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ACYCLIC DECOMPOSITIONS OF MANIFOLDS

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This paper develops techniques for generating decompositions of manifolds into homologically acyclic compacta. For $n \geq 3$ it presents examples of decompositions of the n-sphere S^n that are totally acyclic, in the sense that each decomposition element is an acyclic but non-cell-like set. One class of examples yields non-ANR's as quotient spaces; another class (for $n > 3$) yields ANR's. The distinction essentially depends upon whether the decomposition elements are nearly 1-movable.

Every since their introduction in the 1930's, homology manifolds have held an important position in topology. In low dimensions (≤ 2) their local algebraic properties are sufficiently strong to imply that they are genuine manifolds. In higher dimensions this is not the case; nevertheless, they attract attention for at least two reasons: first, their local algebraic properties subject them to the same global algebraic properties possessed by manifolds (like Poincaré Duality), and second, they arise naturally as fixed point sets of certain group actions on genuine manifolds.

Homology manifolds, sometimes called generalized mnaifolds, occur in two distinct forms, the ANR's and the non-ANR's. Current practice, it seems, reserves the term "generalized manifold" for the ANR form. In accordance with that practice, we shall speak of a locally compact metric space X as a *generalized n-manifold* if X is an ANR and, for each $x \in X$,

$$H_*(X, X - \{x\}; Z) \cong H_*(E^n, E^n - \{\text{point}\}; Z);$$

and we shall speak of X as a *homology n-manifold* if, for each $x \in X$,

$$\check{H}_*(X, X - \{x\}; Z) \cong H_*(E^n, E^n - \{\text{point}\}; Z),$$

where $\check{H}_*$ denotes Čech homology. If we change from integer coefficients to some other module R and if

$$\check{H}_*(X, X - \{x\}; R) \cong H_*(E^n, E^n - \{\text{point}\}; R),$$

we shall call X an *R-homology n-manifold.*

Within the past few years these generalized manifolds have moved into a central position in geometric topology. It is a consequence of the classical Vietoris-Begle Mapping Theorem [**Be**] that any finite dimensional space X which is the cell-like image of an n-manifold is, in fact, a generalized n-manifold. In 1977 R. D. Edwards [**E$_1$**] proved that such a space X, the cell-like image of a manifold, is always a factor of some manifold (more remarkably, it is a manifold itself provided $n \geq 5$ and X satisfies the minimal general positioning embodied in the Disjoint Disks Property). This line of research has culminated with the announcement by F. Quinn [**Q**] that for $n \geq 5$ a finite dimensional space X is a generalized n-manifold iff it is the cell-like image of an n-manifold.

Beginning with the pioneering work of R. H. Bing [**Bi**] in the 1950's, much of the early work concerning cell-like decompositions or cell-like mappings was directed at discovering non-manifold generalized manifolds. Comparable efforts still go on at present, aided by the result of Edwards. In contrast, very little has been done with decompositions of manifolds into Z-acyclic compact sets, other than the classical result [**Be**] that the associated decomposition space turns out to be a homology manifold having the same (homology) dimension as the source. Now, in view of Quinn's result, it is worth investigating which homology n-manifolds arise as the image of some n-manifold under a Z-acyclic mapping. We shall supply some data for such investigations here, primarily through the introduction of machinery for producing unusual and nontrivial decompositions of manifolds into Z-acyclic but not cell-like sets.

Besides this machinery, we identify a property of such decompositions crucial for distinguishing whether the decomposition space is a generalized manifold, rather than the more pathological homology manifold. It is the shape property called "nearly 1-movable". §5 traces the history establishing the key result that if G is a decomposition of an n-manifold M into nearly 1-movable, Z-acyclic compact sets and if M/G is finite dimensional, then M/G is a generalized n-manifold.

1. Defining sequences and decompositions. Decompositions of manifolds can be prescribed efficiently in terms of defining sequences. In this section we review the general notions of defining sequences to be employed.

