose neighborhoods are the usual ones. $|K|$ is a complete metric space under the metric

$$d(x, y) = \begin{cases} \max(x, y) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Hence $|K|$ is $\mathbf{Z}$-replete [**1**, Theorem 9], so $|f|$ can be extended continuously over the whole of $\upsilon_0 X$ whenever f belongs to the vector space $C(X, K)$ of all continuous functions from X into K.

A set $A \subseteq \upsilon_0 X$ is called bounding if $\|f\|_A := \sup_{x \in A} |f|(x) < \infty$ for all $f \in C(X, K)$. We omit the relatively easy proof of the following:

PROPOSITION 1. *The following are equivalent for $A \subseteq \upsilon_0 X$:*

(i) *A is bounding.*

(ii) *Every $g \in C(\upsilon_0 X, |K|)$ is bounded on A.*

(iii) *If $(U_i)_{i=1}^{\infty}$ is a partition of $\upsilon_0 X$ in open-and-closed subsets, then $U_i \cap A = \varnothing$ for all but finitely many i.*

(iv) *If $g \in C(\upsilon_0 X, |K|)$, then $g(A)$ is compact in $|K|$.*

(v) *If $g \in C(\upsilon_0 X, |K|)$, then $g(A)$ is relatively compact in $|K|$.*

(vi) *$\overline{A}^{\upsilon_0 X}$ is compact.*

Let $\mathcal{P}$ be an arbitrary family of subsets of $v_0 X$ such that $Y_{\mathcal{P}} := \bigcup \mathcal{P}$ is dense in $v_0 X$. Let $C(X, K, \mathcal{P})$ be the Hausdorff locally convex space $C(X, K)$ with topology of uniform convergence on all members of $\mathcal{P}$. Without loss of generality we assume:

(i) If $A, B \in \mathcal{P}$, then $A \cup B \in \mathcal{P}$.

(ii) If $A \in \mathcal{P}$, $B \subset A$, then $B \in \mathcal{P}$.

(iii) If $A \in \mathcal{P}$, then $\overline{A}^{Y_{\mathcal{P}}} \in \mathcal{P}$.

If $\mathcal{P}$ is the set $\mathcal{K}(X)$ (resp. $\mathcal{C}(X)$) of all compact (resp. finite) subsets of X, then we write $C_c(X, K)$ (resp. $C_s(X, K)$) instead of $C(X, K, \mathcal{P})$.

DEFINITION 2. The family $\overline{\mathcal{P}} := \{B \subseteq v_0 X: \exists\, B' \in \mathcal{P}$ with $B \subseteq \overline{B'}^{v_0 X}\}$ is called the extended family of $\mathcal{P}$.

$\mathcal{P}$ and $\overline{\mathcal{P}}$ induce the same topology on $C(X, K)$. $\overline{\mathcal{P}}$ satisfies (i)–(iii) as well as

(iii)′ If $A \in \overline{\mathcal{P}}$, then $\overline{A}^{v_0 X} \in \overline{\mathcal{P}}$.

DEFINITION 3. If $A \subseteq v_0 X$, $A_n \subseteq v_0 X$ for all $n = 1, 2, \ldots$ then $(A_n)_{n=1}^{\infty}$ is A-finite if $A_n \cap A = \varnothing$ for all but finitely many n. $(A_n)_{n=1}^{\infty}$ is $\mathcal{P}$-finite if it is A-finite for all $A \in \mathcal{P}$.

PROPOSITION 4. *Let $Y_{\mathcal{P}} \subseteq Z \subseteq v_0 X$, $A \subseteq Z$. The following are equivalent*:

(a) *Every bounded subset of $C(X, K, \mathcal{P})$ is uniformly bounded on A.*

(b) *If $(A_n)_{n=1}^{\infty}$ is a $\mathcal{P}$-finite sequence of open subsets of Z, then it is A-finite.*

In (b) *we can replace "open" by "clopen" and/or "Z" by "$v_0 X$".*

DEFINITION 5. Let $Y_{\mathcal{P}} \subseteq Z \subseteq v_0 X$. The Z-saturated family $\tilde{\mathcal{P}}^Z$ associated to $\mathcal{P}$ is the set of all $A \subseteq Z$ that satisfy one of the conditions mentioned in Proposition 4. $\mathcal{P}$ is Z-saturated iff $\tilde{\mathcal{P}}^Z = \mathcal{P}$. We write $\tilde{\mathcal{P}}$ instead of $\tilde{\mathcal{P}}^{v_0 X}$.

2. Completeness and quasi-completeness. The results in this section are relatively easy and are stated here mainly for further use.

