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HYPERINVARIANT SUBSPACES AND THE TOPOLOGY ON Lat A

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The lattice of invariant subspaces of an operator is a metric space. We give various topological conditions on a point in the lattice which ensure it is a hyperinvariant subspace for the operator.

Introduction. Let $\mathcal{H}$ be a Hilbert space and A a bounded operator on $\mathcal{H}$. We write Lat A for the lattice of invariant subspaces of A, and Hyp A for the subset of Lat A consisting of the hyperinvariant subspaces (i.e. subspaces which are invariant for every operator B on $\mathcal{H}$ commuting with A). In [6] Rosenthal showed that if $M \in \text{Lat } A$ is a *pinch point* of Lat A, i.e. M is comparable to every point of Lat A, then $M \in \text{Hyp } A$. This result was extended by Stampfli who showed for example that if M and N are pinch points and the set $[M, N] = \{L \in \text{Lat } A\colon M \subseteq L \subseteq N\}$ is countable, then $[M, N] \subseteq \text{Hyp } A$ [7]. A related result due to Fillmore [4] says that if S is a countable subset of Lat A, every element of which is comparable to every element of $(\text{Lat } A)\setminus S$, then $S \subseteq \text{Hyp } A$.

In [1] Douglas and Pearcy noticed many of these types of conditions could be viewed as topological conditions, and this enabled them to considerably extend the above results. They define a metric d on Lat A by $d(M, N) = \|P_M - P_M\|$, where P_M denotes the orthogonal projection onto M, and they define a point $M \in \text{Lat } A$ to be *inaccessible* if its path-component in the metric space Lat A is just $\{M\}$. In particular, isolated points of Lat A are inaccessible. They then show that inaccessible points of Lat A must lie in Hyp A. (and in the case where A is normal, that Hyp A consists of the inaccessible (in fact, isolated) points of Lat A). It's trivial to see that if P_M and P_N commute, then $\|P_M - P_N\| = 1$. Thus if Lat A is commutative, then it is discrete, and so Lat A = Hyp A. They also remark that if $M \in \text{Lat } A$ is a pinch point then since P_M commutes with all P_N ($N \in \text{Lat } A$), $d(M, N) = 1$, and so M is isolated in Lat A. Thus they recover Rosenthal's result, and they also show Fillmore's result can be obtained from their topological conditions in [1] and [2]. Finally they point out that inaccessibility is not a necessary condition on $M \in \text{Lat } A$ that M lie in Hyp A (their counterexample involves the lattice of the unilateral shift of multiplicity one).

In this paper we present some refinements of the Douglas-Pearch techniques, and obtain some strengthenings of their results. Also we present some new results on reducing and complemented spaces in Lat A which determine whether these spaces lie in Hyp A.

Throughout, $\mathcal{H}$ will always denote a Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. As in the introduction, for $M, N \in \text{Lat } A$, $[M, N] = \{L \in \text{Lat } A: M \subseteq L \subseteq N\}$. $\mathbf{D}$ denotes the unit disc, $\mathbf{D} = \{\lambda \in \mathbf{C}: |\lambda| \leq 1\}$, and $\mathring{\mathbf{D}}$ its interior. We say $M \in \text{Lat } A$ is *reducing* if $M^\perp \in \text{Lat } A$, and that M is complemented in Lat A if there exists $N \in \text{Lat } A$ such that $M + N = \mathcal{H}$ and $M \cap N = 0$.

1. **Intervals in lattices.** We shall need the following two lemmas.