Let X be a space and $\mathfrak{M}$ a collection of subsets of X. For an arbitrary subset A of X define its *star in* $\mathfrak{M}$ as

$$\operatorname{st}(A, \mathfrak{M}) = \operatorname{st}^1(A, \mathfrak{M}) = A \cup \left(\bigcup \{M \in \mathfrak{M} : M \cap A \neq \varnothing\}\right)$$

and, recursively for any integer $k > 1$, define its kth *star in* $\mathfrak{M}$ as

$$\operatorname{st}^k(A, \mathfrak{M}) = \operatorname{st}(\operatorname{st}^{k-1}(A, \mathfrak{M}), \mathfrak{M})$$

Now suppose that X is a (locally) compact metric space. According to the general definition introduced in [**DW**], a *defining sequence* in X is a sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \dots\}$ satisfying the following two simple axioms:

Axiom 1. For each i, the set $\mathcal{M}_i$ is a (locally) finite cover of X by compact subsets which have nonempty, pairwise disjoint interiors.

Axiom 2. For each i and each $x \in X$,

$$\mathrm{st}^3(x, \mathcal{M}_{i+1}) \subset \mathrm{Int}\, \mathrm{st}^2(x, \mathcal{M}_i).$$

The elements of the *decomposition* G *associated with* $\mathcal{S}$ are the sets $\bigcap_i \mathrm{st}^2(x, \mathcal{M}_i)$, $x \in X$. By Theorem 2.3 of [**DW**], G is upper semicontinuous. This definition is all-encompassing since every upper semicontinuous decomposition of a locally compact metric space arises from such a defining sequence.

More useful for our purposes is a somewhat restricted notion of defining sequence introduced in [**CD**], which inspired the generalization in [**DW**]. It is this more restrictive notion that will be employed here. From now on, a *defining sequence* in an n-manifold T is a sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \dots\}$ satisfying the following conditions:

(i) Disjointness Criterion: for each i, the set $\mathcal{M}_i$ is a locally finite cover by compact, connected n-manfiolds-with-boundary which are embedded locally flatly in T and which have pairwise disjoint interiors;

(ii) Nesting Criterion: for each $i > 1$ and each $A \in \mathcal{M}_i$, there is a unique element $\mathrm{Pre}\, A \in \mathcal{M}_{i-1}$ which properly contains A;

(iii) Boundary Size Criterion: for each $i \geq 1$, each $A \in \mathcal{M}_i$, and each pair of distinct points $x, y \in \partial A$, there is an integer $k > i$ such that no element of $\mathcal{M}_k$ contains both x and y.

As before, the *decomposition* G *associated with* $\mathcal{S}$ is the one having the sets $\bigcap_i \mathrm{st}^2(t, \mathcal{M}_i)$, $t \in T$, for its members. By Theorem 1 of [**CD**], G is upper semicontinuous.

REMARK. In this situation, two additional features fill out the depiction of G: first, no $g \in G$ contains more than one point of $\mathrm{Bd}\, \mathcal{S} = \bigcup \{\partial A: A \in \bigcup \mathcal{M}_i\}$, and, second, if $t \in g \in G$ and if either $t \in \mathrm{Bd}\, \mathcal{S}$ or $g \cap \mathrm{Bd}\, \mathcal{S} = \varnothing$, then $g = \bigcap_i \mathrm{st}(t, \mathcal{M}_i)$ (cf. [**CD**, Addendum to Theorem 1]).

Comparing these two definitions of defining sequences, one sees immediately that the Disjointness Criterion parallels Axiom 1. Less obviously, the Nesting Criterion and the Boundary Size Criterion, taken together, imply a form of Axiom 2. In particular, with the forthcoming constructions, just as with that in [**CD**], the Boundary Size Criterion will be fulfilled so that for each $i \geq 1$ and each $t \in T - \mathrm{Bd}\, \mathcal{S}$ or $t \in \bigcup\{\partial A: A \in \mathfrak{M}_i\}$,

$$\mathrm{st}^2(t, \mathfrak{M}_{i+1}) \subset \mathrm{Int}\, \mathrm{st}^2(t, \mathfrak{M}_i)$$

which, with the Nesting Criterion, implies

$$\mathrm{st}^3(t, \mathfrak{M}_{i+1}) \subset \mathrm{st}^2(t, \mathfrak{M}_i).$$

This then gives the modification of Axiom 2 that, for each $t \in T$, ultimately,

$$\mathrm{st}^3(t, \mathfrak{M}_{i+2}) \subset \mathrm{Int}\, \mathrm{st}^2(t, \mathfrak{M}_i).$$

