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REMARKS ON FENN'S "THE TABLE THEOREM" AND ZAKS' "THE CHAIR THEOREM"

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In 1970 Roger Fenn showed that one could always balance a square table on a hill. Soon after, Joseph Zaks stated that one can always translate a triangular chair to balance on a hill. Recent results have been found by E. H. Kronheimer and P. B. Kronheimer. This paper gives precise statements of the theorems and shows that:

1. **Compact support for the hill in the Chair Theorem cannot be replaced by $\lim_{|x|\to\infty} f(x) = 0$ (answering a question of Zaks).**
2. **The isometry of the Table cannot be replaced by a translation.**
3. **The square Table cannot be replaced by an n-gon table for $n \geq 5$.**
4. **The Table Theorem is still open for cyclic quadrilaterals.**

In 1970 Roger Fenn showed that one could always balance a square table on a hill ([**Fe**]). Soon after, Joseph Zaks stated that one can always translate a triangular chair to balance on a hill ([**Za**]). (For a correct and elegant proof of the latter see E. H. Kronheimer and P. B. Kronheimer, [**KK**]). After giving a precise statement of these theorems we shall discuss four attempts at improving upon these results. We consider relaxing the compactness hypothesis for the support of the hill in the Chair Theorem, strengthening the conclusion of the Table Theorem by using a translation as the Chair Theorem does, using a table of more than four sides, and using a four-sided but not square table.

1. Below are stated the aforementioned Table Theorem and (parenthetically) Chair Theorem.

THE TABLE (CHAIR) THEOREM. *Let D be a compact convex disk in $\mathbf{R}^2$. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be a non-negative map which is zero outside D. Let V be the set of vertices of a square (triangle) with center (arbitrary point) C. Then there is an isometry (translation), τ, of $\mathbf{R}^2$ such that $\tau C \in D$ and f takes a single value on τV.*

The next section answers part of Question 1 of Zaks ([**Za**]).

2. The Chair Theorem is false if we change the condition that f is zero outside D to $\lim_{|x|\to\infty} f(x) = 0$.

To see this let $g(x, y) = |y \bmod 2|$ where for $y \bmod 2$ we choose a value from -1 to 1. Let $V = \{(0,0), (0,1), (\sqrt{3}/2, 1/2)\}$. Let $h(x, y) = (2/3)^{\sqrt{x^2+y^2}}$ and $f(x, y) = g(x, y)h(x, y)$. Then f satisfies the necessary hypotheses. Given a translation τ, suppose the above functions take values g_i, h_i, and f_i on the ith vertex of τV, $i = 1, 2$, or 3. Two of the values taken by g on τV must differ by at least $1/2$, say $g_1 + 1/2 \leq g_2$. Then $g_1 < 1$ and since the vertices of V are a unit apart, $h_1/h_2 \leq 3/2$. So

$$f_1 = g_1 h_1 \leq g_1(3/2)h_2 < (g_1 + 1/2)h_2 \leq g_2 h_2 = f_2$$

and the chair isn't "balanced."

3. The Table Theorem is false if we change the isometry τ to a translation.

Let ∂D be a large (compared to the square) non-circular ellipse and let the graph of $f|D$ be a cone on that ellipse. There will be only one position of the square which gives a solution. To see this, solve the general quadratic through the four points $(\pm 1, \pm 1)$ to show that there are only two ellipses of any given positive eccentricity circumscribing a given square.

Fenn ([**Fe**]) indicated that there is no Table Theorem for a table which is a *regular* n-gon, $n \geq 5$. In fact we have:

4. The Table Theorem is false if we change V to the vertices of any n-gon, $n \geq 5$, where the "center" C can be taken as any point in the interior of the convex hull of the table. (For C elsewhere, there will be a trivial solution with $C \in D$ and V disjoint from the interior of D.)

In general, five points determine a conic, so there is an ellipse such that no similar ellipse can pass through the vertices of the given polygon. Using such an ellipse for the base of a cone we get a hill none of whose level curves can support the table. We must also make the base ellipse large enough so that its curvature is small, to ensure that if the "center" of the polygon is on or inside the base ellipse some vertex is inside the base ellipse.

The above result is quoted in [**Za**] as from a private communication from L. M. Sonneborn, but no details are given.

Fenn noted that a circular cone shows that a four-sided Table Theorem could only hold for cyclic quadrilaterals. The argument used above fails to give us a counterexample for any cyclic quadrilateral since they can be inscribed in ellipses of arbitrary eccentricity. To see this consider a fifth point lying on the circle which circumscribes the quadrilateral. Move this point off to a point at infinity so that the eccentricity of the unique conic through the five points varies continuously from zero to at least one.

5. The Table Theorem is open for tables which are cyclic quadrilaterals.

This is contrary to a statement by Zaks ([**Za**]). Zaks showed that a cyclic quadrilateral $ABCD$ with angles of $(150 + \varepsilon)°$, $90°$, $(30 - \varepsilon)°$, and $90°$ respectively ($\varepsilon > 0$) cannot be inscribed (all 4 vertices touching edges) in an equilateral triangle. He erroneously concluded that such a table can't stand on a hill which is the cone over an equilateral triangle. For while we can't balance the table on any level curve, we can place the table at height zero with all four vertices outside the base equilateral triangle and any "center" of the table at a vertex of the base triangle.

One can also show that rounding the corners of the triangle does not repair this example. However, Zaks' construction does show that the type of Table Theorem proved in [**KK**] for squares cannot be extended to arbitrary cyclic quadrilaterals.

References

[Fe] Roger Fenn, *The table theorem*, Bull. London Math. Soc., **2** (1970), 73–76.

[KK] E. H. Kronheimer and P. B. Kronheimer, *The tripos problem*, J. London Math. Soc., (2) **24** (1981), 182–192.

[Za] Joseph Zaks, *The chair theorem*, Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Louisiana State University, Baton Rouge, 1971), 557–562.

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