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SINGULAR CHARACTERS AND THEIR L^p NORMS ON CLASSICAL LIE GROUPS

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We show that the Weyl's characters formula takes a particular form in the case of representations whose maximal weight is singular. This formula enables us to prove the following statement. Let G be a compact connected Lie group such that the complexification of its Lie algebra is a direct sum of its center with classical Lie algebras; then there exists a sequence $\{\lambda_n\}$ in the dual object $\hat{G}$ such that $d_{\lambda_n} \to \infty$ and $\|\chi_{\lambda_n}\|_p \leq K(p) < \infty$ for all n and for all $p < 3$.

Let G be a compact connected Lie group, with dual object $\hat{G}$. We consider the characters χ_λ of G where $\lambda \in \hat{G}$. It is well known that there exists a constant $p_G \leq 3$ such that $\|\chi_\lambda\|_p$ is uniformly bounded, as λ ranges over $\hat{G}$, if and only if $p < p_G$ (see Clerc [**4**], Dooley [**5**], Giulini, Soardi and Travaglini [**8**]). If G is a compact, simply connected, simple Lie group then $p_G = 2 + l/|P|$ where l is the rank of G and $|P|$ is the cardinality of the set of the positive roots. On the other hand, Giulini, Soardi and Travaglini [**8**] proved that, for general compact connected Lie groups and for all $p \geq 3$:

$$(*) \qquad \|\chi_\lambda\|_p \to \infty \quad \text{as } d_\lambda \text{ increases to infinity.}$$

We wonder whether the latter result is best possible in the following sense: there exists a sequence $\{\lambda_n\}$ in $\hat{G}$ such that $d_{\lambda_n} \to \infty$ and:

$$(**) \qquad \|\chi_{\lambda_n}\|_p \leq K(p) < \infty \quad \text{for all } p < 3 \text{ (and for all } n).$$

In general the answer is negative. In fact, if G is the compact, simply connected, simple Lie group corresponding to the exceptional Lie algebra G_2, then $(*)$ holds for $p > 14/5$ (see [**5**], [**8**]). Nevertheless Rider [**12**] proved that $(**)$ is true in the case $G = \mathrm{SU}_n$.

In this paper we show that the Weyl's characters formula takes a particular form in the case of unitary irreducible representations whose maximal weight is singular. As a consequence we prove that $(*)$ holds for the "classical" compact Lie groups, i.e. for compact connected Lie groups such that $g_{\mathbf{C}}$ is a direct sum of its center with classical Lie algebras A_l, B_l, C_l, D_l. In §5 we describe the role that the estimates for the L^p norms of the characters play in lacunarity and in divergence of Fourier series.

1. Let G be a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and let T denote a maximal torus of G with Lie algebra $\mathfrak{t}$. The complexification $\mathfrak{t}_{\mathbf{C}}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and we denote by Δ the set of the roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. Choose in Δ a system of positive roots P and let $S = \{\alpha_1, \dots, \alpha_l\}$ be the associated simple system. We set $\delta = \frac{1}{2}\Sigma_{\alpha \in P}\, \alpha$.

We transfer the Killing form to a non degenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{t}^*_{\mathbf{C}} \times \mathfrak{t}^*_{\mathbf{C}}$ via the natural isomorphism of $\mathfrak{t}_{\mathbf{C}}$ with its dual space $\mathfrak{t}^*_{\mathbf{C}}$. We can define a partial ordering $\prec$ on $\mathfrak{t}^*_{\mathbf{C}}$ letting $\lambda_1 \prec \lambda_2$ if $\lambda_2 - \lambda_1$ is a sum (possibly empty) of simple roots (see [**14**], p. 314).

An element λ of $\mathfrak{t}^*_{\mathbf{C}}$ is called integral if $\langle \lambda, \alpha_i \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ is an integer for all $i = 1, \dots, l$; it is called dominant if $\langle \lambda, \alpha_i \rangle$ is real and non negative for all $i = 1, \dots, l$. Let L denote the set of all integral elements of $\mathfrak{t}^*_{\mathbf{C}}$, D_P the set of all dominant elements of L and $L(G)$ the set of all integral linear functions on $\mathfrak{t}_{\mathbf{C}}$ which occur as weights of representations of G: it is well known that there is a bijection between $\hat{G}$ and the set $L(G) \cap D_P$ of all dominant weights of G.