2. Defining sequences and acyclicity. From among the various definitions of acyclicity given in the literature, we retrace the formulation given by McMillan [**Mc$_2$**]. The symbol Z_p (p always denotes 0 or a prime) is to be read consistently with fixed p in any discussion, and it denotes the ring of integers mod p, with $Z_0 = Z$ the ring of integers itself. If k is a nonnegative integer, then a compact set X in a manifold T (or, more generally, an ANR T) is said to be *strongly k-acyclic* over Z_p (or to have Property *k-$uv(Z_p)$*) if each neighborhood U of X in T contains another neighborhood V of X such that each k-cycle in V is homologous to zero in U (singular homology with Z_p coefficients). Generally, X is said to be *strongly acyclic over Z_p* if for each $k \geq 0$ it is strongly k-acyclic over Z_p. One should turn to [**L**, §2] and [**Mc$_2$**, §2] for further background and examples.

There is an important relationship between these homological acyclicity properties and the Čech cohomology of X. According to Lacher [**L**, §2.2], for a compact subset X of an ANR T and a positive integer k, (1) if X has properties $(k\text{-}1) - uv(Z_p)$ and k-$uv(Z_p)$, then $\check{H}^k(X: Z_p) = 0$ and (2) if $\check{H}^k(X; Z_p) = \check{H}^{k+1}(X; Z_p) = 0$, then X has property k-$uv(Z_p)$.

We call a decomposition G of an ANR *acyclic* (or, *Z_p-acyclic*) if each $g \in G$ is strongly acyclic over Z (over Z_p). In order that a decomposition G associated with a defining sequence $\mathcal{S}$ be Z_p-acyclic, it is sufficient (but not necessary) that the following condition be satisfied.

(iv) Z_p-Acyclicity Criterion: for each $i \geq 1$ and $A \in \mathfrak{M}_i$, the inclusion map $A \to \mathrm{Pre}\, A \in \mathfrak{M}_{i-1}$ induces the zero homomorphism $H^*(\mathrm{Pre}\, A; Z_p) \to H^*(A; Z_p)$.

PROPOSITION 2.1. *If $\mathcal{S} = \{\mathfrak{M}_1, \mathfrak{M}_2, \dots\}$ is a defining sequence (in an* ANR *T) satisfying the Z_p-Acyclicity Criterion, then each element g of the associated decomposition G is strongly acyclic over Z_p.*

Proof. The case $g \cap \text{Bd}\, \mathcal{S} = \varnothing$ is the easier one and is left to the reader. In the other case, according to the remark in §1 one can assume that there is an $A \in \mathfrak{M}_m$ and $t \in \partial A$ with $g = \bigcap_i \text{st}(t, \mathfrak{M}_i)$. For a neighborhood U of g one can determine $k > m$ such that $\text{st}(t, \mathfrak{M}_k) \subset U$. Let $A_1, \ldots, A_s$ denote the elements of $\mathfrak{M}_{k+1}$ containing t, and for $i = 1, \ldots, s$ let Y_i denote the set $g \cap (\bigcup_{j=1}^{i} A_j)$. A Mayer-Vietoris argument establishes by induction on i that the inclusion of Y_i into $\text{st}(t, \mathfrak{M}_k)$ induces the trivial homomorphism on Čech cohomology; at the heart of the argument is the choice of $t \in g$ so that $Y_{i-1} \cap A_i = \{t\}$ which implies that

$$\check{H}^*(Y_i; Z_p) \cong \check{H}^*(Y_{i-1}; Z_p) \oplus \check{H}^*(g \cap A_i; Z_p)$$

Consequently, $\check{H}^*(g; Z_p) = \check{H}^*(Y_s; Z_p) \cong 0$, and g is Z_p-acyclic [**L**, §2.2].

For a defining sequence $\mathcal{S} = \{\mathfrak{M}_1, \mathfrak{M}_2, \ldots\}$ in a finite dimensional ANR T, a necessary and sufficient condition that the associated decomposition G be Z_p-acyclic is that for each integer i and each $t \in T$ there exists an integer $j > i$ such that

$$H_*(\text{st}^3(x, \mathfrak{M}_j); Z_p) \to H_*(\text{st}^2(x, \mathfrak{M}_i); Z_p)$$

is the trivial homomorphism.