THEOREM 6. *Assume $Y_{\overline{\mathcal{P}}} \subseteq v_K X$ and let $F_{\mathcal{P}}(X, K)$ be the set of all f: $Y_{\overline{\mathcal{P}}} \to K$ that are continuous on every $A \in \overline{\mathcal{P}}$. Then*:

(1) *$F_{\mathcal{P}}(X, K)$ is a vector space over K and contains $C(X, K)$ as a subspace.*

(2) *$F_{\mathcal{P}}(X, K)$ is a locally convex space under the semi-norms $\| \ \|_A$ $(A \in \overline{\mathcal{P}})$ where $\|f\|_A := \sup_{x \in A} |f(x)|$.*

(3) *The natural imbedding* $C(X, K, \mathcal{P}) \to F_{\mathcal{P}}(X, K)$ *is an into homeomorphism.*

(4) $F_{\mathcal{P}}(X, K)$ *is complete and contains* $C(X, K, \mathcal{P})$ *as a dense subspace; hence it is a completion of* $C(X, K, \mathcal{P})$.

THEOREM 7. *Assume* $Y_{\overline{\mathcal{P}}} \subseteq v_K X$. *The following are equivalent.*

(1) $C(X, K, \mathcal{P})$ *is complete.*

(2) $C(X, K, \mathcal{P})$ *is quasi-complete.*

(3) *If* $f\colon Y_{\overline{\mathcal{P}}} \to K$ *is continuous on every* $A \in \overline{\mathcal{P}}$, *then there is a* $g \in C(X, K)$ *such that* $f(x) = g(x)$ *for all* $x \in Y_{\overline{\mathcal{P}}}$.

Proof. (hint for (2) $\Rightarrow$ (3)). Let f be as stated. Choose a sequence $(\lambda_n)_{n=1}^{\infty}$ in K with $|\lambda_n| \underset{n\to\infty}{\to} \infty$. For all n put

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq |\lambda_n|, \\ \lambda_n & \text{if } |f(x)| > |\lambda_n|, \end{cases}$$

$$S_n = \{g \in C(X, K) : |g(x)| \leq |\lambda_n| \text{ for all } x \in X\}.$$

By quasi-completeness there is a function $f_n' \in C(X, K, \mathcal{P})$ with

$$f_n'(X) = \begin{cases} f(x) & \text{if } |f(x)| \leq |\lambda_n| \text{ and } x \in Y_{\overline{\mathcal{P}}}, \\ \lambda_n & \text{if } |f(x)| > |\lambda_n| \text{ and } x \in Y_{\overline{\mathcal{P}}}. \end{cases}$$

Let $S = \{g \in C(X, K)\colon |g(x)| \leq |f(x)| \text{ for all } x \in Y_{\overline{\mathcal{P}}}\}$. By quasi-completeness, the Cauchy-net $(f_n')_{n=1}^{\infty}$ in S has a limit g and $g(x) = f(x)$ for all $x \in Y_{\overline{\mathcal{P}}}$.

REMARK 8. For spherically complete K an example of a quasi-complete, non-complete locally convex space over K may be constructed as in the real case ([**2**, Chap. III, §2.5.]; communicated by N. De Grande-De Kimpe).

3. The bornological space associated to $C(X, K, \mathcal{P})$. J. Schmets [**10**, Théorème III.12] characterized the bornological space associated to $C(X, \mathcal{P})$, the classical (Archimedean) analogue of $C(X, K, \mathcal{P})$. We give an analogous characterization of the bornological space associated to $C(X, K, \mathcal{P})$. We assume the reader consults [**10**] and mainly stress the new features.

It is nice to remark that completeness of K can be dispensed with in this section.

DEFINITION 9. If $f \in C(X, K)$, put $\Delta(f) = \{g \in C(X, K): |g| \leq |f|\}$. In particular, put $\Delta = \Delta(1)$. Let $\tilde{f}$ be the continuous extension of f to a function $\beta_0 X \to \beta_0 K$.

Let D be an absolutely convex absorbing subset of $C(X, K)$ such that $\Delta \subseteq D$. A compact subset A of $\beta_0 X$ is called a carrier of D if $f \in D$ whenever f vanishes on A.

The set of all carriers of D will be denoted by $\mathcal{C}_D$.