We denote by W the Weyl group of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, namely the group generated by the fundamental reflections σ_{α_i} $(i = 1, \dots, l)$ in the hyperplane orthogonal to α_i. If $w \in W$ we denote by $1(w)$ the minimal length of any expression of w as a product of fundamental reflections. The action of W can be transferred on $\mathfrak{t}^*_{\mathbf{C}}$ in a canonical way:

$$w\mu(H) = \mu(w^{-1}H) \quad \text{where} \quad w \in W, \mu \in \mathfrak{t}^*_{\mathbf{C}}, H \in \mathfrak{t}_{\mathbf{C}}.$$

If $w\mu = (-1)^{1(w)}\mu$ for all $w \in W$, then μ is said to be an alternating function. In particular the elementary alternating sum associated to $\mu \in \mathfrak{t}^*_{\mathbf{C}}$:

$$A_\mu(H) = \sum_{w \in W} (-1)^{1(w)} \exp(w\mu(H)) \qquad (H \in \mathfrak{t}_{\mathbf{C}})$$

is obviously an alternating function.

2. We first recall a few facts about certain subgroups of the Weyl group (see e.g. Carter [**2**], p. 27–30). Let λ be a weight of G and $S_\lambda = \{\alpha \in S\colon \alpha \perp \lambda\}$. If λ is singular then $S_\lambda \neq \varnothing$. The subset of Δ spanned by S_λ is still a system of roots and its Weyl group W_λ is the subgroup of W generated by the fundamental reflections σ_α with $\alpha \in S_\lambda$. Obviously an element ϕ of W belongs to W_λ if and only if $\phi(\lambda) = \lambda$. There is a natural way of choosing a system of representatives for the left cosets of W_λ in W: let $D_\lambda = \{w \in W\colon w(\alpha) \in P \text{ for all } \alpha \in S_\lambda\}$, then each $\phi \in W$ has a unique expression $\phi = w \cdot w_\lambda$ where $w \in D_\lambda$ and $w_\lambda \in W_\lambda$.

LEMMA. *If* $w \in D_\lambda$ *then*

$$(1) \qquad |W_\lambda|^{-1} \sum_{\phi \in wW_\lambda} \phi(\delta) + \frac{1}{2} \sum_{\substack{\alpha \in P \\ \alpha \perp w(\lambda)}} \varepsilon_\alpha \alpha \qquad (\varepsilon_\alpha = \pm 1)$$

belongs to L ($|W_\lambda|$ *is the cardinality of* W_λ).

Proof. Let $\phi \in W_\lambda$ and α a positive root not orthogonal to λ, then $\phi(\alpha)$ is a positive root and $\phi(\alpha) \not\perp \lambda$. Moreover there exists an element ψ in W_λ such that $\psi(S_\lambda) \subset -P$. It follows that

$$\phi(\delta) + \phi\psi(\delta) = \sum_{\substack{\alpha \in P \\ \alpha \not\perp \gamma}} \alpha.$$

In general if $\phi \in wW_\lambda$, then

$$\phi(\delta) + \phi\psi(\delta) = \sum_{\substack{\alpha \in P \\ \alpha \not\perp \lambda}} w(\alpha)$$

and (1) becomes:

$$(2) \qquad \frac{1}{2}\left(\sum_{\substack{\alpha \in P \\ \alpha \not\perp \lambda}} w(\alpha) + \sum_{\substack{\alpha \in P \\ \alpha \perp w(\lambda)}} \varepsilon_\alpha \alpha \right) = \frac{1}{2} w\left(\sum_{\alpha \in P} \varepsilon_\alpha \alpha \right)$$

where $\varepsilon_\alpha = +1$ if $\alpha \not\perp \lambda$. Now it is clear that (2) belongs to L. □

Let λ be a singular dominant weight of G. Then we put:

$$Q_\lambda(H) = \sum_{w \in D_\lambda} (-1)^{l(w)} \left(\exp\left(w(\lambda) + |W_\lambda|^{-1} \sum_{\phi \in wW_\lambda} \phi(\delta) \right)(H) \right)$$
$$\times \prod_{\substack{\alpha \in P \\ \alpha \perp w(\lambda)}} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2} \right)$$
$$= \sum_{w \in D_\lambda} (-1)^{l(w)} E_w(H) \qquad (H \in \mathfrak{t}_C).$$

