CIRCLE ACTIONS ON HOMOTOPY SPHERES WITH CODIMENSION 4 FIXED POINT SET

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In this paper we give a complete equivariant classification of smooth S^1 actions on homotopy spheres with codimension 4 fixed point set and point out a relationship with a natural generalization of the twist-spinning process for knots.

Semifree S^1 actions on homotopy spheres with codimension 4 fixed point set have been classified by J. Levine; so we concentrate on actions with exceptional orbits. There are some obvious linear models for these actions. Let ξ be the standard generator of the complex representation ring of S^1. Then in some sense the actions with one exceptional orbit type are modeled after $\xi^k \oplus \xi \oplus 0$ and those with two exceptional orbit types are modeled after $\xi^k \oplus \xi^m \oplus 0$. Let $\mathbb{S}_k^n$ denote the set of diffeomorphism classes of pairs (Σ_k^{n-1}, Δ) where Σ^{n-1} is a homotopy $(n-1)$-sphere and Δ_k is a smooth Z_k-acyclic orientable codimension 2 submanifold with boundary an integral homology sphere. Similarly, for relatively prime integers k and m, let $\mathbb{S}_{k,m}^n$ denote the set of triads $(\Sigma^{n-1}; \Delta_k, \Delta_m)$ where Δ_k and Δ_m are respectively Z_k and Z_m-acyclic orientable codimension 2 smooth submanifolds meeting tangentially such that $\partial\Delta_k = \partial\Delta_m = \Delta_k \cap \Delta_m$ is an integral homology sphere. In these two cases the classification theorem states that actions on homotopy n-spheres with one exceptional orbit type Z_k, or two exceptional orbit types Z_k and Z_m are in 1-1 correspondence with $\mathbb{S}_k^n$ and $\mathbb{S}_{k,m}^n$. These 1-1 correspondences are realized by associating with an S^1 action on a homotopy n-sphere its structured orbit space and viewing Δ_k and Δ_m as the images in the orbit space of the fixed point sets of Z_k and Z_m.

That these two types of actions do not comprise all S^1 actions on (homotopy) spheres with codimension 4 fixed point set was shown by E. V. Stein in answer to a question of Frank Raymond. It turns out that all these other actions correspond in a 1-1 fashion via their structured orbit spaces to the set $\mathbb{T}_{k,m}^n$ of diffeomorphism classes of triads $(\Sigma^{n-1}, \Delta_k, \Delta_m)$ as in the definition of $\mathbb{S}_{k,m}^n$, except that the interiors of Δ_k and Δ_m intersect transversely in a (perhaps disconnected) $n-5$ manifold without boundary. This intersection manifold corresponds to the image in the orbit space of the exceptional orbits of type Z_{km}. The intersecting aspect

of these examples is that they have more orbit types than their slice representation at the fixed point set. Our description gives rise to an easy construction of all these actions.

Our classification of S^1 actions is carried out in terms of weak equivalence. (Recall that a *weak equivalence* of smooth S^1-manifolds M_1 and M_2 is a diffeomorphism $f: M_1 \to M_2$ satisfying $f(tx) = a(t)f(x)$ for $t \in S^1$, $x \in M_1$ where a is an automorphism of S^1.) Since $t \to t^{-1}$ is the only nontrivial automorphism of S^1, a weak equivalence of S^1-manifolds is just an equivariant diffeomorphism up to a change of direction of the action in one of the manifolds. This can be avoided by carrying along a specified orientation on the normal bundle in Σ^{n-1} of $\partial \Delta_k$ (the fixed point set image), see [L]; however, we have opted for the simplicity that goes along with classification up to weak equivalence.

In §4 we use S^1 actions on homotopy spheres to introduce a class of knots which generalize the twist-spin knots of Zeeman [Z]. It is then a corollary of the classification theorem for S^1 actions that the knot complements of knots in this class fiber over the circle with fiber a punctured cyclic branched cover of the original knot. A special case of this is Zeeman's main theorem [Z]. Furthermore, this class of knots gives rise to infinitely many counterexamples to the n-dimensional Smith Conjecture.

The question of which homotopy spheres admit S^1 actions with codimension 4 fixed point set has been answered separately by J. Levine and R. Schultz in the semifree case. Recently Schultz has shown that any homotopy sphere which admits an S^1 action with codimension 4 fixed point set must also admit one which is semifree, and in fact he gives much more precise information [S]. We wish to thank Reinhard Schultz for explaining his results to us and for encouraging the publication of our results. We also wish to thank Allan Edmonds for his excellent advice which has led to the restructing of our original format.

1. Orbit space and orbit structure. In this section we describe the general features of the orbit structure of a smooth S^1-action on a homotopy n-sphere M^n whose fixed point set has codimension 4. We identify the circle group S^1 with the group of complex numbers of unit modulus. If S^1 acts on M and X is a subset of M then X^* denotes its image in the orbit space M^* and $p: M \to M^*$ denotes the orbit map. Given a subset Y^* of M^* we let $Y = p^{-1}(Y^*)$ when this causes no confusion. Let F denote the fixed point set of M and E the union of the exceptional orbits (those with finite nontrivial isotropy group). The union of the exceptional orbits of type Z_k will be called E_k, and F_k will denote the fixed point set of Z_k (so $E_k \subset F_k$).