LEMMA 1 ([1]). *If M_1, M_2 are subspaces of $\mathcal{H}$ and $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ are invertible then*

$$d(A_1M_1, A_2M_2) \leq \|A_1 - A_2\|\left(\|A_1^{-1}\| + \|A_2^{-1}\|\right) + d(M_1, M_2)\left(\|A_1^{-1}\|\,\|A_2\| + \|A_2^{-1}\|\,\|A_1\|\right).$$

LEMMA 2 ([5], p. 112). *If M is a subspace of $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, A nonzero, and if for distinct points λ, $\mu \in \|A\|^{-1}\mathring{\mathbf{D}}$ we have $(1 - \lambda A)M = (1 - \mu A)M$, then $M \in$ Lat A.*

Thus this result says if the map $\phi: \lambda \mapsto (1 - \lambda A)M$ is not injective on $\|A\|^{-1}\mathring{\mathbf{D}}$, then $M \in \text{Lat } A$. In particular, if the set $\{(1 - \lambda A)M: |\lambda| < \|A\|^{-1}\}$ is countable, then $M \in \text{Lat } A$.

We shall be using these two results repeatedly.

Recall that a *disc* in a topological space X is a subset of X homeomorphic to $\mathbf{D}$.

THEOREM 3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $M, N \in$ Hyp A. If $L \in [M, N]$ lies in no disc in $[M, N]$, then $L \in$ Hyp A.*

Proof. If $L \notin \text{Hyp } A$, then there exists $B \in \mathcal{B}(\mathcal{H})$ commuting with A such that $L \notin \text{Lat } B$ and $\|B\| = 1$. Now if $|\lambda| < 1$ then $1 - \lambda B$ is invertible, and since $M, N \in \text{Hyp } A$ we have $M = (1 - \lambda B)M \subseteq (1 - \lambda B)L \subseteq N = (1 - \lambda B)N$. Thus $(1 - \lambda B)L \in [M, N]$. By Lemma 2, the map $\phi: \mathring{\mathbf{D}} \to [M, N], \lambda \to (1 - \lambda B)L$, is injective. By Lemma 1,

$$d(\phi(\lambda), \phi(\mu)) \leq |\lambda - \mu|\left(\|(1 - \lambda B)^{-1}\| + \|(1 - \mu B)^{-1}\|\right),$$

so ϕ is continuous. Then ϕ maps the (compact) disc $\frac{1}{2}\mathbf{D}$ homeomorphically into (the Hausdorff) space $[M, N]$. Hence $L = \phi(0)$ lies on a disc in $[M, N]$. □

If X is a topological space, then an *arc* in X is a subset homeomorphic to $[0, 1]$. Let's generalize this: an *interval* in X is a subset of X homeomorphic to an interval in $\mathbf{R}$ (i.e. a connected subset of $\mathbf{R}$). Thus an interval in X is a connected set which can be embedded in $\mathbf{R}$. Elementary topology shows intervals cannot contain discs.

THEOREM 4. *Let $A \in \mathcal{B}(\mathcal{H})$, $M, N \in \operatorname{Hyp} A$ and $\mathcal{O}$ be an open subset of $[M, N]$.*

(i) *If a path-component C of $\mathcal{O}$ is an interval, then $C \subseteq \operatorname{Hyp} A$.*

(ii) *If L is an isolated or inaccessible point of $\mathcal{O}$, then $L \in \operatorname{Hyp} A$.*

(iii) *If $\mathcal{O}$ is countable, discrete, or totally disconnected, then $\mathcal{O} \subseteq \operatorname{Hyp} A$.*

Proof. (i) Let C be a path-component of $\mathcal{O}$ and suppose C is an interval. If $L \in C$ and $L \notin \operatorname{Hyp} A$, then by Theorem 3, L lies in a disc D in $[M, N]$. Hence $\mathcal{O} \cap D$ is a nonempty $(L \in \mathcal{O} \cap D)$ open subset of a disc, and hence must itself contain a disc, D_1 say, containing L. Thus as D_1 is path-connected and lies in $\mathcal{O}$, $D_1 \subseteq C$, i.e. we have a disc in an interval. This is impossible. Hence $L \in C$ implies $L \in \operatorname{Hyp} A$.

(ii) If L is an isolated or inaccessible point of $\mathcal{O}$ then its path-component in $\mathcal{O}$ is $\{L\}$, which is clearly an interval.