If λ is not singular the set $\{\alpha \in P\colon \alpha \perp w(\lambda)\}$ is empty for all $w \in W$. Nevertheless we can still define Q_λ provided we put

$$\prod_{\alpha \perp w(\lambda)} \left(e^{\alpha/2} - e^{-\alpha/2} \right) = 1.$$

Therefore if λ is not singular $Q_\lambda = A_{\lambda+\delta}$. The following theorem shows that this equality holds in general.

THEOREM. *If λ is a singular dominant weight of G, then:*

$$Q_\lambda(H) = A_{\lambda+\delta}(H)$$

for all $H \in \mathfrak{t}_{\mathbf{C}}$.

Proof. We begin by proving that Q_λ is alternating. Let $\sigma = \sigma_{\alpha_i}$ with $\alpha_i \in S$, then:

$$Q_\lambda = \sum_{w \in D_\lambda} (-1)^{l(w)} \sigma E_w$$

where

$$\sigma E_w = \left(\exp\left(\sigma w(\lambda) + |W_\lambda|^{-1} \sum_{\phi \in w W_\lambda} \sigma\phi(\delta) \right) \right) \prod_{\substack{\alpha \in P \\ \alpha \perp w(\lambda)}} (e^{\sigma\alpha/2} - e^{-\sigma\alpha/2})$$

If we consider a fixed element $w \in D_\lambda$ either $\sigma w(\lambda) = w(\lambda)$ or $\sigma w(\lambda) \neq w(\lambda)$. If the first case happens, σ permutes the elements of the left coset wW_λ. Moreover $\alpha_i \perp w(\lambda)$ and σ transforms α_i in $-\alpha_i$ whereas it permutes the remaining positive roots orthogonal to $w(\alpha)$. Therefore $\sigma E_w = -E_w$ in this case.

If, on the contrary, $\sigma w(\lambda) \neq w(\lambda)$, then σw belongs to D_λ. For, suppose there exists a positive root α, orthogonal to λ, such that $\sigma w(\alpha) \notin P$; then $w(\alpha) = \alpha_i$ and $(\alpha_i, w(\lambda)) = (\alpha, \lambda) = 0$ which is a contradiction. Moreover, if $\alpha \in P$ and $\alpha \perp w(\lambda)$, $\sigma\alpha \in P$. Finally $\sigma E_w = E_{\sigma w}$, but $l(\sigma w) = l(w) \pm 1$ and

$$(-1)^{l(w)} \sigma E_w = -(-1)^{l(\sigma w)} E_{\sigma w}.$$

This proves that Q_λ is alternating.

Now Q_λ belongs to the algebra of all finite linear combinations of exponentials e^γ, with $\gamma \in L$. Then Q_λ is a linear combination of elementary alternating sum:

$$Q_\lambda = \sum_{\mu \in D_P} c_\mu A_\mu.$$

By (2) $\mu = w(\lambda + \frac{1}{2}\Sigma_{\alpha \in P} \varepsilon_\alpha \alpha)$ where $\varepsilon_\alpha = \pm 1$ and $\varepsilon_\alpha = +1$ if $\alpha \not\perp \lambda$. Let ν be the sum of the roots α which appear in the previous expression with $\varepsilon_\alpha = -1$; obviously $\nu = \Sigma_{\alpha \in S_\lambda} m_\alpha \alpha$ where the m_α's are non negative integers.

We suppose that $\nu \neq 0$ and $\mu = w(\lambda + \delta - \nu)$ is dominant for some $w \in D_\lambda$. We have $(\nu, \nu) = \Sigma_{\alpha \in S_\lambda} m_\alpha(\nu, \alpha) > 0$. Hence there exists at least one $\alpha \in S_\lambda$ such that $m_\alpha > 0$ and $(\nu, \alpha) > 0$: thus $\langle \nu, \alpha \rangle$ is a positive integer. Let $\sigma = \sigma_\alpha$, then: $w\sigma w^{-1}(w(\lambda + \delta - \nu)) = w(\lambda + \sigma(\delta - \nu)) = w(\lambda + \delta - \nu) - \langle \delta - \nu, \alpha \rangle w(\alpha)$. Now $w(\alpha) \in P$ and $w(\lambda + \delta - \nu)$ is dominant, therefore $\langle \delta - \nu, \alpha \rangle \geq 0$. But we have also $\langle \delta - \nu, \alpha \rangle = 1 - \langle \nu, \alpha \rangle \leq 0$. Thus $\langle \delta - \nu, \alpha \rangle = 0$ and $\mu = w(\lambda + \delta - \nu)$ is a singular weight: therefore $A_\mu = 0$.