3. A decomposition incorporating components of a compactum. Although our main emphasis is on acyclic decompositions, the main result is substantially more general and can be used to product examples of manifold decompositions, the shape of whose elements reflects that of the components of a prechosen compactum.

MAIN THEOREM. *Let X be a compact k-dimensional subset of the interior of a compact connected PL n-manifold T, $n \geq 3$ and $k \leq n - 2$, let $K(X)$ denote the decomposition of T into points and the components of X with decomposition map π_X: $T \to T/K(X)$, and suppose that $\pi_X(X)$ is a Cantor set. Then there exists a defining sequence $\mathcal{S}$ for an upper semicontinuous decomposition G of T such that*:

(0) Bd $\mathcal{S}$ *does not intersect X*;

(1) *each component of X is contained in some $g \in G$*,

(2) *each $g \in G$ contains a component of X*,

(3) *each set $\pi_X(g)$, $g \in G$, is cell-like and* 1-*dimensional*,

(4) *T/G has dimension $\leq n$.*

Proof. We shall obtain G as the decomposition associated with a defining sequence $\mathcal{S} = \{\mathfrak{M}_1, \mathfrak{M}_2, \dots\}$. We begin with $\mathfrak{M}_1 = \{T\}$ and proceed to give an inductive specification of the rest of $\mathcal{S}$.

Inductive Hypothesis (j). Suppose that $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_j$ have been obtained satisfying the following properties:

(a) $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_j$ fulfill the appropriate finite segment from Criteria (i), (ii), and (iii) for a defining sequence; in particular, for $1 < i \leq j$ and $A \in \mathfrak{M}_i$, the diameter of $A \cap \partial(\text{Pre}\, A)$ is less than $1/i$;

(b) for each $A \in \mathfrak{M}_i$, $1 < i \leq j$, there exists a $(1/i)$-map of $\pi_X(A)$ to an arc; and

(c) for each $A \in \mathfrak{M}_i$, $1 < i \leq j$, $\partial A \cap X = \varnothing$ and $A \cap X \neq \varnothing$.

Description of $\mathfrak{M}_{j+1}$. Assuming Inductive Hypothesis (j) we shall specify $\mathfrak{M}_{j+1}$ so that $\mathfrak{M}_1, \dots, \mathfrak{M}_{j+1}$ fulfill the parallel Inductive Hypothesis $(j + 1)$. In order to do this, we fix an $A \in \mathfrak{M}_j$ and note that it will suffice to describe the elements of $\mathfrak{M}_{j+1}$ which are contained in A. This will be accomplished in five steps.

Step 1. *Splitting components of $A \cap X$.* In $T/K(X)$ the compact set $\pi_X(A \cap X)$ can be covered by a finite number of pairwise disjoint nonempty compact subsets $Q_1, \dots, Q_q$ of $\pi_X(X)$, each having diameter less than $1/(j + 1)$. Let R_i denote the preimage of Q_i in T $(i = 1, \dots, q)$.

Step 2. *Decomposing A into cells.* Let T_∂ denote a triangulation of ∂A. Choose a PL collar neighborhood $(\partial A) \times [0, 1]$ on ∂A in A (which is empty in case $j = 1$ and T has no boundary) disjoint from $\cup R_i$, with ∂A corresponding to $\partial A \times \{0\}$. The product $T_\partial \times [0, 1]$ provides a cell-decomposition of the collar, and it extends to a PL cell-decomposition T_Δ of A. Shortening the collar and subdividing both T_∂ and A minus the collar, where necessary, we may assume that each of the cells $D_1, \dots, D_s$ of T_Δ as well as each of their images in $T/K(X)$ has diameter less than $1/(j + 1)$, that each of the cells $D_1, \dots, D_s$ misses at least one of the R_i's, and that at least q of the cells in T_Δ miss each of $R_1, R_2, \dots, R_q$ (most conveniently, these cells can be chosen to lie in $T_\Delta \times [0, 1]$ unless $j = 2$ and T has no boundary).