LEMMA 10. *Let $A \subseteq \beta_0 X$ be compact. If $f \in D$ whenever f vanishes on a neighborhood of A in $\beta_0 X$, then $A \in \mathcal{C}_D$.*

LEMMA 11. *If $A, B \in \mathcal{C}_D$, then $A \cap B \in \mathcal{C}_D$.*

LEMMA 12. *$\mathcal{C}_D$ contains a smallest element $\mathcal{K}(D)$.*

LEMMA 13. *If $\tilde{f}(x) \in \overline{\{\lambda \in K: |\lambda| \leq 1\}}^{\beta_0 K}$ for all $x \in K(D)$, then $f \in D$.*

LEMMA 14. *The following are equivalent*:
(a) $K(D) \subseteq v_0 X$.
(b) *D is a neighborhood in $C(X, K, \mathcal{K}(v_0 X))$.*
(c) *D is bornivorous in $C(X, K, \mathcal{K}(v_0 X))$.*

The above results are in a form that make them comparable with the Archimedean ones as given in [**10**, Théorème III.1.2 and related results]. The Archimedean analogues go back to [**9**].

THEOREM 15. *$C(X, K, \tilde{\mathcal{P}})$ is the bornological space associated to $C(X, K, \mathcal{P})$.*

Proof. By (a) of Lemma 4, it suffices to prove that $C(X, K, \tilde{\mathcal{P}})$ is bornological. Let D be an absolutely convex bornivorous subset of $C(X, K, \tilde{\mathcal{P}})$. We may assume $\Delta \subseteq D$ (for Δ is bounded). By Lemma 14 we have $K(D) \subseteq v_0 X$; from Lemma 13 we induce that $\{f \in C(X, K): \|f\|_{K(D)} \leq 1\} \subseteq D$. Hence it suffices to prove that $K(D) \in \tilde{\mathcal{P}}$. Suppose not.

By Lemma 4(b) there is a $\mathcal{P}$-finite sequence $(A_n)_{n=1}^{\infty}$ of open subsets of $v_0 X$ that is not $K(D)$-finite. We may assume each A_n to be open-and-closed and $A_n \cap K(D) \neq \varnothing$ for all n. For all n there is an $f_n \in C(X, K)$ with $\tilde{f}_n = 0$ on $\beta_0 X \setminus A_n$ and $f_n \notin D$ (since $\beta_0 X \setminus A_n \notin \mathcal{C}_D$). If $\lambda_0 \in K$, $0 < |\lambda_0| < 1$, then $\bigcup_{n=1}^{\infty} \Delta(\lambda_0^{-n} f_n)$ is bounded in $C(X, K, \mathcal{P})$ so there is a

$\lambda \in K$ with $\bigcup_{n=1}^{\infty} \Delta(\lambda_0^{-n} f_n) \subseteq \lambda D$. Then for all $n, f_n \in \lambda_0^n \lambda D$, a contradiction.

COROLLARY 16. ([**5**, **6**]). *If $\mathcal{P} = \mathcal{K}(X)$ or $\mathcal{P} = \mathcal{Q}(X)$, then $C(X, K, \mathcal{P})$ is bornological iff X is* **Z**-*replete.*

4. Semi-bornological spaces $C(X, K, \mathcal{P})$ and $C(X, \mathcal{P})$. In this section we characterize the semi-bornological spaces $C(X, K, \mathcal{P})$ as well as their Archimedean counterparts $C(X, \mathcal{P})$. In the non-Archimedean setting semi-bornological spaces $C(X, K, \mathcal{P})$ are bornological in most practically occurring cases. In the Archimedean setting this turns out not to be true.

We use the notations of A. C. M. Van Rooij [**11**] on non-Archimedean measure theory. The notations on $C(X, \mathcal{P})$ are taken from J. Schmets [**10**]. In particular, X then denotes a completely regular Hausdorff space. Our main result is the following:

THEOREM 17. *If $C(X, K, \mathcal{P})$ is bornological (equivalently, $\tilde{\mathcal{P}} \subseteq \overline{\mathcal{P}}$), then it is semi-bornological. Conversely, assume that either K is spherically complete or has non-measurable cardinality. Then, if $C(X, K, \mathcal{P})$ is semi-bornological, it is bornological.*

Proof. We prove only the second part. Let $A \in \tilde{\mathcal{P}}$; we may assume A is closed in $\overline{A}^{v_0 X}$, i.e. A is compact. From [**11**, Theorem 7.9] we infer that there is a non-Archimedean measure μ on A such that, for every open-and-closed subset $B \subseteq A$ there is an $f \in C(A, K)$ with $\|f\|_{A \setminus B} = 0$ and $\int_A f\,d\mu \neq 0$. Define L: $C(v_0 X, K) \to K$ by $L(f) = \int_A f\,d\mu$. Then L is linear and $|L(f)| \leq \|f\|_A \cdot \|A\|_\mu$ for $f \in C(v_0 X, K)$. By the assumption on K in the theorem we may assume L is defined on the whole of $C(X, K)$ with $|L(f)| \leq \|f\|_A \cdot \|A\|_\mu$ for all $f \in C(X, K)$.