If $\nu = 0$ then $w(\lambda + \delta)$ is dominant if and only if w is the identity of W. It follows that $Q_\lambda = c_{\lambda+\delta} A_{\lambda+\delta}$ and obviously $c_{\lambda+\delta} = 1$. □

3. In this section let G denote a compact simply connected simple Lie group such that its Lie algebra $\mathfrak{g}$ is a compact real form of one of the classical Lie algebras A_l, B_l, C_l, D_l. For the notations we refer to Bourbaki [**1**], Ch. VI.

For $G = \mathrm{SU}_n$ Rider [**12**] proved that if ω_1 is the fundamental weight not orthogonal to α_1, then the sequence $\lambda_n = n\omega_1$ fulfils inequality (**). For general "classical" G there exists a singular dominant weight λ such that (**) holds for the sequence $\lambda_n = n\lambda$; for the proof we must use the clasification theory and a case by case verification (see Bourbaki [**1**], Ch. VI and Table 1).

TABLE 1

$\mathfrak{g}_{\mathbf{C}}$	$\lambda = \omega_1$	R
$A_l\,(l \geq 1)$	$\frac{1}{l+1}\sum_{r=1}^{l}(l-r+1)\alpha_r$	$\sum_{r=1}^{l}\alpha_r$
$B_l\,(l \geq 2)$	$\sum_{r=1}^{l}\alpha_r$	$\beta_j = \sum_{r=1}^{l}\alpha_r + \sum_{r=j+1}^{l}\alpha_r \quad (1 \leq j \leq l)^*$
$C_l\,(l \geq 2)$	$\sum_{r=1}^{l-1}\alpha_r + \frac{1}{2}\alpha_l$	$\beta_j = \sum_{r=1}^{l-1}\alpha_r + \sum_{r=j}^{l}\alpha_r \quad (1 \leq j \leq l)$
$D_l\,(l \geq 3)$	$\sum_{r=1}^{l-2}\alpha_r + \frac{1}{2}(\alpha_{l-1} + \alpha_l)$	$\beta_j = \sum_{r=1}^{l-2}\alpha_r + \sum_{r=j+1}^{l}\alpha_r \quad (1 \leq j \leq l-1)$
*obviously $\sum_{r=j+1}^{l}\alpha_r = 0$ if $j = l$.		

If we consider $\lambda = \omega_1$, the positive roots $\sum_{r=1}^{j}\alpha_r$ $(j = 1, \ldots, l-1)$ are not orthogonal to λ. Let R denote the set of the remaining positive roots not orthogonal to λ: they can be written as $\sum_{r=1}^{l} m_r\alpha_r$ with $m_1 > 0$ and $m_l > 0$. It is easy to check that one can arrange the elements of R in a finite sequence $\{\beta_i\}$ so that $\beta_i \succ \beta_j$ if $i < j$. We put:

$$\text{if } \mathfrak{g}_{\mathbf{C}} = B_l \text{ or } D_l \quad \beta_{ij} = \beta_i - \beta_j \qquad (i<j);$$

$$\text{if } \mathfrak{g}_{\mathbf{C}} = C_l \quad \beta_{ij} = \beta_i - \beta_j \qquad (1<i<j),$$

$$\beta_{1j} = 2\beta_j - \beta_1 = 2\sum_{r=j}^{l-1} \alpha_r + \alpha_l \qquad (1<j<l), \text{ and}$$

$$\beta_{1l} = 2\beta_l - \beta_1 = \alpha_l.$$

In any case $\beta_{ij} \in \Delta$ and the β_{ij}'s are orthogonal to λ and pairwise distinct.