Let S^1 act smoothly on a homotopy n-sphere M^n ($n \geq 5$), so F is an integral homology sphere [**B**]. If F is codimension 4 then the S^1-action on a slice D^n at a point of F must be equivalent to

$$S^1 \times D^2 \times D^2 \times D^{n-4} \to D^2 \times D^2 \times D^{n-4},$$

$$t \times (z_1, z_2, w) \to (z_1 t^k, z_2 t^m, w),$$

where k and m are relatively prime positive integers. Since F is connected this slice representation is the same at each point of F.

PROPOSITION 1.1. *If* $k \geq 2$ (*resp.* $m \geq 2$) *then* F_k (*resp.* F_m) *is a smooth orientable* Z_k (*resp.* Z_m)-*homology* $n - 2$ *sphere.*

Proof. Suppose $k \geq 2$. It is easily seen that F_k is a smooth submanifold of M^n, and if k is not a 2-power F_k is orientable since it is the fixed point set of a Z_k-action ([**B**, IV.2.1]). For any prime p dividing k we have $F_k \subset F_p$; so F_p is a Z_p-homology $n - 2$ sphere. The connected component of F_k which contains F is a closed $n - 2$ submanifold of F_p; hence $F_k = F_p$. It now follows from the universal coefficient theorem that F_k is a Z_k-homology sphere. If k is a 2-power it follows that F_k is a Z_2-homology sphere and so is orientable.   □

PROPOSITION 1.2. *There are the following possibilities for the appearance of exceptional orbit types:*
   (i) *no exceptional orbit types* (*semifree action*),
   (ii) *one exceptional orbit type,*
   (iii) Z_k *and* Z_m-*orbits,*
   (iv) Z_k, Z_m, *and* Z_{km}-*orbits,*
where in (iii) *and* (iv) k *and* $m \geq 2$ *and are relatively prime.*

Proof. Let $t \times (z_1, z_2, w) \to (z_1 t^k, z_2 t^m, w)$ be the slice representation at a point of F. If p is any prime then F_{p^s} is a Z_p-homology sphere containing F. Thus if $k = m = 1$ then (i) holds. If $k > 1$, $m = 1$ and Z_r is an isotropy type, $r > 1$, then each prime p dividing r also divides k. Let p^s be the highest power of p dividing r, then F_{p^s} is a Z_p homology $n - 2$ sphere; so $F_p = F_{p^s} = F_k$. Suppose $r = p_1^{s_1} \cdots p_u^{s_u}$. Then $F_r = F_{p_1^{s_1}} \cap \cdots \cap F_{p_u^{s_u}} = F_k$. So $r = k$ since Z_r and Z_k are isotropy types.

If $k > 1$ and $m > 1$ and if Z_r is an isotropy type with $r = p_1^{s_1} \cdots p_u^{s_u} > 1$, the above argument shows that each $F_{p_i^{s_i}} = F_k$ or F_m. Thus $F_r = F_k$, F_m, or $F_k \cap F_m = F_{km}$. So $r = k$, m, or km.   □

Note that the submanifolds F_k and F_m meet transversely.

That M^* is a topological manifold follows from slice considerations. We now describe how it can be smoothed. For details see the forthcoming paper of R. Schultz [S]. Consider first the semifree case which is presented in a paper of J. Levine [L]. Let $N(F)$ be an equivariant tubular neighborhood of F and note that $N^*(F^*)$ is the total space of a D^3-bundle over F^*. Since $p \mid F$ is a homeomorphism F^* has an induced smooth structure and $N^*(F^*)$ thus inherits a smooth structure. Since the S^1-action on $M^n - F$ is free, $M^* - F^*$ also inherits a smooth structure. These structures agree on the overlap and so make M^* a smooth manifold.

Consider next the case where the action has one exceptional orbit type Z_k. Then as above there is a smooth structure induced on M/Z_k by the semifree Z_k-action and it can be seen that the induced semifree $S^1/Z_k \cong S^1$ action on M/Z_k is smooth [S]. Hence $M^* = (M/Z_k)/(S^1/Z_k)$ is a smooth manifold.

In case there are two or three exceptional orbit types we use the diagram:

$$
\begin{array}{ccc}
M & \to & M/Z_k \\
\downarrow & & \downarrow \\
M/Z_m & \to & M/Z_{km} \\
& & \searrow \\
& & \qquad M^*
\end{array}
$$

where each map is the orbit map of a semifree action. The diagram can be used in two different ways to make M^* a smooth manifold; but it can be seen that the two smooth structures agree.

The existence of these smooth structures has been known for some time, and the following summary of their properties appears in [S].