Since $A \in \tilde{\mathcal{P}}$, L is bounded (Lemma 4 and Definition 5) so L is continuous. Let $A' \in \mathcal{P}$ be such that L is $\|\cdot\|_{A'}$ continuous; we prove $A \subseteq \overline{A'}^{v_0 X}$.

Suppose not. Let A'' be an open-and-closed subset of $A \setminus A'$. Let $f \in C(A, K)$ be zero on $A \setminus A'$ and $\int_A f\,d\mu \neq 0$. Since A'' is compact, f can be extended to a function in $C(A \cup \overline{A'}^{v_0 X}, K)$ with $\|f\|_{A'} = 0$. By completeness of K and compactness of $A \cup \overline{A'}^{v_0 X}$ it follows from [**11**, Theorem 5.24] that f can be extended to a function in $C(v_0 X, K)$. Hence $\|f\|_{A'} = 0$ and $L(f) \neq 0$, a contradiction.

Surprisingly, the Archimedean analogue of Theorem 17 does not hold; a more complicated theory has to be developed. We make free use of the notations in [**10**].

DEFINITION 18. A Radon measure μ on a compact Hausdorff space A is called strictly positive if $|\mu|(U) \neq 0$ for every non-empty open subset U of A.

The strict family associated to $\mathfrak{P}$ is $\mathfrak{P}_{\text{str}} = \{A \in \mathfrak{P}\colon \overline{A}^{vX}$ carries a strictly positive measure$\}$.

Remark that $\mathfrak{P}_{\text{str}}$ satisfies (i)–(iii) and $Y_{\mathfrak{P}_{\text{str}}} = Y_{\mathfrak{P}}$. We have $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$ where:

(α) A is separable.

(β) A carries a strictly positive measure.

(γ) Every family of nonempty disjoint open subsets of A is countable.

LEMMA 19. *Let L: $C(X, \mathfrak{P}) \to \mathbf{R}$ be bounded and linear. Then it is continuous on $C(X, (\tilde{\mathfrak{P}}^{v})_{str})$.*

Proof. Cf. [7, Lemma 1].

LEMMA 20. *Assume $\mathfrak{P}$ and $\mathfrak{Q}$ satisfy* (i)–(iii) *and $\mathfrak{P} \subseteq \overline{\mathfrak{Q}}$. If $C(X, \mathfrak{Q})$ is semi-bornological, then $(\tilde{\mathfrak{P}}^{v})_{\text{str}} \subseteq \overline{\mathfrak{Q}}$.*

Proof. Let $A \in (\tilde{\mathfrak{P}}^{v})_{\text{str}}$. We may assume A is compact. Let μ be a strictly positive measure on A and put $L(f) = \int_A f\,d\mu$. Then L is bounded on all $\mathfrak{P}$-bounded sets.

Since $\mathfrak{P} \subseteq \overline{\mathfrak{Q}}$ and $C(X, \mathfrak{Q})$ is semi-bornological, it follows that L is continuous on $C(X, \mathfrak{Q})$. Let $B \in \overline{\mathfrak{Q}}$ and $\varepsilon > 0$ be such that $|L(f)| \leq 1$ whenever $\|f\|_B \leq \varepsilon$.

We may assume B is compact; a standard device then shows that $A \subseteq B$. Hence $A \in \overline{\mathfrak{Q}}$.

THEOREM 21. *There is a smallest family $\mathfrak{Q}$ of relatively compact subsets of vX such that* (a)–(c) *hold*:

(a) $\mathfrak{Q}$ *satisfies* (i)–(iv).

(b) $\mathfrak{Q} \supseteq \mathfrak{P}$.

(c) *$C(X, \mathfrak{Q})$ is semi-bornological.*

Actually, $\mathfrak{Q} = \{A \subseteq vX$: there exist $A_1 \in \mathfrak{P}$, $A_2 \in (\tilde{\mathfrak{P}}^{v})_{\text{str}}$ such that $A \subset \overline{A_1 \cup A_2}^{vX}\}$.

Proof. From Lemmas 19 and 20.

COROLLARY 22. *$C(X, \mathfrak{P})$ is semi-bornological iff $(\tilde{\mathfrak{P}}^{v})_{\text{str}} \subseteq \overline{\mathfrak{P}}$.*

REMARKS 23. (a) Let X be a compact Hausdorff space that carries no strictly positive measure and put $\mathcal{P} = (\mathcal{K}(X))_{\mathrm{str}}$. Then $(\tilde{\mathcal{P}}^{v})_{\mathrm{str}} \subseteq (\mathcal{K}(X))_{\mathrm{str}} = \mathcal{P} \subseteq \overline{\mathcal{P}}$ so $C(X, \mathcal{P})$ is semi-bornological. On the other hand, $C(X, \mathcal{P})$ is not bornological; in fact $C_c(X)$ is the bornological space associated to $C(X, \mathcal{P})$.