THEOREM. *Let G be a compact simply connected simple Lie group such that $\mathfrak{g}_{\mathbf{C}}$ is a classical Lie algebra. Then there exists a sequence $\{\lambda_n\} \subset \hat{G}$ with $d_{\lambda_n} \to \infty$ and*

$$\|\chi_{\lambda_n}\|_p \leq K(p) < \infty \quad \textit{for all } p < 3.$$

Proof. Of course we suppose $2 < p < 3$. Let $\lambda = \omega_1$ and $\lambda_n = n\lambda$. By Weyl's integration and characters formulas

$$\|\chi_{\lambda_n}\|_p^p = \int_G |\chi_{\lambda_n}(g)|^p \, dg$$

$$= |W|^{-1} \int_T \frac{|A_{\lambda+\delta}(\log t)|^p}{|A_\delta(\log t)|^p} |A_\delta(\log t)|^2 \, dt.$$

Let Q be a fundamental domain of $\mathfrak{t}$ centered at the origin:

$$\|\chi_{\lambda_n}\|_p^p \leq C(p) \int_Q \sum_{w \in D_\lambda} \left| \prod_{\substack{\alpha \in P \\ \alpha \perp w(\lambda)}} \sin \frac{i\alpha(H')}{2} \right|^p |A_\delta(H')|^{2-p} \, dH'$$

$$= C(p) \sum_{w \in D_\lambda} \int_Q \left| \prod_{\substack{\alpha \in P \\ \alpha \perp \lambda}} \sin \frac{i\alpha(w^{-1}H')}{2} \right|^p |A_\delta(H')|^{2-p} \, dH'.$$

Changing the variables $H' = wH$ in each term, we obtain

$$\|\chi_{\lambda_n}\|_p^p \leq C(p) |D_\lambda| \int_Q \left| \prod_{\substack{\alpha \in P \\ \alpha \perp \lambda}} \sin \frac{i\alpha(H)}{2} \right|^p |A_\delta(H)|^{2-p} \, dH.$$

We recall that

$$|A_\delta(H)| = \prod_{\alpha \in P} |e^{\alpha(H)/2} - e^{-\alpha(H)/2}| = 2^{|P|} \prod_{\alpha \in P} \left| \sin \frac{i\alpha(H)}{2} \right|.$$

Then:

$$\|\chi_{\lambda_n}\|_p^p \leq C(p)\int_Q \frac{\prod_{\alpha\in P,\alpha\perp\lambda}\left|\sin\frac{i\alpha(H)}{2}\right|^2}{\prod_{\alpha\in P,\alpha\not\perp\lambda}\left|\sin\frac{i\alpha(H)}{2}\right|^{p-2}}\,dH.$$

Let $\{B_k\}$ be a finite covering of $\bar{Q}$ consisting of open balls centered at suitable points X_k of $\bar{Q}$, such that the following estimates hold:

$$\text{(3')}\qquad \left|\sin\frac{i\alpha(H)}{2}\right| \geq C > 0 \quad \text{for } H\in B_k,\text{ if } \alpha(X_k)\notin 2\pi i\mathbf{Z},$$

$$\text{(3'')}\qquad \left|\sin\frac{i\alpha(H)}{2}\right| \geq C|\alpha(H)-\alpha(X_k)| \quad \text{for } H\in B_k,\ \text{if } \alpha(X_k)\in 2\pi i\mathbf{Z}.$$

Of course $X_k = 0$ for a suitable k. By (3′) and (3″) it follows that:

$$\left|\sin\frac{i\alpha(H)}{2}\right| \geq C|\alpha(H)-\alpha(X_k)| \quad \text{for } H\in B_k \text{ and } \alpha\in P.$$

Let $J_k = \{\alpha\in P\colon\ \alpha(X_k)\in 2\pi i\mathbf{Z}\}$ and $R_k = J_k\cap R = \{\beta_{i_r}\}$ $(r = 1,\ldots,m)$. If $\beta_{i_r}, \beta_{i_s}\in R_k$ and $i_r < i_s$, we denote by M_k the set of the roots $\beta_{i_r i_s} = \tilde{\beta}_{rs}$. Since $M_k\subset J_k$, we have

$$\text{(3''')}\qquad \left|\sin\frac{i\tilde{\beta}_{rs}(H)}{2}\right| \leq \frac{1}{2}|\tilde{\beta}_{rs}(H)-\tilde{\beta}_{rs}(X_k)| \quad \text{for } H\in B_k \text{ and } \tilde{\beta}_{rs}\in M_k.$$

