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CHARACTERIZATIONS OF $\aleph$-SPACES

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Two simultaneous generalizations of metric spaces and $\aleph_0$-spaces, the $\aleph$-spaces introduced by O'Meara and the cs-σ-spaces of Guthrie, are shown to be the same.

It was shown by Guthrie [2] that a regular space is an $\aleph_0$-space if and only if it has a countable cs-network (see definitions below). We show here that, in parallel manner, O'Meara's $\aleph$-spaces may be characterized as the regular spaces admitting σ-locally finite cs-networks; that is, the classes of $\aleph$-spaces and cs-σ-spaces coincide. While this equivalence has been proved by Guthrie [3] for paracompact spaces, the fact that these classes contain non-paracompact examples [6] makes our result an honest improvement.

DEFINITION 1. A collection $\mathcal{P}$ of subsets of a topological space X is a *k-network* for X if, given any compact subset C of X and any neighborhood U of C, there is a finite subcollection $\mathcal{P}^*$ of $\mathcal{P}$ so that $C \subset \bigcup \mathcal{P}^* \subset U$. A collection $\mathcal{P}$ is a cs-*network* for X if, given any sequence σ converging to $x \in X$ and any neighborhood U of x, there is a $P \in \mathcal{P}$ so that $P \subset U$ and σ is eventually in P. A regular space is an $\aleph_0$-*space* [5] ($\aleph$-*space* [6], [7], cs-σ-*space* [3]) if it has a countable k-network (σ-locally finite k-network, σ-locally finite cs-network); because of regularity, these collections can be chosen to consist of closed sets.

We say that a subset W of a topological space X is a *sequential neighborhood* of a subset F of W if every sequence converging to a member of F is eventually in W.

LEMMA 2. *A discrete family* $\{F_\alpha: \alpha \in A\}$ *of subsets of an* $\aleph$-*space* X *admits a pairwise disjoint family* $\{W_\alpha: \alpha \in A\}$ *of sequential neighborhoods.*

Proof. For every $n < \omega$, let $\mathcal{P}_n$ be a locally finite collection of closed sets so that $\bigcup_{n<\omega} \mathcal{P}_n$ is a k-network for X. For $n < \omega$ and $B \subset A$, let

$$T(n, B) = \bigcup \{P \in \mathcal{P}_n : P \cap \bigcup \{F_\alpha : \alpha \in B\} = \varnothing\}.$$

For every $\alpha \in A$, let

$$W_\alpha = \bigcup_{n<\omega} [T(n, A\backslash\{\alpha\})\backslash T(n, \{\alpha\})].$$

It is simple to verify that the W_α's are pairwise disjoint. To see that W_α is a sequential neighborhood of F_α, note that for a sequence σ converging to a member of F_α there is an $n < \omega$ so that σ is eventually in $T(n, A\backslash\{\alpha\})$; hence σ is eventually in $T(n, A\backslash\{\alpha\})\backslash T(n, \{\alpha\}) \subset W_\alpha$.

LEMMA 3. *Assume X has a point-countable k-network $\mathcal{P}$ of closed sets so that $\mathcal{P}$ is closed under finite intersections. If $x \in X$, if W is a sequential neighborhood of x, and if σ is a sequence converging to x, then there is a finite subset $\mathcal{P}^*$ of $\mathcal{P}$ so that $\bigcup\mathcal{P}^* \subset W$ and $\bigcup\mathcal{P}^*$ contains a tail of σ.*

Proof. Let $\{\mathcal{P}_n: n < \omega\}$ be the family of all finite subsets $\mathcal{P}^*$ of $\mathcal{P}$ such that $x \in \bigcap\mathcal{P}^*$ and σ is eventually in $\bigcup\mathcal{P}^*$. If no finite subset of $\mathcal{P}$ satisfies the conclusion of the lemma, then we could find a $y_n \in \bigcap_{i\leq n}(\bigcup\mathcal{P}_i)\backslash W$ for every $n < \omega$. This sequence $\{y_n: n < \omega\}$ converges to x; indeed, if U is a neighborhood of x, we could find a $\mathcal{P}_m$ so that $\{y_n: n \geq m\} \subset \bigcup\mathcal{P}_m \subset U$. The convergence of $\{y_n: n < \omega\}$ contradicts that W is a sequential neighborhood of x.