(b) Clearly $\mathcal{P} \subseteq \overline{\mathcal{P}} \subseteq \tilde{\mathcal{P}}^{v}$ and $\mathcal{P}_{\mathrm{str}} \subseteq \mathcal{P}$. Less trivially we have $(\mathcal{P}_{\mathrm{str}})^{v} \supseteq \tilde{\mathcal{P}}^{v}$; this is proved by an argument involving the fact that separable members of $\mathcal{P}$ belong to $\mathcal{P}_{\mathrm{str}}$.

THEOREM 24. *Let* $\mathcal{P} = \mathcal{K}(X)$ *or* $\mathcal{P} = \mathcal{Q}(X)$. *The following are equivalent*:

(1) $C(X, \mathcal{P})$ *is bornological.*

(2) $C(X, \mathcal{P})$ *is semi-bornological.*

(3) $X = vX$.

Proof. The equivalence of (1) and (3) is known ([**10**]), while (1) $\Rightarrow$ (2) is trivial.

To prove (2) $\Rightarrow$ (3) remark that $\mathcal{Q}(X)^{v} = \mathcal{Q}(vX)$ ([**10**, III.4.3]) so $(\mathcal{Q}(X)^{v})_{\mathrm{str}} = \mathcal{Q}(vX)$. If $\mathcal{P} = \mathcal{K}(X)$, see [**10**, III.2.4].

5. Reflexivity and semi-reflexivity for non spherically complete *K*.

In this section we assume $Y_{\overline{\mathcal{P}}} \subseteq v_K X$. Since $Y_{\overline{\mathcal{P}}}$ is dense in $v_0 X$ the dual $C(X, K, \mathcal{P})'$ of $C(X, K, \mathcal{P})$ separates the points of $C(X, K, \mathcal{P})$. Let b be the strong topology on $C(X, K, \mathcal{P})'$. There is a natural injection from $C(X, K, \mathcal{P})$ into $(C(X, K, \mathcal{P})'_b)'$. $C(X, K, \mathcal{P})$ is called semi-reflexive if this injection is onto, and reflexive if it is a homeomorphism onto $(C(X, K, \mathcal{P})'_b)'_b$.

LEMMA 25. *If* $L \in C(X, K, \mathcal{P})'$, *then there is a compact subset* $A_L \subset v_0 X$ *such that*:

(1) *For all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $|L(f)| \leq \varepsilon$ *whenever* $f \in C(X, K)$ *and* $\|f\|_{A_L} \leq \delta$.

(2) *If* A *is a compact subset of* $v_0 X$ *and* L *is bounded on* $\|\cdot\|_A$*-bounded subsets of* $C(X, K)$, *then* $A_L \subset A$.

(3) *If* A *is a compact subset of* $v_0 X$ *and* $L(f) = 0$ *for all* $f \in C(X, K)$ *for which* $\|f\|_A = 0$, *then* $A_L \subset A$.

(4) *If* $f \in C(X, K)$ *and* $\|f\|_{A_L} = 0$, *then* $L(f) = 0$.

(5) $A_L \in \overline{\mathcal{P}}$.

The set A_L *is called the carrier of* L.

REMARK 26. By [**11**, Theorem 7.18] there is a non-Archimedean measure μ on A_L such that $L(f) = \int_{A_L} f\,d\mu$ for all $f \in C(X, K)$.

COROLLARY 27. *Let $\mathcal{P}$ be directed by $\leq$ where $A \leq B$ iff $\overline{A}^{v_0 X} \subseteq \overline{B}^{v_0 X}$. If $A \leq B$ put*

$$h_{B,A}\colon C_c(\overline{A}^{v_0 X}, K)' \to C_c(\overline{B}^{v_0 X}, K)',$$

where $(h_{B,A}(L))(f) = L(f|_{\overline{A}^{v_0 X}})$ whenever $L \in C_c(\overline{A}^{v_0 X}, K)'$ and $f \in C_c(\overline{B}^{v_0 X}, K)$.

For $A \in \mathcal{P}$ put

$$h_A\colon C_c(\overline{A}^{v_0 X}, K)' \to C(X, K, \mathcal{P})'$$

where $(h_A(L))(f) = L(f|_{\overline{A}^{v_0 X}})$ whenever $L \in C(\overline{A}^{v_0 X}, K)'$ and $f \in C(X, K)$. Then $C(X, K, \mathcal{P})'$ is the algebraic inductive limit of $(C_c(\overline{A}^{v_0 X}, K)')_{A \in \mathcal{P}}$ with respect to the above $h_{B,A}$ and h_A.