Combining (3′), (3″) and (3‴) we obtain

$$\|\chi_{\lambda_n}\|_p^p \leq C(p)\sum_k\int_{B_k}\frac{1}{\prod_{j=1}^{l-1}|\sum_{r=1}^{j}\alpha_r(H-X_k)|^{p-2}} \times\frac{\prod_{\tilde{\beta}_{rs}\in M_k}|\tilde{\beta}_{rs}(H-X_k)|^2}{\prod_{\beta\in R_k}|\beta(H-X_k)|^{p-2}}\,dH.$$

If we translate the variable at the origin in each integral, we need only to estimate:

$$\text{(4)}\qquad \int_{|H|<1}\frac{1}{\prod_{j=1}^{l-1}|\sum_{r=1}^{j}\alpha_r(H)|^{p-2}}\,\frac{\prod_{\tilde{\beta}_{rs}\in M_k}|\tilde{\beta}_{rs}(H)|^2}{\prod_{\beta\in R_k}|\beta(H)|^{p-2}}\,dH \quad \text{for all } k.$$

As $\mathrm{Sup}_{\beta \in R_k, |H|<1} |\beta(H)| < \infty$, it is easy to prove by induction that:

$$(5)\quad \prod_{\tilde{\beta}_{rs} \in M_k} |\tilde{\beta}_{rs}(H)|^2 = \prod_{r<s} |\beta_{i_r}(H) - \beta_{i_s}(H)|^2$$

$$= \left| \mathrm{Det} \begin{vmatrix} 1 & \cdots & 1 \\ \beta_{i_1}(H) & \cdots & \beta_{i_m}(H) \\ \vdots & & \vdots \\ (\beta_{i_1}(H))^{m-1} & \cdots & (\beta_{i_m}(H))^{m-1} \end{vmatrix} \right|^2$$

$$\leq C\left(\sum_{r=1}^{m} |\beta_{i_1}(H) \cdots \beta_{i_{r-1}}(H)\beta_{i_{r+1}}(H) \cdots \beta_{i_m}(H)| \right)^2$$

$$\leq mC \sum_{r=1}^{m} |\beta_{i_1}(H) \cdots \beta_{i_{r-1}}(H)\beta_{i_{r+1}}(H) \cdots \beta_{i_m}(H)|^2.$$

Note that (5) needs some slight but inessential modifications if $\mathfrak{g}_{\mathbf{C}} = C_l$. After substitution of inequality (5) in the integral (4), it suffices to prove:

$$\int_{|H|<1} \frac{1}{\prod_{j=1}^{l-1} |\sum_{r=1}^{j} \alpha_r(H)|^{p-2} |\beta(H)|^{p-2}} dH < \infty$$

for all $\beta \in R_k$ and for all k. If we put $i\alpha_r(H) = K_r$ $(r = 1,\dots,l)$ and $K = (K_1,\dots,K_l)$, the previous integrals are of the same kind as the following:

(6)

$$\int_{|K|<1} \frac{dK}{|K_1(K_1 + K_2) \cdots (K_1 + \cdots + K_{l-1})(m_1K_1 + \cdots + m_lK_l)|^{p-2}}$$

where $m_l \neq 0$, and (6) is finite if and only if $p < 3$.

Of course if $R_k = \varnothing$ the last factor at the denominator does not exist, hence the result continues to hold true. □

COROLLARY. *The statement of the previous theorem continues to hold for arbitrary "classical" compact connected Lie group.*

Proof. By structure theorem, if G is a compact connected Lie group,

$$G = \left(T \times \prod_{i=1}^{n} G_i \right) / F$$

where T is a torus, G_i are compact simply connected simple Lie groups and F is a discrete subgroup of the center of the product. Further χ_λ is an irreducible character of $T \times \prod_{i=1}^n G_i$ if and only if $\chi_\lambda = \chi_T \prod_{i=1}^n \chi_{\lambda_i}$, where χ_T is a character of T and χ_{λ_i} is an irreducible character of G_i; moreover

$$d_\lambda = \prod_{i=1}^n d_{\lambda_i} \quad \text{and} \quad \|\chi_\lambda\|_p = \prod_{i=1}^n \|\chi_{\lambda_i}\|_p.$$