PROPOSITION 1.3. *Let M^* have the smooth manifold structure described above. Then:*

(a) *$p\colon M \to M^*$ is a smooth map.*

(b) *$p \mid F$ is a diffeomorphism and $F \to M^*$ is a smooth embedding.*

(c) *If $k > 1$ $(m > 1)$ then F_k^* (F_m^*) is smoothly embedded in M^* extending the embedding of F^*.*

(d) *If the slice representation at F is equivalent to*

$$
S^1 \times \mathbf{C} \times \mathbf{C} \times R^{n-4} \to \mathbf{C} \times \mathbf{C} \times R^{n-4},
$$

$$
t \times (z_1, z_2, w) \to \left(z_1 t^k, z_2 t^m, w\right)
$$

then:

(i) *at F, p is smoothly equivalent to*

$$\mathbf{C} \times \mathbf{C} \times R^{n-4} \to \mathbf{C} \times R \times R^{n-4},$$

$$\left(z_1, z_2, w\right) \to \left(z_1^k z_2^{-m}, |z_1|^{2k} - |z_2|^{2m}, w\right).$$

(ii) *at E_k, p is smoothly equivalent to*

$$\mathbf{C} \times R^{n-3} \times R \to \mathbf{C} \times R^{n-3},$$

$$\left(z, w, s\right) \to \left(z^k, w\right),$$

and a similar statement holds for E_m.

(iii) *at E_{km} ($k, m > 1$), p is smoothly equivalent to*

$$\mathbf{C} \times \mathbf{C} \times R^{n-5} \times R \to \mathbf{C} \times \mathbf{C} \times R^{n-5},$$

$$\left(z_1, z_2, w, s\right) \to \left(z_1^k, z_2^m, w\right).$$

Furthermore, any two smooth manifold structures $\mathcal{A}$ and $\mathcal{B}$ on M^* which satisfy (a)–(d) are equivalent in the sense that there is a diffeomorphism $(M^*, \mathcal{A}) \to (M^*, \mathcal{B})$ which is topologically isotopic to the identity. □

PROPOSITION 1.4. (a) *M^* is a homotopy $n - 1$ sphere.*

(b) *F_k^*, F_m^*, F_{km}^*, F^* (and $F_k^* \cup F_m^*$ in case (iii) of Prop. 1.2) are orientable smooth submanifolds of M^*, and $\partial F_k^* = \partial F_m^* = F^*$.*

(c) *F^* is an integral homology S^{n-4}; F_k^* (resp. F_m^*) is Z_k-acyclic (resp. Z_m-acyclic). (So in case (iii) of Prop. 1.2, $F_k^* \cup F_m^*$ is a rational homology S^{n-3}.)*

(d) *In case (iv) of Prop. 1.2 E_{km}^* is an orientable closed $n - 5$ submanifold of M^* which is the transverse intersection of E_k^* and E_m^*.*

Proof. Part (a) follows from [CF]. Except for the orientability (b) follows from slice considerations and Prop. 1.3, and (c) follows from Smith theory and Prop. 1.1. To see that F_k^* is orientable, let γ^* be any loop in Int $F_k^* = F_k^* - F^*$. Since E_{km}^* is codimension 2 in F_k^* an isotopy moves γ^* into E_k^*. For a tubular neighborhood N^* of γ^* we have $p^{-1}(N^*) = N \cong N^* \times S^1$. But N is orientable since F_k is orientable, thus N^* is orientable and F_k^* is also. To prove (d) note that the Z_{km}-action on a slice $D^2 \times D^2 \times D^{n-5}$ at a point of E_{km} must be

$$e^{2\pi i/km} \times \left(z_1, z_2, w\right) \to \left(z_1 e^{2\pi i/m}, z_2 e^{2\pi i/k}, w\right).$$

All of (d) follows from this except for the statement about orientability. Since k and m are relatively prime, we may suppose that m is odd. Now E_{km} is a component of the fixed point set of the induced Z_m action on the orientable manifold F_k. Thus E_{km} is orientable ([**B**, IV.2.1]), and the argument given above to show F_k^* is orientable also shows tht E_{km}^* is orientable.   □

2. Actions with one exceptional orbit type. Consider a smooth S^1 action on a homotopy n-sphere M^n ($n \geq 5$) with codimension 4 fixed point set and one exceptional orbit type Z_k. Then F_k^* is a smooth codimension 2 submanifold of M^* with boundary F^*, where M^* is a homotopy $n - 1$ sphere, F_k^* is an orientable Z_k-homology $n - 3$ disk and F^* is an integral homology $n - 4$ sphere. Call the pair (M_k^*, F^*) the *structured orbit space* of the action. In view of the above facts we define the set of potential structured orbit spaces $\mathbb{S}_k^n$ to be the set of diffeomorphism classes of pairs (Σ^{n-1}, Δ_k), where Δ_k is a smooth orientable $n - 3$ dimensional Z_k-acyclic submanifold of the homotopy $n - 1$ sphere Σ^{n-1} and $\partial \Delta_k$ is an integral homology $n - 4$ sphere.