(iii) If $\mathcal{O}$ is countable, discrete, or totally disconnected, then all its path-components are singleton sets, and so intervals. □

COROLLARY 5. *Let $A \in \mathcal{B}(\mathcal{H})$, and $M, N \in \operatorname{Hyp} A$.*

(i) *If a path-component C of $[M, N]$ is an interval, then $C \subseteq \operatorname{Hyp} A$.*

(ii) *If L is isolated or inaccessible in $[M, N]$, then $L \in \operatorname{Hyp} A$.*

(iii) *If $[M, N]$ is countable, discrete, or totally disconnected, then $[M, N] \subseteq \operatorname{Hyp} A$.*

COROLLARY 6. *Let $A \in \mathcal{B}(\mathcal{H})$.*

(i) *If a path-component C of* Lat A *is an interval, then $C \subseteq \operatorname{Hyp} A$.*

(ii) *If L is isolated or inaccessible in* Lat A, *then $L \in \operatorname{Hyp} A$.*

(iii) *If* Lat A *is countable, discrete, or totally disconnected, then* Lat $A = \operatorname{Hyp} A$.

Proof. Simply take $M = 0$ and $N = \mathcal{H}$ in Corollary 5. □

REMARK. Parts (ii) and (iii) of Corollary 6 are not new, and can be found in [**1**], [**2**], [**5**], and [**7**]. These papers also contain some related results not covered by the above theorems.

Recall that a metric space X is an *n-manifold* if for each $x \in X$ there is an open neighbourhood U of x homeomorphic to $\mathbf{R}^n$.

COROLLARY 7. *If* $A \in \mathcal{B}(\mathcal{H})$ *and the open set* $\mathcal{O}$ *in* Lat A *is a* 1*-manifold, then* $\mathcal{O} \subseteq \operatorname{Hyp} A$.

Proof. If $L \in \mathcal{O}$, then there is an open set U in $\mathcal{O}$ containing L which is homeomorphic to $\mathbf{R}$. Hence the path-component of L in U is an interval. So by Theorem 4(i), $L \in \operatorname{Hyp} A$. □

REMARK. We know from Theorem 3, that if Lat A contains no disc, then Lat $A = \operatorname{Hyp} A$. The converse is false. For if A denotes the unilateral shift of multiplicity 1, then Lat $A = \operatorname{Hyp} A$ (see for example [**1**]). Also if $|\lambda| < 1$, then $A - \lambda$ is bounded below, so $(A - \lambda)\mathcal{H} \in \operatorname{Lat} A$. Moreover if λ, μ are distinct points of $\mathring{\mathbf{D}}$, then $(A - \lambda)\mathcal{H} \neq (A - \mu)\mathcal{H}$. (For otherwise, if $x \in \mathcal{H}$, then $(A - \lambda)x = (A - \mu)y$ for some $y \in \mathcal{H}$. Hence $(\mu - \lambda)x = (A - \mu)y - (A - \mu)x \in (A - \mu)\mathcal{H}$. Therefore $x \in (A - \mu)\mathcal{H}$, and so $A - \mu$ is onto. But this is impossible since $\mu \in \sigma(A)$, the spectrum of A.) It's easy to see that the map $\phi\colon \lambda \mapsto (A - \lambda)\mathcal{H}$ is continuous from $\mathring{\mathbf{D}}$ to Lat A, from which one can deduce that $A\mathcal{H} = \phi(0)$ lies in a disc in Lat A, i.e. Lat A contains discs. Essentially this example was also used in [**1**].

We finish this section with some short observations on the finite-dimensional case.

THEOREM 8 (*Fillmore. See* [**5**], *p.* 113). *If* $\mathcal{H}$ *is finite dimensional, and* $A \in \mathcal{B}(\mathcal{H})$, *then the hyperinvariant subspaces of* A *are precisely the ranges and null spaces of polynomials in* A.

COROLLARY 9. ($\dim \mathcal{H} < \infty$). *The following conditions are equivalent.*

(i) Lat $A = \operatorname{Hyp} A$.