Finally, by (28.10) of **[10]**, we can identify the annihilator

$$A\left(\left(T \times \prod_{i=1}^n G_i\right)^{\wedge}, F\right)$$

of F in the dual object of the product with $\hat{G}$. We observe that, if G is not simply connected, a weight ω of $\mathfrak{g}_{\mathbf{C}}$ is not necessarily a weight of G; however $k\omega$ is (for a suitable positive integer k). Now the corollary follows by the previous theorem and standard representation theory.

4. If G is a compact connected Lie group:

(7) $$\|\chi_\lambda\|_3 \to \infty \quad \text{as } d_\lambda \to \infty \text{ (see [8])}.$$

Therefore the theorem of §3 shows that this result is the best possible for a large class of groups G.

Nevertheless if G is the compact simply connected simple Lie group corresponding to the exceptional Lie algebra G_2, it is possible to improve (7) by replacing the L^3 norm with the L^p norm with $p > 14/5$ (see **[5]**, **[8]**).

In this case it is very easy to exhibit a sequence $\{\lambda_n\} \subset \hat{G}$ such that $d_{\lambda_n} \to \infty$ and $\|\chi_{\lambda_n}\|_p \leq K(p) < \infty$ for all $p < 14/5$. Imitating the proof of the previous theorem we choose $\lambda_n = n\omega_1$ where $\omega_1 = 2\alpha_1 + \alpha_2$ and we obtain:

$$\|\chi_{\lambda_n}\|_p^p \leq C\int_Q \frac{|\sin\frac{1}{2}i\alpha_2(H)|^2}{\left|\prod_{\alpha\in P,\alpha\not\perp\omega_1}\sin\frac{i\alpha(H)}{2}\right|^{p-2}}dH.$$

It is not difficult to see that for this integral the "most" singular point of Q is the origin. Let $i\alpha_1(H) = x$, $i\alpha_2(H) = y$; then $\|\chi_{\lambda_n}\|_p \leq K(p) < \infty$ if the following integral converges:

$$\int_{\substack{|x|<1\\|y|<1}} \frac{y^2}{|x(x+y)(2x+y)(3x+y)(3x+2y)|^{p-2}}dx\,dy.$$

By Fubini and by a change of variable the previous integral turns out to be:

$$(8) \qquad 2\int_0^1 x^{13-5p}\,dx \int_{-1/x}^{1/x} \frac{t^2}{|(1+t)(2+t)(3+t)(3+2t)|^{p-2}}\,dt$$

and if $11/4 < p < 3$, (8) is less than $\int_0^1 x^{13-5p}\,dx$ (up to a multiplicative constant) and the last integral converges if and only if $p < 14/5$.

If G is a compact simply connected simple Lie group such that $\mathfrak{g}_{\mathbf{C}}$ is an exceptional Lie algebra different from G_2, it seems reasonable that the statement of §3 holds, but the technique of the proof fails in this case.

5. Results about norms of characters may also be seen as lacunarity results.

DEFINITION. Let $p > 1$. A set $E \subset \hat{G}$ is called a $\Lambda(p)$ set if there exists a constant k such that

$$(9) \qquad \|f\|_p < k\|f\|_1$$

for every trigonometric polynomial f spectral on E.

If (9) holds for every central trigonometric polynomial f spectral on E, then E is called a central $\Lambda(p)$ set. If (9) holds for every character $f = \chi_\lambda$ $(\lambda \in E)$, then E is called a local central $\Lambda(p)$ set.

It is well known that a compact connected semisimple Lie group does not admit infinite $\Lambda(p)$ set for any p ([**9**]). In the central case Cecchini [**3**] showed, for compact Lie groups, that $\operatorname{Sup}_{\lambda \in E} d_\lambda < +\infty$ is a necessary (and sufficient) condition for $E \subset \hat{G}$ to be a local central $\Lambda(4)$ set. Price [**11**] proved that every subset of $\hat{G}$ is a local central $\Lambda(2)$ set. Actually the results of Clerc [**4**], Dooley [**5**], Giulini, Soardi and Travaglini [**8**] can be written as follows. If G is a compact connected Lie group, (a) $\hat{G}$ is a local central $\Lambda(p)$ set if and only if $p < p_G$; (b) every infinite subset of $\hat{G}$ is not a local $\Lambda(3)$ set.