Let B be bounded in $C(X, K, \mathcal{P})$, $B_A (A \in \mathcal{P})$ the set of all restrictions to $\overline{A}^{v_0 X}$ of functions from B. Every B_A $(A \in \mathcal{P})$ is bounded in $C_c(\overline{A}^{v_0 X}, K)$ and

$$B^0 = \bigcup_{A \in \mathcal{P}} h_A\big((B_A)^0\big)$$

where B^0 (resp. B_A^0) is the polar of B (resp. B_A) in $C(X, K, \mathcal{P})'$ (resp. $C_c(\overline{A}^{v_0 X}, K)'$).

LEMMA 28. *Let $\varphi \in (C(X, K, \mathcal{P})'_b)'$. For every $A \in \mathcal{P}$ there is a $\varphi_A \in C_c(\overline{A}^{v_0 X}, K)''$ such that*

$$\varphi(h_A(L)) = \varphi_A(L) \quad \text{for all } L \in C_c(\overline{A}^{v_0 X}, K)'.$$

Proof. By assumption there is a bounded set B in $C(X, K, \mathcal{P})$ such that $|\varphi(L_0)| \leq 1$ whenever $L_0 \in B^0$. If $A \in \mathcal{P}$, then $|\varphi(h_A(L))| \leq 1$ whenever $L \in B_A^0$. Since $\varphi \circ h_A$ is linear, we infer $\varphi \circ h_A \in C_c(\overline{A}^{v_0 X}, K)''$.

PROPOSITION 29. *Assume K non-spherically complete, K and Y nonmeasurable, $\varphi \in (C(X, K, \mathcal{P})'_b)'$. Then there is a function $f\colon Y_{\overline{\mathcal{P}}} \to K$ such that:*

(1) $f|_A$ is continuous on every $A \in \overline{\mathcal{P}}$.

(2) $\varphi(h_A(L)) = L(f|_A)$ for all $A \in \overline{\mathcal{P}}$ and $L \in C_c(\overline{A}^{v_0 X}, K)'$.

Proof. For every $A \in \mathcal{P}$ the space $C_c(\overline{A}^{v_0 X}, K)$ is isomorphic to a space of type $c_0(I)$ [**11**, 5.23]. Since K and X are nonmeasurable, I is nonmeasurable. By [**11**, Theorem 4.21] $c_0(I)$ is reflexive. Hence there is an $f_A \in C_c(\overline{A}^{v_0 X}, K)$ such that $\varphi(h_A(L)) = L(f_A)$ for all $L \in C_c(\overline{A}^{v_0 X}, K)'$.

Let $A, B \in \mathcal{P}$, $A \leq B$. For $L \in C_c(\overline{A}^{v_0 X}, K)'$ we have

$$L(f_A) = \varphi(h_A(L)) = \varphi(h_B(h_{B,A}(L))) = (h_{B,A}(L))(f_B).$$

In particular, if $L = \delta_a$ ($a \in \overline{A}^{v_0 X}$, δ_a the evaluation in a), then

$$f_A(a) = \delta_a(f_A) = (f_{B,A}(\delta_a))(f_B) = \delta_a(f_B) = f_B(a).$$

Hence there is an $f\colon Y \to K$ such that $f|_{\overline{A}^{v_0 X}} = f_A$ for all $A \in \mathcal{P}$; clearly (1) and (2) hold.

THEOREM 30. *Assume K non-spherically complete, K and X nonmeasurable. The following are equivalent*:

(1) *$C(X, K, \mathcal{P})$ is complete.*

(2) *$C(X, K, \mathcal{P})$ is quasi-complete.*

(3) *$C(X, K, \mathcal{P})$ is semi-reflexive.*

(4) *If $f\colon Y_{\overline{\mathcal{P}}} \to K$ is continuous on every $A \in \overline{\mathcal{P}}$, then there is a $g \in C(X, K)$ such that $g = f$ on $Y_{\overline{\mathcal{P}}}$.*

Proof. By the assumptions $v_K X = v_0 X$. The equivalence of (1), (2) and (4) is Theorem 7. Furthermore (3) $\Rightarrow$ (2) by a standard argument (remark that $C(X, K, \mathcal{P})'$ separates the points of the completion of $C(X, K, \mathcal{P})$ by virtue of Theorem 6).