(ii) Lat A *is finite.*

(iii) Lat A *is discrete.*

Proof. From Theorem 8, $\operatorname{Hyp} A = \{N((A - \lambda_1) \cdots (A - \lambda_n))\colon \lambda_1, \ldots, \lambda_n \in \sigma(A)\} \cup \{R((A - \lambda_1) \cdots (A - \lambda_n))\colon \lambda_1, \ldots, \lambda_n \in \sigma(A)\} \cup \{0, \mathcal{H}\}$ and this is clearly a finite set. The corollary now follows using

Corollary 6(iii). ($N(A)$ and $R(A)$ denote respectively the null space and range of A.) □

2. Special points in lattices.

DEFINITION. Let X be a topological space, and P a topological property (such as connectedness). If the set of points x in X such that $X \setminus \{x\}$ has property P is countable, we call these points *special* points of P in X. A point in X which is special for some topological property we call a *special point of* X.

For example, a point x in X is a *cut point* of X if $X \setminus \{x\}$ is disconnected, otherwise x is a *non-cut-point*. (This is a standard topological definition.) Thus in $[0, 1]$, 0 and 1 are non-cut-points, every other point is a cut point. Hence 0, 1 are special points of $[0, 1]$.

Clearly every countable topological space consists of special points. **R** has no special points, neither does any other uncountable homogeneous space.

Here's an example of an uncountable space X with a dense countable subset of special points: $X = [0, 1] \cup \{(k/n, 1/n): 0 \le k \le n, n = 2, 3, 4, \ldots\}$ in the plane. The "snowflakes" $(k/n, 1/n)$ can easily be shown to be special in X.

THEOREM 10. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $M, N \in \operatorname{Hyp} A$. *If* C *is a path-component of* $[M, N]$ *then its special points lie in* $\operatorname{Hyp} A$. *In particular, if* C *has a dense set of special points, then* $C \subseteq \operatorname{Hyp} A$.

Proof. Let L be a special point of C, and suppose B is an operator commuting with A and assume that $\|B\| = 1$. Then we've seen already in the proof of Theorem 3 that the map $\mathring{\mathbf{D}} \to [M, N]$, $\lambda \mapsto (1 - \lambda B)L$, is continuous, hence since $\mathring{\mathbf{D}}$ is connected we deduce that $(1 - \lambda B)L \in C$. From this we can conclude that for each $\lambda \in \mathring{\mathbf{D}}$, the homeomorphism $[M, N] \to [M, N]$, $L_1 \mapsto (1 - \lambda B)L_1$, maps the path-component C onto itself. Denote by ϕ_λ the restriction of this homeomorphism to C, ϕ_λ: $C \to C$. Now there is some topological property P such that L is special for P and only countably many other points of C are special for P. But each $\phi_\lambda(L)$ is also special for P, since ϕ_λ is a homeomorphism, and if $C \setminus \{L\}$ has property P, so does $\phi_\lambda C \setminus \{\phi_\lambda L\}$. Hence $\{\phi_\lambda(L): |\lambda| < 1\}$ is countable, i.e. $\{(1 - \lambda B)L: |\lambda| < \|B\|^{-1}\}$ is countable. We now deduce that $L \in \operatorname{Lat} B$.

Thus special points of C are in $\operatorname{Hyp} A$.

If C has a dense set D of special points, then $D \subseteq \operatorname{Hyp} A$. But it is trivially seen that $\operatorname{Hyp} A$ is closed, so $C = \overline{D} \subseteq \operatorname{Hyp} A$. □

COROLLARY 11. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then the special points of each path-component* C *of* Lat A *lie in* Hyp A. *If* C *has a dense set of special points, then* $C \subseteq \operatorname{Hyp} A$.

EXAMPLE. If the path component C of Lat A has only countably many cut points, they lie in Hyp A. Similarly if C has only countably many non-cut-points, they lie in Hyp A. In particular if $M \in \operatorname{Lat} A$ is inaccessible then $M \in \operatorname{Hyp} A$, as we've seen already.