THEOREM 4. *The following are equivalent for a regular space X.*
(a) *X has a σ-discrete* cs-*network.*
(b) *X has a σ-discrete k-network.*
(c) *X has a σ-locally finite* cs-*network.*
(d) *X has a σ-locally finite k-network.*

Proof. It is clear that (a) implies (c) and (b) implies (d). As Guthrie observed in [3], his proof of the countable case in [2] can be adapted to show (c) implies (d), and the same is true for (a) implies (b). It therefore suffices to show (d) implies (a).

For every $m < \omega$ let $\mathcal{P}_m$ be a locally finite collection of closed sets (our only use of regularity) which is closed under finite intersections, so that $\mathcal{P}_m \subset \mathcal{P}_{m+1}$ and $\mathcal{P} = \bigcup_{m<\omega}\mathcal{P}_m = \{P_\alpha: \alpha \in A\}$ is a k-network for X.

For each m let $\mathcal{U}_m$ be an open cover of X that witnesses the local finiteness of $\mathcal{P}_m$. Since a space X satisfying (d) is clearly subparacompact [1], it follows from [1] that $\mathcal{U}_m$ has a σ-discrete closed refinement $\bigcup_{n<\omega}\{F_\beta: \beta \in B_{m,n}\}$, where $\{F_\beta: \beta \in B_{m,n}\}$ is discrete for each n. It follows that, if $\beta \in \bigcup_{n<\omega} B_{m,n}$, then $F_\beta \cap P_\alpha \neq \varnothing$ for only finitely many $P_\alpha \in \mathcal{P}_m$.

By Lemma 2 we can find, for every $\langle m, n\rangle \in \omega^2$, a pairwise disjoint family $\{W_\beta: \beta \in B_{m,n}\}$ of sequential neighborhoods for $\{F_\beta: \beta \in B_{m,n}\}$.

For every pair $\langle m, n\rangle \in \omega^2$ let

$$C_{m,n} = \{\langle \alpha, \beta\rangle : P_\alpha \in \mathcal{P}_m, \beta \in B_{m,n}, P_\alpha \cap F_\beta \neq \varnothing\}.$$

Let us check that the collection $\{P_\alpha \cap W_\beta: \langle \alpha, \beta\rangle \in C_{m,n}\}$ is star-finite. Indeed, if $\langle \alpha, \beta\rangle \in C_{m,n}$ and $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \varnothing$ (where $\langle \gamma, \delta\rangle \in C_{m,n}$), the fact that β and δ are in $B_{m,n}$ with $W_\beta \cap W_\delta \neq \varnothing$ forces $\beta = \delta$. Consequently, $\langle \gamma, \beta\rangle \in C_{m,n}$; it follows that $P_\gamma \cap F_\beta \neq \varnothing$. So P_γ is one of the finitely many members of $\mathcal{P}_m$ which meets F_β. So there are only finitely many pairs $\langle \gamma, \delta\rangle \in C_{m,n}$ for which $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \varnothing$.

Fix $\langle m, n\rangle \in \omega^2$. Now if $\langle \alpha, \beta\rangle \in C_{m,n}$ and $r < \omega$, let

$$S(\alpha, \beta, r) = \bigcup \{P_\alpha \cap P_\gamma: P_\gamma \in \mathcal{P}_r \text{ and } P_\gamma \subset W_\beta\}$$

and

$$\mathcal{S}(m, n, r) = \{S(\alpha, \beta, r): \langle \alpha, \beta\rangle \in C_{m,n}\}.$$

Since $S(\alpha, \beta, r) \subset P_\alpha \cap W_\beta$ for every $r < \omega$, the collections $\mathcal{S}(m, n, r)$ inherit the star-finite property from $\{P_\alpha \cap W_\beta: \langle \alpha, \beta\rangle \in C_{m,n}\}$. Note too that every member of $\mathcal{S}(m, n, r)$ is the union of a subcollection of the locally finite collection $\{P_\alpha \cap P_\gamma: P_\alpha \in \mathcal{P}_m, P_\gamma \in \mathcal{P}_r\}$ and thus $\mathcal{S}(m, n, r)$ is closure-preserving. Because a star-finite collection of sets is σ-disjoint and because a disjoint and closure-preserving collection of closed sets is discrete, we have that $\mathcal{S}(m, n, r)$ is σ-discrete.