Step 3. *Pairing the cells with parts of $R_r \cap X$.* Because of these last assumptions, to each n-cell D_i of T_Δ we can associate some set $R_{r(i)}$ so that $R_{r(i)} \cap D_i = \varnothing$ and so that each R_r is associated with some $D_i \in T_\Delta$. The hypothesis that $\pi_X(X)$ is a Cantor set means that each R_r can be fractured into exactly the same number of pairwise disjoint nonempty $K(X)$-saturated closed sets as the number of n-cells D_i of T_Δ for which

$R_{r(i)} = R_r$. In order to simplify notation, we shall assume that the association of the n-cells of T_Δ with the sets R_r, pairing $R_{r(i)}$ and D_i, is a bijection.

Step 4. *Connecting* D_i *to* $R_{r(i)}$. Since $\dim X \leq n-2$, for $i = 1,\ldots,s$ we can thread an arc α_i in $\pi_X(\operatorname{Int} A - \operatorname{Int} D_i)$ through the points of $\pi_X(R_{r(i)} \cap X)$ so that $\alpha_i \cap \pi_X(D_i)$ is an endpoint of α_i in $\pi_X(\partial D_i - X)$ and α_i meets no point of $\pi_X(X - R_{r(i)})$. Let $\beta_i = \pi_X^{-1}(\alpha_i)$. After slight general position adjustments of $\beta_i - X$, we can assume that $\beta_i \cap \beta_m = \varnothing$ whenever $i \neq m$.

Step 5. *Defining* $A_1,\ldots,A_s$. Note that there is a $1/(j+1)$-retraction of $\pi_X(D_i \cup \beta_i \cup R_{r(i)}) = \pi_X(D_i) \cup \alpha_i$ to α_i, sending $\pi_X(D_i)$ to an endpoint. Determine a fine simplicial subdivision T_1 of T_Δ such that the sets $\beta_1 \cup R_{r(1)},\ldots,\beta_s \cup R_{r(s)}$ have pairwise disjoint connected simplicial neighborhoods $N_1',\ldots,N_s'$, respectively, in $\operatorname{Int} A$ so close to the core sets $\beta_i \cup R_{r(i)}$ that $\pi_X(D_i \cup N_1')$ admits a $1/(j+1)$-map to α_i and that no $D_m \in T_\Delta$ is contained in or separated by any N_i'. If T_2 is a pth barycentric subdivision of T_1 for sufficiently large p, then the sets $N_i = \operatorname{st}(N_i', T_2)$ will possess the same properties but will be PL n-manifolds. The set A_i is defined as $(D_i \cup N_i) - \bigcup_{m \neq i} N_m$, and it is also a PL n-manifold ($i = 1,\ldots,s$).

This completes the inductive specification of the defining sequence $\mathcal{S}$. The three conditions of Inductive Hypothesis $(j+1)$ can be verified directly by inspection of this construction. The only item warranting special mention concerns (a), namely, that $A_i \cap \partial A = D_i \cap \partial A$, which has diameter $< 1/(j+1)$ according to Step 2.

As discussed in §1 (cf. [**CD**, Theorem 1]), $\mathcal{S}$ is a defining sequence for an upper semicontinuous decomposition G of T. Clearly, $\operatorname{Bd} \mathcal{S} \cap X = \varnothing$. Moreover, by the remark in §1, each point of T/G has arbitrarily small neighborhoods with frontiers in the image of $\operatorname{Bd} \mathcal{S}$, which is naturally homeomorphic to $\operatorname{Bd} \mathcal{S}$, showing T/G to have dimension equal to either $n-1$ or n.

In order to see that the other conclusions of the Main Theorem hold, one can use the connectedness of elements of $\mathcal{M}_j$ and Condition (c) of the Inductive Hypothesis to show that to each component Y of X there exists a unique $A_j \in \mathcal{M}_j$ containing Y, thereby establishing

$$Y \subset \bigcap_j A_j \subset \bigcap_j \operatorname{st}^2(y, \mathcal{M}_j) \in G, \quad \text{for each } y \in Y.$$