For a smooth S^1 action on a homotopy n-sphere M^n ($n \geq 5$) with codimension 4 fixed point set and one exceptional orbit type Z_k, the structured orbit space $(M^*, F_k^*) \in \mathbb{S}_k^n$. We shall show in this section that for each $\sigma \in \mathbb{S}_k^n$ there is up to weak equivalence a unique action $M(\sigma)$ of S^1 on a homotopy n-sphere with structured orbit space σ.

Given $\sigma = (\Sigma^{n-1}, \Delta_k) \in \mathbb{S}_k^n$ we now proceed with the construction of $M(\sigma)$. According to [**L**] if we fix an orientation of the normal bundle of $\partial \Delta_k$ in Σ^{n-1} we determine up to equivariant diffeomorphism a semifree S^1 action on a homotopy n-sphere N^n with fixed point set $\overline{F} \cong \partial \Delta_k$ and smooth orbit map $\mu \colon N^n \to \Sigma^{n-1}$ with $\mu(\overline{F}) = \partial \Delta_k$. Let $\overline{F}_k = \mu^{-1}(\Delta_k)$; so $\overline{F}_k$ is a smooth codimension 2 submanifold of N^n.

Claim. $\overline{F}_k$ is an orientable Z_k-homology sphere.

Proof. $\mu|\overline{F}_k \to \Delta_k$ is the orbit map of a semifree S^1 action with codimension 2 fixed point set. It follows from [**B**, V.10.1] that $\overline{F}_k$ is *homeo*morphic to $\Delta \times S^1/(x, t) \sim (x, t')$ for $t, t' \in S^1$. I.e. $\overline{F}_k$ is homeomorphic to $\Delta_k \times S^1 \cup \partial \Delta_k \times D^2 \cong \partial(\Delta_k \times D^2)$; so $\overline{F}_k$ is orientable since Δ_k is orientable, and duality implies that $\overline{F}_k$ is a Z_k-homology sphere.   □

Since $H_1(N^n - \overline{F}_k; Z_k) \approx H^{n-2}(\overline{F}_k; Z_k) \approx Z_k$ there is a k-fold cyclic branched cover M^n of N^n branched over $\overline{F}_k$. According to [**DK**] M^n has a unique smooth manifold structure such that $\nu \colon M^n \to N^n$ is the smooth

orbit map of a smooth semifree Z_k action, and such that the smooth structure on N^n which is imposed via the semifree action (as in §1) is just the original smooth structure on N^n.

Claim. The S^1 action on N^n lifts to a smooth S^1 action on M^n containing the Z_k action.

Proof. By [**B**, I.9] there is a unique S^1 action on $M - \nu^{-1}(\overline{F}_k)$ covering the S^1 action on $N - \overline{F}_k$; i.e. $\nu(t \cdot x) = t^k \cdot \nu(x)$ for all $x \in M - \nu^{-1}(\overline{F}_k)$ and $t \in S^1$. This action is smooth because the smooth structure of $M - \nu^{-1}(\overline{F}_k)$ is lifted from $N - \overline{F}_k$ via the cover. The normal bundle of $\overline{F}_k$ in N^n is trivial [**MS**, §11]; thus it follows from [**B**, VI.11.1] that the S^1 action on an equivariant normal bundle of $\overline{F}_k$ in N is equivalent to

$$S^1 \times \overline{F}_k \times D^2 \to \overline{F}_k \times D^2,$$

$$t \times (x, z) \to (t \cdot x, zt^u)$$

for some integer u, where $t \cdot x$ denotes the action of S^1 on $\overline{F}_k$. Now M is diffeomorphic to the union of $M - \nu^{-1}(\overline{F}_k)$ with the tubular neighborhood T of $\nu^{-1}(\overline{F}_k)$ in M, and T can be identified with $\overline{F}_k \times D^2$ (see [**DK**; p. 160]). Under this identification $\nu \mid \overline{F}_k \times D^2$ is $\nu(x, z) = (x, z^k)$. Define the S^1 action on $T \cong \overline{F}_k \times D^2$ by $t \times (x, z) \to (t^k \cdot x, zt^u)$. This covers the S^1 action on $\overline{F}_k \times D^2 \subset N$; so by uniqueness of covering action on $T - \nu^{-1}(\overline{F}_k)$ the actions on T and $M - \nu^{-1}(\overline{F}_k)$ patch together to give a smooth S^1 action on M containing the given Z_k action. $\square$

Claim. $\pi_1(M) = 0$.