Thus $\mathcal{S} = \bigcup\{\mathcal{S}(m, n, r): \langle m, n, r\rangle \in \omega^3\}$ is σ-discrete; write $\mathcal{S} = \bigcup_{k<\omega} \mathcal{S}_k$ so that every $\mathcal{S}_k$ is a discrete collection of closed sets and $\mathcal{S}_j \cap \mathcal{S}_k = \varnothing$ if $j \neq k$. Let

$$\mathbf{F} = \{\mathcal{F}: \mathcal{F} \text{ is a finite subset of } \mathcal{S}, \bigcap \mathcal{F} \neq \varnothing\},$$

and for every finite subset Φ of ω, let

$$\mathbf{F}_\Phi = \{\mathcal{F} \in \mathbf{F}: \{k < \omega: \mathcal{F} \cap \mathcal{S}_k \neq \varnothing\} = \Phi\}.$$

Note that for a particular $k < \omega$, a collection $\mathcal{F} \in \mathbf{F}$ may contain at most one member of $\mathcal{S}_k$, as $\mathcal{S}_k$ is pairwise disjoint.

Now for a given finite subset Φ of ω consider the collection $\{\bigcap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$. It is locally finite because it is comprised of finite intersections of the locally finite family $\bigcup_{k\in\Phi} \mathcal{S}_k$. It is also pairwise disjoint: if $\mathcal{F}_1 \neq \mathcal{F}_2$ are members of $\mathbf{F}_\Phi$, then $\mathcal{F}_1 \cap \mathcal{S}_k \neq \mathcal{F}_2 \cap \mathcal{S}_k$ for some $k \in \Phi$; i.e. if $\{S_1\} = \mathcal{F}_1 \cap \mathcal{S}_k$ and $\{S_2\} = \mathcal{F}_2 \cap \mathcal{S}_k$, then $S_1 \neq S_2$. Pairwise disjointness of $\mathcal{S}_k$ gives $S_1 \cap S_2 = \varnothing$, and thus $(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) = \varnothing$. So $\{\bigcap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$ is both pairwise disjoint and a locally finite collection of closed sets; therefore it is discrete.

Again we apply Lemma 2 to find, for every finite subset Φ of ω, a pairwise disjoint family $\{V(\mathcal{F}): \mathcal{F} \in \mathbf{F}_\Phi\}$ of sequential neighborhoods of $\{\bigcap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$. For $j < \omega$ and $\mathcal{F} \in \mathbf{F}_\Phi$ let

$$V(\mathcal{F}, j) = \bigcup \left\{S \cap P_\delta: S \in \mathcal{F}, P_\delta \in \mathcal{P}_j, P_\delta \subset V(\mathcal{F})\right\}.$$

Now $V(\mathcal{F}, j) \subset V(\mathcal{F})$, so for a fixed $j < \omega$ the collection

$$\mathcal{V}(\Phi, j) = \{V(\mathcal{F}, j): \mathcal{F} \in \mathbf{F}_\Phi\}$$

is pairwise disjoint. Further, every $V(\mathcal{F}, j) \in \mathcal{V}(\Phi, j)$ is the union of a subcollection of the locally finite family of closed sets $\{S \cap P_\delta: S \in \bigcup_{k \in \Phi} \mathcal{S}_k, P_\delta \in \mathcal{P}_j\}$. Hence

$$\begin{aligned}\mathcal{V} &= \{V(\mathcal{F}, j): \mathcal{F} \in \mathbf{F}, j < \omega\} \\ &= \bigcup \{\mathcal{V}(\Phi, j): \Phi \text{ is a finite subset of } \omega, j < \omega\}\end{aligned}$$

is σ-discrete. We will now verify that $\mathcal{V}$ is a cs-network for X.