On the other hand, given $g \in G$, one can name $t \in g$ and then produce $A_j \in \mathcal{M}_j$ such that $A_j \subset \operatorname{st}(t, \mathcal{M}_j)$ and $A_j = \operatorname{Pre} A_{j+1}$ ($j = 1,2,\ldots$); one can use Condition (c) again to show that $\bigcap_j A_j$ contains a component of X and, of course, $\bigcap_j A_j \subset g$. Finally, the 1-dimensionality and cell-likeness of $\pi_X(g)$ follows directly (as below) from Condition (b) of the Inductive Hypothesis in case $g \in G$ misses Bd $\mathcal{S}$; in the other case, when $g = \bigcap_i \operatorname{st}(t, \mathfrak{M}_i)$ where $t \in \partial A'$ and $A' \in \mathfrak{M}_m$ (see the remark in §1), for $j \geq m$ and each $A \in \mathfrak{M}_j$ containing t, there exists a $(1/j)$-map f_A of $\pi_X(A)$ to an arc α_A. The various arcs α_A can be wedged at the points $f_A \pi_X(t)$ to form a contractible 1-complex Γ, and $\pi_X(g)$ can be $(2/j)$-mapped to Γ by the natural compilation of the f_A's (restricted). Thus, $\pi_X(g)$ is a treelike continuum, which is another way of saying that it is cell-like and 1-dimensional.

REMARK. The assumption that dim $X \leq n - 2$ is more restrictive than necessary but it permits a significantly simpler proof. We state, leaving the interested reader to perform the not entirely trivial modifications to the proof, that it suffices to assume that, for each open connected subset $U \subset T$ with $\operatorname{Fr} U \cap X = \varnothing$, the set $U - X$ is also connected (of course, continuing to assume that X is a subset of the interior of T and $\pi_X(X)$ is a Cantor set).

4. The preservation of shape properties. The elements of the decomposition constructed in the preceding section will possess certain of the shape properties of the components of the compactum X. Singled out in what follows are several properties, including those with which we shall be subsequently concerned. The first result is quite general; the forward implication is due to McMillan [**Mc**$_3$; Theorem 1] assuming only that f is a continuous surjection while the reverse implication is due to Dydak [**D**$_1$].

A compactum C is *nearly* 1-*movable* if for some (and hence for every) embedding of C in an ANR X, the following holds: for each neighborhood U of C in X there exists another neighborhood V of C in X, with $V \subset U$, such that "V nearly 1-moves toward C in U", meaning that for every loop $f\colon \partial B^2 \to V$ and for every neighborhood W of C in X, there exists a finite collection of pairwise disjoint disks $\{B_i\}$ in Int B^2 and an extension F of f to

$$F\colon \left(B^2 - \bigcup \operatorname{Int} B_i,\ \bigcup \partial B_i\right) \to (U, W).$$

PROPOSITION. 4.1 (*McMillan, Dydak*). *Let $f\colon X \to Y$ be a map between compact spaces with each point inverse $f^{-1}(y)$ nearly* 1-*movable. Then X is nearly* 1-*movable if and only if Y is nearly* 1-*movable.*

Recall that a compact subset Z of an ANR T is said to have *Property* UV^k provided for each neighborhood U of Z there is a neighborhood V of Z such that each map of the j-sphere ($j = 0, 1, \ldots, k$) into V extends to a map of the ($j + 1$)-ball into U.

PROPOSITION 4.2. *Suppose X is a compact subset of a finite dimensional ANR T, $\pi\colon T \to T/K(X)$ is the decomposition map for the decomposition $K(X)$ into points and the components of X, and g is a compact subset of T with $\pi(g)$ 1-dimensional and cell-like and $\pi^{-1}\pi(g) = g$. Then*:

(1) *g has Property UV^k if and only if each component of X contained in g has Property UV^k.*

(2) *g is cell-like if and only if each component of X contained in g is cell-like.*

(3) *g is strongly Z_p-acyclic if and only if each component of X contained in g is strongly Z_p-acyclic.*

Proof. Each of the reverse implications is known with that in (3) following from the Vietoris-Begle Mapping Theorem [**Be**], with that in (2) contained in [**Sh**] or [**K**], and with that in (1) contained in [**Mo**]. We shall prove the forward implication in (1); a similar argument establishes (3) while (2) follows from (1) and the finite dimensionality of g.