Proof. By Van Kampen's theorem $\pi_1(M)$ is the free product of $\pi_1(M - \nu^{-1}(\overline{F}_k))$ and $\pi_1(T)$ amalgamated over $\pi_1(T - \nu^{-1}(\overline{F}_k))$. Now $M - \nu^{-1}(\overline{F}_k) = (\mu\nu)^{-1}(\Sigma^{n-1} - \Delta_k)$. But Δ_k is codimension 2 in Σ^{n-1} and $\partial\Delta_k \neq 0$; so the usual piping argument shows that $\pi_1(\Sigma^{n-1} - \Delta_k) = 0$. Hence the exact sequence

$$\pi_1(S^1) \to \pi_1\big((\mu\nu)^{-1}\big(\Sigma^{n-1} - \Delta_k\big)\big) \to \pi_1\big(\Sigma^{n-1} - \Delta_k\big) = 0$$

shows that $\pi_1(M - \nu^{-1}(\overline{F}_k))$ is generated by a circle orbit. Since the S^1 action on M has a nontrivial fixed point set $\pi_1(M - \nu^{-1}(\overline{F}_k)) \to \pi_1(M)$ is the 0-map. Also, since $\partial T \to \nu^{-1}(\overline{F}_k)$ is an S^1 bundle projection, the homomorphism $\pi_1(T - \nu^{-1}(\overline{F}_k)) = \pi_1(\partial T) \to \pi_1(\nu^{-1}(\overline{F}_k)) = \pi_1(T)$ is onto. It follows that $\pi_1(M) = 0$. $\square$

Now for $R = Q$ or Z_p, p prime not dividing k, we have

$$H_*(M; R) \approx H_*(M; R)^{Z_k} \approx H_*(N; R) \approx H_*(S^n; R)$$

where the first isomorphism is due to the fact that Z_k action is contained in the S^1 action and the second isomorphism follows from [**B**, III.2.4] since p is prime to k.

Claim. $H_*(M^n; Z_k) \approx H_*(S^n; Z_k)$.

Proof. Since Δ_k is Z_k acyclic, Alexander duality implies that $\Sigma^{n-1} - \Delta_k$ is Z_k acyclic. But $M - \nu^{-1}(\bar{F}_k) = (\mu\nu)^{-1}(\Sigma^{n-1} - \Delta_k)$ is a circle bundle over $\Sigma^{n-1} - \Delta_k$; hence $M - \nu^{-1}(\bar{F}_k)$ has the Z_k-homology of a circle. We have seen above that the tubular neighborhood T of $\nu^{-1}(\bar{F}_k) = F_k$ is $T \cong F_k \times D^2$, and $F_k \cong \bar{F}_k$ is a Z_k homology $n - 2$ sphere. The claim now follows from a simple Mayer-Vietoris argument. $\square$

It now follows from the universal coefficient theorem that $H_*(M, Z) \approx H_*(S^n; Z)$; so M is a homotopy n sphere. Letting $M(\sigma) = M$ we have proved the realization theorem:

THEOREM 2.1. Given $\sigma \in \mathcal{S}_k^n$ there is a smooth S^1 action on a homotopy n-sphere $M(\sigma)$ with structured orbit space σ. $\square$

In order to prove that this action is unique we need the next lemma.

LEMMA 2. Let X and Y be S^1 spaces and consider the actions of Z_k embedded in the S^1 actions. Suppose we have the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi_X \downarrow & & \downarrow \pi_Y \\
X/Z_k & \xrightarrow{g} & Y/Z_k
\end{array}
$$

where f is Z_k-equivariant and g is S^1-equivariant with respect to the induced actions. Then f is S^1-equivariant.

Proof. Let $t \in S^1$ and $x \in X$. Then

$$\pi_Y(f(t \cdot x)) = g(\pi_X(t \cdot x)) = g(t \cdot \pi_X(x)) = t \cdot g(\pi_X(x))$$
$$= t \cdot \pi_Y(f(x)) = \pi_Y(t \cdot f(x)).$$

Thus there is an $s \in Z_k$ such that $t \cdot f(x) = s \cdot f(t \cdot x)$. But $s = 1$ when $t = 1$, so a continuity-connectedness argument implies that s is always 1; so f is S^1-equivariant.   □

THEOREM 2.3. *Let M_1^n be a smooth S^1-manifold with codimension 4 fixed point set and one exceptional orbit type Z_k. Suppose that M_1^n has structured orbit space $\sigma \in \mathbb{S}_k^n$. Then M_1^n is weakly equivalent to $M(\sigma)$; in particular M_1^n is a homotopy sphere.*

Proof. Let f^* be a diffeomorphism of pairs $(M_1^*, F_k^*(M_1)) \to (\Sigma^{n-1}, \Delta_k) = \sigma$. As in §1, M_1/Z_k has an inherited smooth structure such that M_1 is the smooth k-fold cyclic branched cover of M_1/Z_k branched over $\bar{F}_k(M_1)$. The induced S^1/Z_k action on M_1/Z_k is semifree and (after if necessary a change in the orientation of the S^1 action on M_1) Levine's theorem [L] yields an S^1-equivariant diffeomorphism $\bar{f}$

$$
\begin{array}{ccc}
M_1/Z_k & \overset{\bar{f}}{\to} & N \\
\mu_1 \downarrow & & \downarrow \mu \\
M_1^* & \overset{f^*}{\to} & \Sigma^{n-1}
\end{array}
$$