Suppose U is open and σ is a sequence converging to $x \in U$. Because $\mathcal{P}$ is a k-network for X, we can find an $m < \omega$ and a finite subset $\mathcal{P}_m^*$ of $\mathcal{P}_m$ so that $\bigcup \mathcal{P}_m^* \subset U$, σ is eventually in $\bigcup \mathcal{P}_m^*$, and, because the members of $\mathcal{P}$ are closed, we may choose such a $\mathcal{P}_m^*$ so that $x \in \bigcap \mathcal{P}_m^*$.

Since $X \subset \bigcup_{n<\omega}\{F_\beta: \beta \in B_{m,n}\}$, we can find an $n < \omega$ and a $\beta \in B_{m,n}$ so that $x \in F_\beta$. Now W_β is a sequential neighborhood of F_β, hence of x, so by applying Lemma 3 we can find an $r < \omega$ and a finite subset $\mathcal{P}_r^{**}$ of $\mathcal{P}_r$ so that $\bigcup \mathcal{P}_r^{**} \subset W_\beta$ and σ is eventually in $\bigcup \mathcal{P}_r^{**}$. Because the members of $\mathcal{P}$ are closed, necessarily $x \in \bigcup \mathcal{P}_r^{**}$.

If $P_\alpha \in \mathcal{P}_m^*$, the fact that $x \in P_\alpha \cap F_\beta$ implies $\langle \alpha, \beta \rangle \in C_{m,n}$. If, in addition, $P_\gamma \in \mathcal{P}_r^{**}$, then $P_\gamma \in \mathcal{P}_r$ and $P_\gamma \subset W_\beta$, and thus $P_\alpha \cap P_\gamma \subset S(\alpha, \beta, r)$. From this we see that $(\bigcup \mathcal{P}_m^*) \cap (\bigcup \mathcal{P}_r^{**}) \subset \bigcup \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$ and, because there is a γ so that $x \in P_\gamma \in \mathcal{P}_r^{**}$, that $x \in \bigcap \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$. Let $\mathcal{F} = \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$ (a finite subset of $\mathcal{S}$). The previous sentence implies σ is eventually in $\bigcup \mathcal{F}$ (since σ is eventually in $\bigcup \mathcal{P}_m^* \cap \bigcup \mathcal{P}_r^{**}$) and $\bigcap \mathcal{F} \neq \varnothing$ (since $x \in \bigcap \mathcal{F}$). In particular, $\mathcal{F} \in \mathbf{F}$.

As $V(\mathcal{F})$ is a sequential neighborhood of $\bigcap \mathcal{F}$, hence of x, Lemma 3 enables us to find a $j < \omega$ and a finite subset $\mathcal{P}_j^{***}$ of $\mathcal{P}_j$ so that $\mathcal{P}_j^{***} \subset V(\mathcal{F})$ and σ is eventually in $\bigcup \mathcal{P}_j^{***}$.

Now if $P_\delta \in \mathcal{P}_j^{***}$, then $P_\delta \in \mathcal{P}_j$ and $P_\delta \subset V(\mathcal{F})$; therefore for any $S \in \mathcal{F}$ we have $S \cap P_\delta \subset V(\mathcal{F}, j)$. It follows that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{P}_j^{***}) \subset V(\mathcal{F}, j)$. As a result, σ is eventually in $V(\mathcal{F}, j)$.

In addition,

$$V(\mathcal{F}, j) \subset \cup \mathcal{F} = \cup \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\} \subset \cup \mathcal{P}_m^* \subset U.$$

So $\mathcal{V}$ is a cs-network for X.

Our Theorem 4, taken with Theorem 2 of [3], gives the following answer to Michael's question in [4].

COROLLARY 5. *If X is an $\aleph_0$-space and Y is an $\aleph$-space, then the space of continuous functions from X to Y equipped with the compact-open topology is an $\aleph$-space.*

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