Choose a point $y \in \pi(g)$ and let U be a neighborhood of $\pi^{-1}(y)$. The 0-dimensionality of $\pi(X)$ and the 1-dimensionality of $\pi(g)$ imply the existence of a compact 0-dimensional set A in $g - X$ separating $\pi^{-1}(y)$ from $g - U$; explicitly, $g - A$ is expressed as the union of mutually separated sets E and F containing $\pi^{-1}(y)$ and $g - U$, respectively.

Name disjoint open subsets W_E and W_F containing E and F, respectively, with $W_E \subset U$ such that $\mathrm{Cl}\, W_E \cap \mathrm{Cl}\, W_F \subset A$. Furthermore, use the fact that A is 0-dimensional to name an open set W_A, with $A \subset W_A \subset \mathrm{Cl}\, W_A \subset U - \pi^{-1}(y)$, such that each component of W_A is contractible in U. Because g satisfies Property UV^k by hypothesis, there is a neighborhood V' of g such that any map of a j-sphere ($j = 0, 1, \ldots, k$) into V' extends to a map of the ($j + 1$)-ball into $W_E \cup W_A \cup W_F$. Let $V = V' - \mathrm{Cl}(W_A \cup W_F)$.

Let f denote a map of the j-sphere S^j into V ($1 \leq j \leq k$). It extends to a map f of the ($j + 1$)-ball B^{j+1} into $W_E \cup W_A \cup W_F$. In order to excise its image from the part outside U, $f^{-1}(\mathrm{Cl}\, W_F)$ can be separated from S^j by a finite collection of connected locally flat j-manifolds $\{J_i\}$ in $f^{-1}(W_A)$ bounding pairwise disjoint connected ($j + 1$)-manifolds $\{D_i\}$ in Int B^{j+1}, and the choice of W_A can be used to redefine the map f on each D_i, sending it into U. Upon reassembly, one has a map F of B^{j+1} into

$$f\left(B^{j+1} - \bigcup D_i\right) \cup U \subset (W_E \cup W_A) \cup U \subset U.$$

In case $k = 0$, replacing V by the component of V containing $\pi^{-1}(y)$, we have as well that maps of 0-spheres in V extend to maps of 1-cells into U.

5. The preservation of ANR's. The purpose of this brief section is to emphasize a somewhat neglected fact, which provides a mild condition under which a Z-acyclic decomposition of an ANR yields another ANR.

THEOREM 1. (*Dydak, Hurewicz.*) *Suppose G is a Z-acyclic decomposition of an* ANR *X into nearly* 1-*movable compacta, and suppose that X/G is finite dimensional. Then X/G is an* ANR.

Proof. The Vietoris-Begle Mapping Theorem [**Be**], coupled with the hypothesis that elements of G are Z-acyclic, implies that X/G is homologically locally k-connected (as measured by Čech homology) over Z for each integer $k \geq 0$. According to a result of Dydak [$\mathbf{D_1}$, Theorem 1], the hypothesis that elements of G are nearly 1-movable implies that X/G is LC^1. Work of Hurewicz [**H**] (cf. [$\mathbf{D_2}$, Theorem 3.2]) then implies that X/G is LC^k for each $k \geq 0$. Finally, the finite dimensionality of X/G yields that X/G is an ANR.

COROLLARY 5.2. *If G is a Z-acyclic decomposition of an n-manifold M into nearly* 1-*movable compacta and if M/G is finite dimensional, then M/G is a generalized n-manifold.*

REMARK 1. If G is a Z-acyclic decomposition of a finite dimensional space X, must X/G be finite dimensional? This turns out to be equivalent to the question of whether cell-like maps can raise dimension [$\mathbf{E_2}$].

REMARK 2. If G is a Z-acyclic decomposition of an ANR X such that X/G is an ANR, must the elements of G be nearly 1-movable? It seems reasonable to believe they must be. In case G has just a single nondegenerate element, Shrikhande [**Sk**] has shown it to be nearly 1-movable.

6. Several examples.

EXAMPLE 1. Suppose T is a compact PL n-manifold, $n \geq 3$. There exists a defining sequence $\mathcal{S}$ for a Z-acyclic (hence, Z_p-acyclic for all $p > 0$) upper semicontinuous decomposition G of T such that no $g \in G$ is cell-like and T/G fails to be an ANR. In fact, at each point T/G fails to be locally 1-connected.