Note that

$$\bar{f}\left(\bar{F}_k(M_1)\right) = \mu^{-1} f^* \mu_1\left(\bar{F}_k(M_1)\right) = \mu^{-1} f^*\left(F_k^*(M_1)\right) = \mu^{-1}(\Delta_k) = \bar{F}_k.$$

By the uniqueness of smooth cyclic branched covers [DK] there is a Z_k-equivariant diffeomorphism f:

$$
\begin{array}{ccc}
M_1 & \overset{f}{\to} & M(\sigma) \\
\downarrow & & \downarrow \\
M_1/Z_k & \overset{\bar{f}}{\to} & N
\end{array}
$$

and f is S^1-equivariant by Lemma 2.2.   □

3. Actions with 2 or 3 exceptional orbit types.

The classification theorem of the previous section is the main tool used in studying smooth S^1 actions on homotopy n-spheres with codimension 4 fixed point set and two or three exceptional orbit types. For $n \geq 5$ and k, m relatively prime positive integers define $\mathbb{S}_{k,m}^n$ to be the set of diffeomorphism classes of triads $(\Sigma^{-1}; \Delta_k, \Delta_m)$ where Σ^{n-1} is a homotopy sphere, Δ_k is a smooth orientable Z_k-acyclic $n - 3$ submanifold, Δ_m is a smooth orientable Z_m

acyclic $n - 3$ submanifold, $\Delta_k \cap \Delta_m = \partial\Delta_k = \partial\Delta_m$ is an integral homology $n - 4$ sphere, and $\Delta_k \cup \Delta_m$ is a smooth submanifold of Σ^{n-1}.

Define $\mathfrak{T}_{k,m}^n$ to be the set of diffeomorphism classes of triads $(\Sigma^{n-1}, \Delta_k, \Delta_m)$ as above except that Int Δ_k and Int Δ_m meet transversely in Q a (perhaps disconnected) $n - 5$ dimensional smooth submanifold, and $\Delta_k \cup \Delta_m - Q$ is a smooth submanifold of Σ^{n-1}.

If S^1 acts smoothly on a homotopy n-sphere M^n with codimension 4 fixed point set and two or three exceptional orbit types we call $(M^*; F_k^*, F_m^*)$ the *structured orbit space* of the action. If $n \geq 5$ it follows from §1 that $(M^*; F_k^*, F_m^*) \in \mathfrak{S}_{k,m}^n$ or $\mathfrak{T}_{k,m}^n$.

THEOREM 3.1. *Suppose that the S^1-manifolds M_1 and M_2 have the same structured orbit space $\sigma \in \mathfrak{S}_{k,m}^n$ or $\mathfrak{T}_{k,m}^n$. Then M_1 and M_2 are weakly equivalent.*

Proof. Let f^* be a diffeomorphism $(M_1^*; F_k^*(M_1), F_m^*(M_1)) \to (M_2^*; F_k^*(M_2), F_m^*(M_2))$. The induced S^1 action on M_i/Z_m has one exceptional orbit type Z_k and structured orbit space $(M_i^*, F_k^*(M_i))$, so by Theorem 2.3 there is a weak equivalence $f: M_1/Z_m \to M_2/Z_m$ covering f^*. Hence $f(F_m(M_1)/Z_m) = F_m(M_2)/Z_m$. Now M_i is the smooth m-fold cyclic branched cover of M_i/Z_m branched over $F_m(M_i)/Z_m$; so we may apply the uniqueness of branched covers and Lemma 2.2 to finish the argument as in the proof of Theorem 2.3.   □

The rest of this section is devoted to showing that for each $\sigma \in \mathfrak{S}_{k,m}^n$ or $\mathfrak{T}_{k,m}^n$ there is a smooth S^1-action $M(\sigma)$ on a homotopy n-sphere with structured orbit space σ. So let $\sigma = (\Sigma^{n-1}; \Delta_k, \Delta_m) \in \mathfrak{S}_{k,m}^n$ or $\mathfrak{T}_{k,m}^n$. Proceeding as in §2 we may orient the normal bundle of $\partial\Delta_k = \partial\Delta_m$ in Σ^{n-1} and obtain a smooth semifree S^1 action on a homotopy sphere N^n with orbit map $\mu: N \to \Sigma^{n-1}$ and fixed point set $\mu^{-1}(\partial\Delta_k)$. Let $\bar{F}_k = \mu^{-1}(\Delta_k)$ and $\bar{F}_m = \mu^{-1}(\Delta_m)$. As in §2 the respective k and m-fold cyclic branched covers M_k and M_m are homotopy n-spheres with smooth S^1 actions and structured orbit spaces (Σ^{n-1}, Δ_k) and (Σ^{n-1}, Δ_m). Now form the pullback diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\nu_m'} & M_k \\
{\scriptstyle \nu_k'} \downarrow & & \downarrow {\scriptstyle \nu_k} \\
M_m & \xrightarrow{\nu_m} & N
\end{array}
$$

$$\searrow \mu$$

$$\Sigma^{n-1}$$

The pullback M is a topological manifold because $\overline{F}_k$ and $\overline{F}_m$ meet transversely in N. Also v'_k and v'_m are branched k and m-fold covering projections. So M inherits a smooth structure as in §2 in two different ways and the S^1-actions lift to S^1 actions on M, smooth in their respective structures. However since both actions have the same structured orbit space σ, by Theorem 3.1 they are weakly equivalent. In particular the above two smooth manifold structures on M are diffeomorphic.

