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THE p-EQUIVALENCE OF SO$(2n+1)$ AND Sp(n)

ALBERT T. LUNDELL

Elementary homotopy methods are used to construct homotopy equivalences of the localized spaces $\mathrm{SO}(2n+1)_{\mathcal{P}}$ and $\mathrm{Sp}(n)_{\mathcal{P}}$, where $\mathcal{P}$ is the set of odd primes. The equivalences are H-maps.

Serre [**1**] conjectured a $\mathcal{C}$-isomorphism $\pi_k(\mathrm{Sp}(n)) \approx \pi_k(\mathrm{SO}(2n+1))$ where $\mathcal{C}$ is the class of 2-primary abelian groups. This was proved by Harris [**3**]. Since the development of localization techniques for spaces [**4**, **8**], other proofs of equivalence via decomposition as products have been given [**6**]. Friedlander [**2**] has proved the p-equivalence of $\mathrm{BSO}(2n+1)$ and $\mathrm{BSp}(n)$, for odd primes p, by the use of etale homotopy theory. None of these methods prove the equivalence by actually giving a map.

The purpose of this note is to use the results of Harris [**3**], a map described in [**5**], and elementary homotopy theory to construct homotopy equivalences of the localized spaces $\mathrm{SO}(2n+1)_{\mathcal{P}}$ and $\mathrm{Sp}(n)_{\mathcal{P}}$, where $\mathcal{P}$ is the set of odd primes. These equivalences are H-maps, but the author does not know if they can be delooped to obtain Friedlander's result.

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1. Notation. The unitary group $U(n)$ is the group of non-singular complex $n \times n$ matrices with inverse the conjugate transpose. The orthogonal group $O(n)$ is the subgroup of $U(n)$ left pointwise fixed under complex conjugation, i.e. the subgroup of real matrices. We denote by $\mathrm{SO}(n) \subset O(n)$ and $\mathrm{SU}(n) \subset U(n)$ the subgroups of elements of determinant 1, and by α: $O(n) \to U(n)$ (or α: $\mathrm{SO}(n) \to \mathrm{SU}(n)$) the inclusion monomorphism.

If $J \in \mathrm{SU}(n)$ is the matrix with 2×2 blocks $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ down the diagonal, then $\mathrm{Sp}(n)$ is the subgroup of $\mathrm{SU}(2n)$ left pointwise fixed by the automorphism $g \to J\bar{g}J^{-1}$, where $\bar{g}$ is the complex conjugate matrix of g (i.e. $\overline{(g_{ij})} = (\bar{g}_{ij})$). We denote the inclusion monomorphism by β: $\mathrm{Sp}(n) \to \mathrm{SU}(2n)$.

The monomorphisms α, β are natural with respect to inclusions $U(n-k) \to U(n)$ described in matrix notation by $A \to \left(\begin{smallmatrix} A & 0 \\ 0 & I_k \end{smallmatrix}\right)$ where I is the $k \times k$ identity.

If $\mathbb{S}$ is a set of prime numbers and X is a space which admits localizations, then $X_{\mathbb{S}}$ will denote a localization of X at $\mathbb{S}$ and $e_{\mathbb{S}}$: $X \to X$ a localization map.

2. The map ϕ. In [5] the author defined a map ϕ: $O(n) \to U(n-1)$ so that the diagram

$$\begin{array}{ccc} O(n) & \xrightarrow{\ \phi\ } & U(n-1) \\ & \searrow^{\alpha} & \downarrow j \\ & & U(n) \end{array}$$

homotopy commutes. For the reader's convenience we repeat the definition here.

Let u be a complex number with $|u| = 1$ and define a cross-section σ_u: $S^{2n-1} - \{ue_n\} \to U(n)$ by the formula

$$\sigma_u(x_1, x_2, \ldots, x_n) = \left[\begin{array}{c|c} \left[\delta_{pq} - x_p Q^{-1} \bar{x}_q\right] & \begin{matrix} x_2 \\ \vdots \\ x_{n-1} \end{matrix} \\ \hline P\bar{x}_1\, P\bar{x}_2 \cdots P\bar{x}_{n-1} & x_n \end{array}\right],$$

where $Q = 1 - \bar{x}u$ and $P = u\bar{Q}Q^{-1}$. Taking $u = i$ in this formula $j\phi(x) = [\sigma_i p\alpha(x)]^{-1}\alpha(x)$, where p: $U(n) \to S^{2n-1}$ is the bundle projection which picks out the last column of a matrix in $U(n)$. A proof of the homotopy commutativity of the diagram above as well as other properties of ϕ can be found in [5].