To prove that (4) $\Rightarrow$ (3) assume $\varphi \in (C(X, K, \mathcal{P})_b')'$. Let f be as in Proposition 29. By (4) there is a $g \in C(X, K)$ such that $g(x) = f(x)$ for all $x \in Y_{\overline{\mathcal{P}}}$. Let $\varphi_g\colon C(X, K, \mathcal{P})' \to K$ be defined by $\varphi_g(h_A(L)) = L(g|_A)$ for $A \in \overline{\mathcal{P}}$ and $L \in C(\overline{A}^{v_0 X}, K)'$. Then $\varphi = \varphi_g$ so $C(X, K, \mathcal{P})$ is semi-reflexive.

THEOREM 31. *Assume K non-spherically complete, K and X nonmeasurable. Then $C(X, K, \mathcal{P})$ is reflexive iff both* (a) *and* (b) *hold*:

(a) *$C(X, K, \mathcal{P})$ is semi-reflexive.*

(b) $\tilde{\mathcal{P}}^{Y_{\overline{\mathcal{P}}}} \subset \overline{\mathcal{P}}$.

Proof. If $C(X, K, \mathcal{P})$ is reflexive, then we prove (b). Suppose $A \in \tilde{\mathcal{P}}^{Y_{\overline{\mathcal{P}}}}$ and put $B_A = \{f \in C(X, K)\colon \|f\|_A \leq 1\}$. Let B_A^0 be the polar of B_A in $C(X, K, \mathcal{P})'$ and B_A^{00} the bipolar in $C(X, K, \mathcal{P})$. A routine argument shows that $B_A^{00} = B_A$. If C is any bounded set in $C(X, K, \mathcal{P})$, then, by Proposition 4, there is a $\lambda \in K \backslash \{0\}$ such that $C \subseteq \lambda B_A$, so $B_A^0 \subseteq \lambda C^0$; this proves B_A^0 is bounded in $C(X, K, \mathcal{P})_b'$. By reflexivity, $B_A = B_A^{00}$ is a neighbourhood in $C(X, K, \mathcal{P})$ which implies $A \in \overline{\mathcal{P}}$.

Now assume (a) and (b) and let B be bounded in $C(X, K, \mathcal{P})'_b$. Then B^0 is absolutely convex and absorbs all bounded sets, hence is a neighbourhood in the bornological space associated to $C(X, K, \mathcal{P})$, i.e. in $C(X, K, \tilde{\mathcal{P}})$ (Theorem 15). Let $A \in \tilde{\mathcal{P}}$ and $\lambda \in K \backslash \{0\}$ be such that $\{f \in C(X, K)\colon \|f\|_A \leq |\lambda|\} \subseteq B^0$. Let $A' = A \cap Y_{\overline{\mathcal{P}}}$. If $L \in B$ and $f \in C(X, K)$, $\|f\|_A = 0$, then $L(f) = 0$; hence $A_L \subseteq A$ by Lemma 25. On the other hand $A_L \subseteq Y_{\overline{\mathcal{P}}}$, so $A_L \subseteq A'$.

Let $f \in C(X, K)$ be arbitrary with $\|f\|_{A'} \leq |\lambda|$. Let $g \in C(X, K)$ be such that $\|g\|_A \leq |\lambda|$ and $f = g$ on A'. Then for $L \in B$ we have

$$|L(f)| \leq \max(|L(g)|, |L(f-g)|).$$

Since $f|_{A_L} = g|_{A_L}$, $L(f-g) = 0$, so $|L(f)| \leq |L(g)| \leq 1$. Now $\{f \in C(X, K)\colon \|f\|_{A'} \leq |\lambda|\} \subseteq B^0$ and $A' \in \tilde{\mathcal{P}}^{Y_{\overline{\mathcal{P}}}} \subseteq \overline{\mathcal{P}}$, so B^0 is a neighbourhood in $C(X, K, \mathcal{P})$.

REMARK 32. Two counterexamples prove that (a) and (b) of Theorem 31 are independent.

(1) Let $X = [0, \Omega[$, the first uncountable ordinal, $\mathcal{P} = \mathcal{K}(X)$. By Theorem 7 and local compactness of X, $C_c(X, K)$ is complete. However, $\tilde{\mathcal{P}} = \mathcal{K}([0, \Omega])$ and so $[0, \Omega[\in \tilde{\mathcal{P}}^{Y_{\overline{\mathcal{P}}}} \backslash \overline{\mathcal{P}}$.

(2) Let $X = [0, \Omega]$, $\mathcal{P} = \mathcal{C}(X)$. Then $\tilde{\mathcal{P}}^{Y_{\overline{\mathcal{P}}}} = \tilde{\mathcal{P}} = \mathcal{P} = \overline{\mathcal{P}}$ (Corollary 16). However, X is not discrete and so condition (3) of Theorem 7 is not fulfilled.