3. Reducing spaces and complemented spaces.

THEOREM 12. *Let* $A \in \mathcal{B}(\mathcal{H})$.

(i) *If* M, N *are reducing spaces in* Lat A *and* $d(M, N) < 1/2$ *then* $M \in \operatorname{Hyp} A$ *if and only if* $N \in \operatorname{Hyp} A$.

(ii) *If* Γ *is a path of reducing spaces in* Lat A *one point of which lies in* Hyp A *then* $\Gamma \subseteq \operatorname{Hyp} A$.

Proof. (i) With little extra effort we can and will show that there is a path of reducing spaces in Lat A joining N to M.

If $0 \leq t \leq 1$, let $X_t = 1 + t(2P_MP_N - P_M - P_N)$. Then $\|X_t - 1\| < 1$, since $\|P_M - P_N\| < 1/2$. Thus X_t is invertible. Also, since P_M and P_N commute with A and A^* (because M and N are reducing), so X_t commutes with A and A^*. Thus $X_tN \in \operatorname{Lat} A \cap \operatorname{Lat} A^*$, i.e. X_tN is reducing for A. Finally a simple computation shows $P_MX_1 = X_1P_N$, so $X_1N = M$. Clearly $X_0N = N$. The map $t \mapsto X_tN$ from $[0, 1]$ into Lat A is continuous, since the map $t \mapsto X_t$ is continuous, and by Lemma 1,

$$d(X_tN, X_sN) \leq \|X_t - X_s\|\left(\|X_t^{-1}\| + \|X_s^{-1}\|\right).$$

Thus $t \mapsto X_tN$ is a path in Lat A of reducing spaces from N to M.

Now suppose $N \in \operatorname{Hyp} A$. Then if B is an operator commuting with A, $X_1^{-1}BX_1$ also commutes with A, and so $X_1^{-1}BX_1N \subseteq N$, i.e. $BX_1N \subseteq X_1N$, or $BM \subseteq M$. Thus $M \in \operatorname{Hyp} A$.

(ii) Suppose $M \in \Gamma$ lies in Hyp A and let $N \in \Gamma$. Then there exists a continuous map α: $[0, 1] \to \Gamma$, $\alpha(0) = M$ and $\alpha(1) = N$. Now α is uniformly continuous so there exists $\delta > 0$ such that if $|t - s| < \delta$ then $d(\alpha t, \alpha s) < 1/2$. We can choose $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $|t_i - t_{i+1}| < \delta$ $(i = 0, 1, \ldots, n - 1)$, and then $d(\alpha t; \alpha t_{i+1}) < 1/2$. Now $M = \alpha(t_0) \in \operatorname{Hyp} A$, hence by (i) above, $\alpha(t_1) \in \operatorname{Hyp} A$, hence $\alpha(t_2) \in \operatorname{Hyp} A$, etc. Thus $\alpha(t_n) = N \in \operatorname{Hyp} A$. We've shown $\Gamma \subseteq \operatorname{Hyp} A$. □

Recall that $A \in \mathcal{B}(\mathcal{H})$ is called a *reductive* operator if all its invariant subspaces are reducing. (Whether every such operator is necessarily normal is equivalent to the invariant subspace problem, Dyer-Porcelli [**3**]).

THEOREM 13. *If* $A \in \mathcal{B}(\mathcal{H})$ *is a reductive operator, then* $\operatorname{Hyp} A$ *is clopen (closed and open) in* $\operatorname{Lat} A$. *So if a component* C *of* $\operatorname{Lat} A$ *has a point in* $\operatorname{Hyp} A$, *then* $C \subseteq \operatorname{Hyp} A$.

Proof. That $\operatorname{Hyp} A$ is closed is trivial. By Theorem 12(i) we see $\operatorname{Hyp} A$ is open. □

We can now give a partial extension of these results to the case of complemented spaces.