We first remark that det $\sigma_u(x) = -P$, so if we multiply $\sigma_u(x)$ on the right by the matrix

$$\begin{pmatrix} I_{n-2} & & \\ & -\bar{P} & \\ & & 1 \end{pmatrix}$$

we obtain a cross-section σ'_u: $S^{2n-1} - \{ue_n\} \to \mathrm{SU}(n)$. For $x \in \mathrm{SO}(n)$, the map ϕ': $\mathrm{SO}(n) \to \mathrm{SU}(n-1)$ such that $j\phi'(x) = [\sigma'_i p\alpha(x)]^{-1}\alpha(x)$ factors α: $\mathrm{SO}(n) \to \mathrm{SU}(n)$ through $\mathrm{SU}(n-1)$ up to homotopy and has properties analogous to ϕ. From now on we will suppress primes, writing $\sigma_u = \sigma'_u$ and $\phi = \phi'$.

PROPOSITION 2.1. *The map ϕ and its complex conjugate $\bar{\phi}$ are homotopic maps* $\mathrm{SO}(n) \to \mathrm{SU}(n-1)$.

Proof. One easily sees that for the complex conjugate, $\overline{\sigma_i p\alpha(x)} = \sigma_{-i}p\alpha(x)$, and that

$$\bar{\phi}(x) = [\sigma_{-i}p\alpha(x)]^{-1}[\sigma_i p\alpha(x)]\phi(x).$$

For $y \in S^{2n-1} - \{\pm ie_n\}$, we have $[\sigma_{-i}(y)]^{-1}[\sigma_i(y)] \in \mathrm{SU}(n-1)$, and if we set

$$h(x,t) = \left(\cos\frac{\pi t}{2}\right)p\alpha(x) + i\left(\sin\frac{\pi t}{2}\right)e_{n-1},$$

and $H(x,t) = [\sigma_{-i}h(x,t)]^{-1}[\sigma_i h(x,t)]\phi(x)$, we have $H\colon \mathrm{SO}(n)\times I \to \mathrm{SU}(n-1)$ with $H(x,0) = \bar{\phi}(x)$, $H(x,1) = \phi(x)$. □

3. Construction of the map. We will be concerned with the fibre bundles

$$(*) \qquad \mathrm{SO}(2n+1) \xrightarrow{\alpha} \mathrm{SU}(2n+1) \xrightarrow{p_1} \mathrm{SU}(2n+1)/\mathrm{SO}(2n+1)$$

and

$$(**) \qquad \mathrm{Sp}(n) \xrightarrow{\beta} \mathrm{SU}(2n) \xrightarrow{p_2} \mathrm{SU}(2n)/\mathrm{Sp}(n).$$

Harris [3] showed that the maps

$$q_1\colon \mathrm{SU}(2n+1)/\mathrm{SO}(2n+1) \to \mathrm{SU}(2n+1)$$

and

$$q_2\colon \mathrm{SU}(2n)/\mathrm{Sp}(n) \to \mathrm{SU}(2n)$$

defined by $q_1p_1(x) \to x\cdot x^t$ and $q_2p_2(x) = x\cdot J\cdot x^t\cdot J^{-1}$ have the property that p_1q_1 and p_2q_2 induce $\mathcal{C}$ isomorphisms in homotopy, where $\mathcal{C}$ is the Serre class of 2-primary abelian groups. If we let $\mathcal{P}$ be the set of odd prime integers, the result of Harris implies that after $\mathcal{P}$-localization of spaces and maps, p_1q_1 and p_2q_2 induce isomorphisms of homotopy groups and are therefore homotopy equivalences [7, p. 405]. Let h_i be a ($\mathcal{P}$-local) homotopy inverse of the $\mathcal{P}$-localization of p_iq_i. Of course the localized maps q_ih_i can be deformed to cross-sections of the $\mathcal{P}$-local versions of $(*)$ and $(**)$.