6. Reflexivity and semi-reflexivity for spherically complete *K*. In this section we assume $Y_{\overline{\mathcal{P}}} \subseteq v_K X$. A locally convex space over the spherically complete field K is c-Montel if it is a barrelled space in which all absolutely convex, closed, bounded sets are c-compact [**12**, Definition 3.8.b]. From [**3**, Proposition 2] it follows that "c-Montel" is equivalent with "Montel" if K is a local field.

Our main result is the following substantial generalization of [**4**, Theorem III.45] (See also [**13**]).

THEOREM 33. *Let K be spherically complete. The following are equivalent*:

(1) $C(X, K, \mathcal{P})$ *is a c-Montel space.*

(2) $C(X, K, \mathcal{P})$ *is reflexive.*

(3) $C(X, K, \mathcal{P})$ *is semi-reflexive.*

(4) *For every* $f\colon Y_{\overline{\mathcal{P}}} \to K$ *there is a* $g \in C(X, K)$ *such that* $f = g$ *on* $Y_{\overline{\mathcal{P}}}$.

Proof. (1) $\Rightarrow$ (2). See [**12**, Corollaire 1 of Théorème 4.28].

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). First we show that every $A \in \mathfrak{P}$ is finite. If not, then there is a sequence $(x_n)_{n=1}^{\infty}$ in $v_K X$ and $x \in v_K X$ such that $x_n \neq x$ for all n and $x \in \overline{\{x_1, x_2, \ldots\}}^{v_0 X}$. For all n choose $f_n \in C(X, K)$ so that $f_n(x_i) = 1$ if $i \leq n$, $f_n(x_i) = 0$ if $i > n$, and $|f_n(y)| \leq 1$ for all $y \in X$.

Let B_n be the convex hull of $\{f_n, f_{n+1}, \ldots\}$. The set $\{f \in C(X, K)$: $|f(y)| \leq 1$ for all $y \in X\}$ is absolutely convex, weakly bounded and weakly closed in a semi-reflexive space, hence weakly c-compact [**12**, Théorème 4.25(2)].

Let f be a weak adherence point of the convex filter $(B_n)_{n=1}^{\infty}$. Let n be arbitrary. If $i \leq n$ and $g \in B_n$, then $g(x_i) = 1$; hence $f(x_i) = 1$. Since $f(x_i) = 1$ for all i, we have $f(x) = 1$. On the other hand, $g(x) = 0$ for all $n \in N$ and $g \in B_n$; hence $f(x) = 0$, a contradiction.

Since every $A \in \mathfrak{P}$ is finite, $\mathfrak{P}$ is the family of finite subsets of $Y_{\mathfrak{P}}$. Let T be an arbitrary bounded closed subset of $C(X, K, \mathfrak{P})$ and $\mathrm{Co}(T)$ its absolutely convex closed hull. By [**12**, Théorème 4.25, 2°] $\mathrm{Co}(T)$ is weakly c-compact. From [**3**, Proposition 3(a)] and [**8**, §5, Proposition 4] we infer that $\mathrm{Co}(T)$ is c-compact; by [**3**, Proposition 1] $\mathrm{Co}(T)$ is complete. As a closed subset of a complete set, T is complete.

We conclude that $C(X, K, \mathfrak{P})$ is quasi-complete, hence complete. Since every f: $Y_{\overline{\mathfrak{P}}} \to K$ may be pointwisely approximated by functions from $C(v_k X, K)$, (4) follows.

(4) $\Rightarrow$ (1). If (4) holds, then $\mathfrak{P} = \mathfrak{A}(Y_{\mathfrak{P}})$ and $Y_{\mathfrak{P}}$ is discrete so $C(X, K, \mathfrak{P})$ can be identified with $K^{Y_{\mathfrak{P}}}$. Since K is a c-Montel space, the result follows as in [**4**, Theorem III.45].

COROLLARY 34. *If K is spherically complete and $Y_{\overline{\mathfrak{P}}} = X$, then the following are equivalent*:

(1) *$C(X, K, \mathfrak{P})$ is a c-Montel space (reflexive, semi-reflexive).*

(2) *$C_c(X, K)$ is a c-Montel space (reflexive, semi-reflexive).*

(3) *$C_s(X, K)$ is a c-Montel space (reflexive, semi-reflexive).*

(4) *X is discrete.*

REMARK 35. If K is spherically complete, $X = [0, \Omega]$, then $C_c(X, K)$ is complete (Theorem 7) but not semi-reflexive (Theorem 33).

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