THEOREM 3.2. *For each $\sigma \in \mathbb{S}_{k,m}^n$ or $\mathbb{T}_{k,m}^n$ there is a homotopy n-sphere $M(\sigma)$ with a smooth S^1 action with σ as structured orbit space.*

Proof. Let $M(\sigma) = M$ above. As in §2 for $R = Q$ or Z_p for p any prime not dividing both k and m (hence for any prime p) $H_*(M; R) \approx H_*(S^n; R)$; thus $H_*(M, Z) = H_*(S^n; Z)$. So to show that $M = M(\sigma)$ is a homotopy sphere we must show that M is simply connected.

To this end consider a loop λ in M^n ($n \geq 5$) based at a point in a principal orbit, and by general position suppose λ misses $E \cup F$. Let p: $M \to \Sigma$ be the orbit map. The based loop $p\lambda$ can be homotoped in $\Sigma - E^* \cup F^*$ to an embedding, and this homotopy can be lifted to M. Hence we may suppose that λ and $p\lambda$ are embedded loops. Since $\pi_1\Sigma = 0$, $p\lambda$ bounds a transversely immersed 2-disk D which meets $E^* \cup F^*$ transversely in a finite number of points and which misses $E_{k,m}^*$. By sliding intersections of D with $E^* \cup F^*$ to F^* we may arrange that $D \cap (E^* \cup F^*) = D \cap F^*$. If $D \cap F^* = \varnothing$ introduce an intersection point of D and F^* by pushing D until it meets F^* tangentially at a regular point of D. Now $p^{-1}(D) \cong D \times S^1/\sim$ where $\sim$ identifies $\{x\} \times S^1$ to $(x, 1)$ for each $x \in D \cap F^*$; so the map $p_{\sharp}: \pi_1(p^{-1}(D)) \to \pi_1(D)$ is an isomorphism, and λ is nullhomotopic. $\square$

COROLLARY 3.3. *Let M^n be a smooth S^1-manifold with structured orbit space $\sigma \in \mathbb{S}_{k,m}^n$ or $\mathbb{T}_{k,m}^n$. Then M is a homotopy sphere.* $\square$

In the introduction we mentioned a question posed by Frank Raymond [**M**; p. 353]. In our context it is equivalent to asking whether each F_q is connected when S^1 acts on S^n. Our theory provides infinitely many counterexamples in each dimension ≥ 5 for a given slice representation (1.3(d)) at the fixed point set determined by the integers k and m. In fact we have described how all such examples with codimension 4 fixed point set can occur. As we have noted the first such example was constructed by Elliott Stein [**St**]. In his example the structured orbit space is $(S^4; D_k^2, D_m^2) \in \mathbb{T}_{k,m}^5$ where $D_k^2 \cup D_m^2$ is the standard immersed S^2 in S^4 with one double point.

Finally, we should mention that in the case $n = 4$ analogous results hold. Essentially $\mathbb{S}_k^4$ is the set of homotopy 3-spheres and $\mathbb{S}_{k,m}^4$ is the set of diffeomorphism classes of knots in homotopy 3-spheres; $\mathfrak{T}_{k,m}^4 = \varnothing$. See [**F**] and [**P**].

4. Twist-spinning knots. By a "knot" we shall mean a smooth pair $N^{n-3} \subset M^{n-1}$ of homotopy spheres. The k *twist-spin* ($k > 1$) of the knot $N^{n-3} \subset M^{n-1}$ may be obtained as follows. Let $(D^{n-1}, D^{n-3}) \subset (M, N)$ be a standard disk pair and consider $\sigma = (M^{n-1}, D^{n-3}) \in \mathbb{S}_k^n$. We shall find it more convenient to use the notation $\sigma = (M^{n-1}, D^{n-3}, N^{n-3} - \text{Int } D^{n-3}) \in \mathbb{S}_{k,1}^n$. Write $F_1 = p^{-1}(N^{n-3} - \text{Int } D^{n-3})$. By Theorem 2.3, $M(\sigma)$ is a homotopy sphere, and following the proof of the first claim of §2 we see that F_1 is also a homotopy sphere. The homotopy sphere pair $(M^n(\sigma), F_1^{n-2})$ is called the k twist-spin of (M^{n-1}, N^{n-3}). Since F_1^{n-2} carries a semifree S^1 action with codimension 2 fixed point set, F_1^{n-2} is diffeomorphic to S^{n-2} [**H**] provided $n \geq 7$. It is an easy exercise to see that if M^{n-1} is diffeomorphic to S^{n-1} then $M(\sigma)$ is diffeomorphic to S^n.