Construction. Let ξ denote the strongly Z-acyclic but not cell-like continuum (ξ fails to have Property UV^1) described by Case-Chamberlin **[CC]**. The product X of ξ with a Cantor set can be regarded as a subset of Int T, since X is 1-dimensional. Let π_X: $T \to T/K(X)$ denote the natural map, with the notation set forth in the statement of the Main Theorem.

There is a decomposition G satisfying the conclusions of the Main Theorem. In particular we have a diagram

$$\begin{array}{ccc} t & \overset{\pi_X}{\to} & T/K(X) \\ \pi\downarrow & \swarrow\tilde{\pi} & \\ T/G & & \end{array}$$

and it follows from conclusions (1) and (3) that $\tilde{\pi} = \pi \circ \pi_X^{-1}$ is a well-defined cell-like map.

The decomposition G is Z-acyclic if and only if π is a Z-acyclic map, which necessarily holds because π is expressed as the composition of two Z-acyclic maps. However, no element of G can be cell-like (or have Property UV^1), for otherwise conclusion (1) of Proposition 4.2 (with conclusion (3) of the Main Theorem) would force a component $\xi \times$ point of X in g (guaranteed by conclusion (2) of the Main Theorem) to have Property 1-UV, which runs contrary to properties of ξ.

If T/G were locally 1-connected at some point $\pi(g)$, then Theorem 3 of [**Mc$_2$**] would show that g has Property UV^1. Hence, T/G fails to be an ANR.

EXAMPLE 2. Suppose p is a prime and T is a compact PL n-manifold, $n \geq 3$. There exists a defining sequence $\mathcal{S}$ for a Z_p-acyclic upper semicontinuous decomposition G of T such that no $g \in G$ is strongly Z-acyclic and T/G fails to be an ANR (at each point T/G fails to be locally 1-connected).

Construction. Proceed exactly as in the construction of Example 1, only use the p-adic solenoid in place of the Case-Chamberlin continuum ξ. That solenoid is strongly Z_p-acyclic but not strongly Z-acyclic. Conclusion (3) of Proposition 4.2 can be used to establish that no $g \in G$ can be strongly Z-acyclic, for otherwise a subset of g homeomorphic to that solenoid would be strongly Z-acyclic as well.

The next examples display a sharp contrast to McMillan's result [**Mc$_2$**] that the elements of a Z-acyclic decomposition of S^3 whose associated decomposition space is an ANR are necessarily cell-like.

EXAMPLE 3. Suppose T is a compact PL n-manifold, $n \geq 4$. There exists a defining sequence $\mathcal{S}$ for a Z-acyclic (hence, Z_p-acyclic for all $p > 0$) upper semicontinuous decomposition G such that T/G is an ANR but no $g \in G$ is cell-like.

Construction. For $n \geq 5$, consider any PL homology $(n-2)$-cell H (to prevent ambiguity, H is a compact, Z-acyclic $(n-2)$-manifold) that is not simply connected. Let X be the image of any embedding in T of the product of H with a Cantor set. There is such an embedding since the homology n-cell $H^* = H \times [0, 1] \times [0, 1]$ embeds in S^n (this holds essentially because $\pi_1(H^*)$ is generated by the image of $\pi_1(\partial H^*)$, so the union of H^* and a contractible n-manifold bounded by ∂H^*, attached together along ∂H^*, is a homotopy n-sphere.)

The construction of G proceeds as in the construction of Example 1. No element $g \in G$ can be cell-like, for that would force a component Y of X in g to be cell-like, in contradiction to $Y \cong H$ being a noncontractible ANR.

Since H is an ANR, it is nearly 1-movable and condition (3) of the Main Theorem and Proposition 4.1 insure that each $g \in G$ is nearly 1-movable. By Condition (4) of the Main Theorem, T/G is finite dimensional. Hence, Theorem 5.1 establishes that T/G is an ANR.

In dimension $n = 4$, we start with a PL homology 3-cell H which is not simply connected such that $H \times [0, 1]$ embeds in S^4 (the examples of Mazur [**Ma**] have this property since each is the boundary of a contractible 4-manifold whose double is homeomorphic to S^4). Choose a 2-dimensional subpolyhedron P of H to which H collapses, and set X equal to the image of an imbedding in T of the product $P \times [0, 1]$.

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