THEOREM 14. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* E, F *indempotent operators commuting with* A, *such that* $\|E - F\| < \frac{1}{2}(\max(\|E\|, \|F\|))^{-1}$. *Then* $E\mathcal{H} \in \operatorname{Hyp} A$ *if and only if* $F\mathcal{H} \in \operatorname{Hyp} A$.

Proof. The reasoning is quite similar to that in Theorem 12(i). Put $X = 1 + 2EF - E - F$. Then $\|X - 1\| < 1$ from the inequality in the hypothesis. Thus X is invertible and commutes with A. So if B is an operator commuting with A, $X^{-1}BX$ commutes with A. An elementary computation shows $EX = XF$, hence $E\mathcal{H} = XF\mathcal{H}$. Thus if $F\mathcal{H} \in \operatorname{Hyp} A$ then $X^{-1}BXF\mathcal{H} \subseteq F\mathcal{H}$, and therefore $BE\mathcal{H} \subseteq E\mathcal{H}$. □

THEOREM 15. *Let* $t \mapsto E_t$ *be a path in* $\mathcal{B}(\mathcal{H})$ *of idempotents commuting with the operator* $A \in \mathcal{B}(\mathcal{H})$. *Suppose* $E_0\mathcal{H} \in \operatorname{Hyp} A$. *Then* $E_t\mathcal{H} \in \operatorname{Hyp} A$ *for* $0 \leq t \leq 1$.

Proof. W.l.o.g. we show only $E_1\mathcal{H} \in \operatorname{Hyp} A$. As $t \mapsto \|E_t\|$ is continuous on the compact set $[0, 1]$, there exists $\varepsilon > 0$, $\|E_t\| < \varepsilon$ for all $t \in [0, 1]$. Also, as $t \mapsto E_t$ is uniformly continuous, there exists $\delta > 0$, $|t - s| < \delta$ implies $\|E_t - E_s\| < 1/2\varepsilon$, and hence $\|E_t - E_s\| < \frac{1}{2}(\max(\|E_t\|, \|E_s\|))^{-1}$. Choose $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $|t_i - t_{i+1}| < \delta$ $(i = 0, 1, 2, \ldots, n-1)$. Then by Theorem 14, since $E_0(\mathcal{H}) \in \operatorname{Hyp} A$, we have $E_{t_1}(\mathcal{H}) \in \operatorname{Hyp} A$ and hence $E_{t_2}(\mathcal{H}) \in \operatorname{Hyp} A$, etc. Thus $E_1(\mathcal{H}) = E_{t_n}(\mathcal{H}) \in \operatorname{Hyp} A$. □

REMARK. In [**1**] it is shown that if M and N are subspaces of $\mathcal{H}$ and $\|P_M - P_N\| < 1$, then M and $N^{\perp}$ are complementary subspaces of $\mathcal{H}$. It follows that if N is a reducing subspace for an operator A on $\mathcal{H}$ and

$M \in \text{Lat}\, A$ satisfies $d(M, N) < 1$, then M is complemented in Lat A (by $N^{\perp}$). Thus reducing spaces in Lat A are interior points in the set of all complemented subspaces in Lat A.

It would be of interest to know if Theorem 12 is valid for complemented subspaces of Lat A. The author wishes to thank the referee for the following example which shows that Theorem 12 is not valid for arbitrary elements of Lat A. Let $A = U \oplus U$ where U is the unilateral shift of multiplicity one, let $M = \mathfrak{M} \oplus \mathfrak{M}$ and $N = \mathfrak{M}_\lambda \oplus \mathfrak{M}$, where $\mathfrak{M}$ and $\mathfrak{M}_\lambda$ are as in Theorem 5 of [**1**], $0 < \lambda < 1$. Then $M \varepsilon \,\text{Hyp}\, A$, $N \varepsilon \,\text{Lat}\, A \backslash \text{Hyp}\, A$, and $d(M, N) \leq (2\lambda - \lambda^2)/(1 - \lambda)$. This example also shows that Theorem 13 is not valid for arbitrary operators (since for A in the example, Hyp A is not clopen).

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