LEMMA 3.1. *If W is a connected CW-complex, the maps of based homotopy sets*

$$\alpha_{\mathcal{P}*}\colon [W, \mathrm{SO}(2n+1)_{\mathcal{P}}] \to [W, \mathrm{SU}(2n+1)_{\mathcal{P}}]$$
$$\beta_{\mathcal{P}*}\colon [W, \mathrm{Sp}(n)_{\mathcal{P}}] \to [W, \mathrm{SU}(2n)_{\mathcal{P}}]$$

are monomorphisms of groups.

Proof. We give the proof for $\beta_{\mathcal{P}*}$; the proof for $\alpha_{\mathcal{P}*}$ is similar. We consider a portion of the long exact homotopy sequence of (∗∗):

$$\cdots \to \left[\sum W, \mathrm{SU}(2n)_{\mathcal{P}}\right] \underset{q_{2,\mathcal{P}*}}{\overset{p_{2,\mathcal{P}*}}{\rightleftarrows}} \left[\sum W, (\mathrm{SU}(2n)/\mathrm{Sp}(n))_{\mathcal{P}}\right],$$

$$\overset{d_*}{\to} [W, \mathrm{Sp}(n)_{\mathcal{P}}] \overset{\beta_{\mathcal{P}*}}{\to} [W, \mathrm{SU}(2n)_{\mathcal{P}}].$$

Since $p_{2,\mathcal{P}}q_{2,\mathcal{P}}$ is a homotopy equivalence, d_* is the trivial map and $\beta_{\mathcal{P}*}$, is injective. □

Let ψ be the composite monomorphism ψ: $\mathrm{Sp}(n) \overset{\beta}{\to} \mathrm{SU}(2n) \overset{j}{\to} \mathrm{SU}(2n+1)$, and $J' = j(J)$ so that $\overline{\psi(x)} = J' \cdot \psi(x) \cdot J'^{-1} = \psi(\bar{x})$.

PROPOSITION 3.2. *Let $\mathcal{P}$ be the set of odd primes.*

(i) *There is a map* Φ: $\mathrm{SO}(2n+1) \to \mathrm{Sp}(n)_{\mathcal{P}}$ *such that* $\beta_{\mathcal{P}}\Phi$ *is homotopic to* $\mathrm{SO}(2n+1) \overset{\phi}{\to} SU(2n) \overset{e_{\mathcal{P}}}{\to} SU(2n)_{\mathcal{P}}$.

(ii) *There is a map* Ψ: $\mathrm{Sp}(n) \to \mathrm{SO}(2n+1)_{\mathcal{P}}$ *such that* $\alpha_{\mathcal{P}}\Psi$ *is homotopic to* $\mathrm{Sp}(n) \overset{\psi}{\to} \mathrm{SU}(2n+1) \overset{e_{\mathcal{P}}}{\to} \mathrm{SU}(2n+1)_{\mathcal{P}}$.

Proof. Using a path in SU(2*n*) from J to the identity and the homotopy of ϕ with $\bar{\phi}$ of Proposition 2.1, we have

$$q_2 p_2 \phi = \phi \cdot J \cdot \phi^t \cdot J^{-1} \simeq \phi \cdot \phi^t \simeq \bar{\phi} \cdot \phi^t = I_{2n} \quad \text{(constant)}.$$

Thus (partially suppressing the subscript $\mathcal{P}$),

$$p_2 e_{\mathcal{P}} \phi \simeq h_2 p_2 q_2 p_2 e_{\mathcal{P}} \phi \simeq h_2 p_2 e_{\mathcal{P}}(q_2 p_2 \phi) \simeq \text{constant}.$$

By the covering homotopy property, there is a map Φ: $\mathrm{SO}(2n+1) \to \mathrm{Sp}(n)_{\mathcal{P}}$ so that $\beta_{\mathcal{P}}\Phi$ is homotopic to $e_{\mathcal{P}}\phi$.

Similarly,

$$q_1 p_1 \psi = \psi \cdot \psi^t \simeq \psi \cdot J' \cdot \psi^t \cdot J'^{-1} = I_{2n+1} \quad \text{(constant)}.$$

An analogous argument completes the proof of (ii). □

Note that this proposition implies that $\alpha_{\mathcal{P}}\Psi_{\mathcal{P}} \simeq \psi_{\mathcal{P}}$ and $\beta_{\mathcal{P}}\Phi_{\mathcal{P}} = \phi_{\mathcal{P}}$.