The classification theorems of §3 point to an obvious generalization of this procedure. Given the knot $N^{n-3} \subset M^{n-1}$ ($n \geq 4$) let $k \geq 2$ and $m \geq 1$ be relatively prime integers. We have (using the same notation as in the last paragraph) $\tau = (M^{n-1}; D^{n-3}, N^{n-3} - \text{Int } D^{n-3}) \in \mathbb{S}_{k,m}^n$. The (k, m) twist-spin of (M, N) is the knot F_m^{n-2} in the homotopy sphere $M(\tau)$. The (k, m) and (m, k) twist-spins of (M, N) form a pair of knotted $n - 2$ spheres in $M(\tau)$ which meet transversely in the $n - 4$ sphere F. Of course $(k, 1)$ twist-spinning is just k twist-spinning and in fact we have seen in §3 that the result of (k, m) twist-spinning may be obtained by constructing an m-fold branched cover over a k twist-spun knot.

For a closed manifold X, let X_0 be the manifold obtained by removing a point from X. Zeeman's main theorem on twist-spun knots [**Z**] states that the complement of the k twist-spin of (M^{n-1}, N^{n-3}) fibers over the circle with fiber V_0 where V is the k-fold cyclic branched cover of M branched over N. More specifically, it says that the complement of the twist-spun knot is $V_0 \times_\phi S^1$, the mapping torus of the canonical deck transformation of V which rotates the normal disk of N^{n-3} through the angle $2\pi/k$.

Our generalization to (k, m) twist-spinning was motivated by a question of Zeeman [**Z**, p. 493] as to whether it is possible to introduce a new factor into the twist-spinning process so that the complement of the knot thus obtained is $V_0 \times_{\phi^m} S^1$ for $1 < m < k$. The next theorem shows that this is exactly what we have done and also gives a streamlined proof of Zeeman's main theorem.

THEOREM 4.1. *Let* $(M(\sigma),\ F_m)$ *be the* (k, m) *twist-spin of the knot* (M^{n-1}, N^{n-3}) $(n \geq 4)$, *and let* $\mathcal{V}$ *be the* k-*fold cyclic branched cover of* (M^{n-1}, N^{n-3}) *with canonical deck transformation* ϕ. *Then* $M(\sigma) - F_m$ *is diffeomorphic to* $V_0 \times_{\phi^m} S^1$.

Proof. S^1 viewed as R/kZ acts on $V \times_{\phi^m} S^1$ by $s \cdot \langle y, t \rangle = \langle y, t + s \rangle$. If we view the homotopy sphere N^{n-3} as the branch set of $\mathcal{V}^{n-1} \to M^{n-1}$ then for this action $E_k = N^{n-3} \times_{\phi^m} S^1$ and all the other orbits are principal. The orbit space is $V/\phi^m = V/\phi = M^{n-1}$, and $E_k^* = N^{n-3}$.

Let $x \in E_k$. We may identify a tube about the orbit of x with $D^{n-3} \times D^2 \times S^1 \subset S^n = \partial(D^{n-3} \times D^2 \times D^2)$ with the standard linear S^1 action $t \cdot (y, z, w) = (y, zt^m, st^k)$. Perform equivariant surgery on the orbit of x by replacing this tube with its complementary $\partial(D^{n-3} \times D^2) \times D^2$ in S^n, and let W^n be the resulting S^1-manifold. The result of this surgery on the orbit space M^{n-1} is to remove a $D^{n-3} \times D^2$ whose intersection with E_k^* is $D^{n-3} \times 0$ and replace it with its complementary $\partial D^{n-3} \times D^2 \subset S^{n-1} = S^n/S^1$. Since the gluing map extends over an $n-1$ disk, the orbit space of W^n is still M^{n-1}; and it is easily seen that $E^* \cup F^* = N^{n-3}$ where F_m^* is an $n - 3$ disk contained in $D^{n-3} \times 0 \subset N^{n-3}$. Thus the structured orbit space of W is in $\mathcal{S}_{k,m}^n$ and $(W^*, E^* \cup F^*) = (M, N)$. So it follows from Corollary 3.3 that W^n is a homotopy sphere, and the knot (W^n, F_m) is the (k, m) twist spin of (M, N). Furthermore, F_m is the cocore of the surgery on $V \times_{\phi^m} S^1$ by which W was constructed. Thus $W - F_m = V_0 \times_{\phi^m} S^1$. $\qquad\square$

Restricting the S^1 action to Z_k or Z_m actions we obtain counterexamples to the n-dimensional Smith conjecture whenever F_k or F_m is actually knotted. For $m = \pm 1 \pmod{k}$ the examples of Giffen [Gi] and Gordon [Go] are obtained.

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