In the same way, our theorem asserts that if G is a "classical" compact connected Lie group and $p < 3$, then there exists in $\hat{G}$ an infinite local central $\Lambda(p)$ set.

Moreover, techniques like Rider's ones [**12**] allow to prove:

(a′) suppose G is a compact connected Lie group and let $p < p_G$; then every infinite subset of $\hat{G}$ contains an infinite central $\Lambda(p)$ set [**6**];

(b′) suppose G is a "classical" compact connected Lie group and let $p < 3$; then there exists in $\hat{G}$ an infinite central $\Lambda(p)$ set (this is a particular case of the Lemma 3 in [7]).

Estimates of norms of characters play a relevant role also in the divergence of Fourier series.

Let $\{\chi_{\lambda_j}\}_{j=1}^{\infty}$ be a sequence of distinct characters of G. Using symmetric sums (see [8], Proposition 1), by (7) one can obtain

$$\left|\left|\left| \sum_{j=1}^{N} d_{\lambda_j} \chi_{\lambda_j} \right|\right|\right|_3^3 \geq K \log N \tag{10}$$

where K is a constant depending only on G, and $|||\, g \,|||_3$ denotes the norm of the L^3-convolutor associated to g.

Suppose $\{\Sigma_N\}_{N=1}^{\infty}$ is an increasing sequence of finite subsets of $\hat{G}$ such that $\bigcup_{N=1}^{\infty} \Sigma_N = \hat{G}$. We define the partial sums relative to $\{\Sigma_N\}_N$ by

$$S_N f = \sum_{\lambda \in \Sigma_N} d_\lambda \chi_\lambda * f \qquad \forall f \in L^1(G).$$

It follows easily by (10) that for $p \geq 3$ (or $p \leq 3/2$) and for every choice of $\{\Sigma_N\}_N$ there exists a central $f \in L^p(G)$ such that

$$\mathrm{Sup}_N \|S_N f\|_p = +\infty.$$

The Theorem of this paper shows that the symmetric sums technique, above described, does not allow to improve the previous divergence result.

The interested reader can find convergence results for instance in [13].

REFERENCES

[1] N. Bourbaki, *Groupes et Algèbres de Lie*, Ch. 4,5 et 6, Hermann, 1968.

[2] R. W. Carter, *Simple Groups of Lie Type*, John Wiley & Sons, 1972.

[3] C. Cecchini, *Lacunary Fourier series on compact Lie groups*, J. Functional Analysis, **11** (1972), 191–203.

[4] J. L. Clerc, *Localisation des sommes de Riesz sur un groupe de Lie compact*, Studia Math., t.**LV** (1976), 21–26.

[5] A. H. Dooley, *Norms of characters and lacunarity for compact Lie groups*, J. Functional Analysis **32** (1979), 254–267.

[6] ______, *Central lacunary sets for Lie groups*, (preprint).

[7] S. Giulini, *Cohen type inequalities and divergence of Fourier series on compact Lie groups*, (preprint).

[8] S. Giulini, P. M. Soardi and G. Travaglini, *Norms of characters and Fourier series on compact Lie groups*, J. Functional Analysis, **46** (1982), 88–101.

[9] S. Giulini and G. Travaglini, *L^p estimates for matrix coefficients of irreducible representations of compact groups*, Proc. Amer. Math. Soc., **80** (1980), 448–450.

[10] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. II, Springer-Verlag, 1970.

[11] J. F. Price, *On local central lacunary sets for compact Lie groups*, Monatsh. Math., **80** (1975), 201–204.

[12] D. Rider, *Norms of characters and central Λ_p sets for $U(n)$*, Conference on Harmonic Analysis, Lecture Notes in Math. 266, Springer-Verlag, 1971.
[13] R. J. Stanton and P. A. Tomas, *Polyhedral summability of Fourier series on compact Lie groups*, Amer. J. Math., **100** (1978), 477–493.
[14] V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Prentice Hall, 1974.

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