Since ψ: $\mathrm{Sp}(n) \to \mathrm{SU}(2n+1)$ is a homomorphism and the localization of an *H*-space is an *H*-space, we obtain

PROPOSITION 3.3. *The map* Ψ: $\mathrm{Sp}(n) \to \mathrm{SO}(2n+1)_{\mathcal{P}}$ *is an H-map of H-spaces.*

Proof. Let $\mu_G\colon G \times G \to G$ be multiplication. Then since $\alpha_{\mathcal{P}}$, $e_{\mathcal{P}}$ and ψ are H-maps,

$$\alpha_{\mathcal{P}}\Psi\mu_{\mathrm{Sp}} \simeq e_{\mathcal{P}}\psi\mu_{\mathrm{Sp}} \simeq \mu_{\mathrm{SU},\mathcal{P}}(e_{\mathcal{P}}\psi \times e_{\mathcal{P}}\psi) \simeq \mu_{\mathrm{SU},\mathcal{P}}(\alpha_{\mathcal{P}}\Psi \times \alpha_{\mathcal{P}}\Psi)$$
$$\simeq \alpha_{\mathcal{P}}\mu_{\mathrm{SO},\mathcal{P}}(\Psi \times \Psi).$$

Since $\alpha_{\mathcal{P}*}$ is injective, taking $W = \mathrm{Sp}(n) \times \mathrm{Sp}(n)$ in 3.1, we have $\Psi\mu_{\mathrm{Sp}} \simeq \mu_{\mathrm{SO},\mathcal{P}}(\Psi \times \Psi)$. □

COROLLARY 3.4. *The localized map* $\Psi_{\mathcal{P}}\colon \mathrm{Sp}(n)_{\mathcal{P}} \to \mathrm{SO}(2n+1)_{\mathcal{P}}$ *is an H-map of H-spaces.*

Proof. This follows by localizing the homotopy of 3.3. □

A proof for Φ analogous to the one above fails because ϕ is not a group homomorphism.

We are now ready to state the main result.

THEOREM 3.5. *If* $\mathcal{P}$ *is the set of odd primes there exist maps* Φ: $\mathrm{SO}(2n+1) \to \mathrm{Sp}(n)_{\mathcal{P}}$ *and* $\Psi\colon \mathrm{Sp}(n) \to \mathrm{SO}(2n+1)_{\mathcal{P}}$ *whose* $\mathcal{P}$*-localizations*

$$\Phi_{\mathcal{P}}\colon \mathrm{SO}(2n+1)_{\mathcal{P}} \to \mathrm{Sp}(n)_{\mathcal{P}},$$
$$\Psi_{\mathcal{P}}\colon \mathrm{Sp}(n)_{\mathcal{P}} \to \mathrm{SO}(2n+1)_{\mathcal{P}}$$

are inverse homotopy equivalences and H-maps.

Proof. By Proposition 3.2 we have a commutative diagram of homotopy sets

$$\begin{array}{ccc} [W, \mathrm{Sp}(n)_{\mathcal{P}}] & \xrightarrow{\beta_{\mathcal{P}*}} & [W, \mathrm{SU}(2n)_{\mathcal{P}}] \\ {\scriptstyle \Psi_{\mathcal{P}*}}\downarrow\uparrow{\scriptstyle \Phi_{\mathcal{P}*}} & & \downarrow {\scriptstyle j_{\mathcal{P}*}} \\ [W, \mathrm{SO}(2n+1)_{\mathcal{P}}] & \xrightarrow{\alpha_{\mathcal{P}*}} & [W, \mathrm{SU}(2n+1)_{\mathcal{P}}]. \end{array}$$

We have

$$\alpha_{\mathcal{P}} \simeq j_{\mathcal{P}}\phi_{\mathcal{P}} \simeq j_{\mathcal{P}}\beta_{\mathcal{P}}\Phi_{\mathcal{P}} \simeq \psi_{\mathcal{P}}\Phi_{\mathcal{P}} \simeq \alpha_{\mathcal{P}}\Psi_{\mathcal{P}}\Phi_{\mathcal{P}}.$$

Taking $W = \mathrm{SO}(2n+1)_{\mathcal{P}}$ and using brackets to denote homotopy class, we have $\alpha_{\mathcal{P}*}[1_{\mathrm{SO}(2n+1)_{\mathcal{P}}}] = \alpha_{\mathcal{P}*}[\Psi_{\mathcal{P}}\Phi_{\mathcal{P}}]$, or $\Psi_{\mathcal{P}}\Phi_{\mathcal{P}} \simeq 1_{\mathrm{SO}(2n+1)_{\mathcal{P}}}$, since $\alpha_{\mathcal{P}*}$ is injective by 3.1. Thus $\Psi_{\mathcal{P}*}\Phi_{\mathcal{P}*}$ is the identity, $\Psi_{\mathcal{P}*}$ is surjective and $\Phi_{\mathcal{P}*}$ is injective.

Now

$$j_{\mathcal{P}}\beta_{\mathcal{P}} \simeq \psi_{\mathcal{P}} \simeq \alpha_{\mathcal{P}}\Psi_{\mathcal{P}} \simeq j_{\mathcal{P}}\phi_{\mathcal{P}}\Psi_{\mathcal{P}} \simeq j_{\mathcal{P}}\beta_{\mathcal{P}}\Phi_{\mathcal{P}}\Psi_{\mathcal{P}}.$$

Take $W = S^k$ so the sets are homotopy groups, and

$$j_{\mathcal{P}*}\beta_{\mathcal{P}*} = j_{\mathcal{P}*}\beta_{\mathcal{P}*}(\Phi\Psi)_{\mathcal{P}*}\colon \pi_k(\mathrm{Sp}(n)_{\mathcal{P}}) \to \pi_k(\mathrm{SU}(2n+1)_{\mathcal{P}}).$$

Since $\beta_{\mathcal{P}*}$ is a monomorphism and $j_{\mathcal{P}*}$ is an isomorphism for $k < 4n$, $(\Phi_{\mathcal{P}}\Psi_{\mathcal{P}})_*$ is the identity on homotopy groups in dimensions $k < 4n$. By the results of Harris [3], $\pi_k(\mathrm{SO}(2n+1)_{\mathcal{P}})$ and $\pi_k(\mathrm{Sp}(n)_{\mathcal{P}})$ are finite groups of the same order in dimension $k \geq 4n$. Since $\Psi_{\mathcal{P}*}$ is an epimorphism, it is an isomorphism (as is $\Phi_{\mathcal{P}*}$). Thus $\Phi_{\mathcal{P}}$ and $\Psi_{\mathcal{P}}$ induce isomorphisms on homotopy groups, and are therefore homotopy equivalences. But $\Phi_{\mathcal{P}}$ is a right homotopy inverse for the homotopy equivalence $\Psi_{\mathcal{P}}$, hence is a left homotopy inverse for $\Psi_{\mathcal{P}}$ and $\Phi_{\mathcal{P}}\Psi_{\mathcal{P}} \simeq 1_{\mathrm{Sp}(n)_{\mathcal{P}}}$.

Finally, since $\Phi_{\mathcal{P}}$ is a homotopy inverse for $\Psi_{\mathcal{P}}$ and $\Psi_{\mathcal{P}}$ is an H-map, $\Phi_{\mathcal{P}}$ is an H-map. □

REMARKS. Since $\Phi = \Phi_{\mathcal{P}}e_{\mathcal{P}}$ and both $\Phi_{\mathcal{P}}$ and $e_{\mathcal{P}}$ are H-maps, Φ is an H-map.

By Theorem 6.6 of [4], there exist maps $\Phi'\colon \mathrm{SO}(2n+1) \to \mathrm{Sp}(n)$ and $\Psi'\colon \mathrm{Sp}(n) \to \mathrm{SO}(2n+1)$ so that $\Phi'_{\mathcal{P}}$ and $\Psi'_{\mathcal{P}}$ are homotopy equivalences. We do not know if Φ' and Ψ' can be chosen to be H-maps or if they can be delooped to maps on the classifying spaces.

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UNIVERSITY OF COLORADO
BOULDER, CO 80309