PACIFIC JOURNAL OF MATHEMATICS
Vol. 110, No. 1, 1984

# COLORINGS OF HYPERMAPS AND A CONJECTURE OF BRENNER AND LYNDON

GIULIANA BIANCHI AND ROBERT CORI

**In this paper the following result is obtained: Let $\alpha$ and $\beta$ be two permutations such that $\alpha\beta$ is transitive and $\alpha^p = \beta^q = 1$ (where $p$ and $q$ are distinct primes). Then the set of all permutations commuting both with $\alpha$ and $\beta$ is either reduced to the identity or one of the three cyclic groups $C_p$, $C_q$ or $C_{pq}$.**

**Introduction.** In this paper we answer a question raised by J. L. Brenner and R. C. Lyndon in [**1**]. They consider a pair of permutations $(\alpha, \beta)$ acting on a finite set of $n$ elements such that $\alpha^3 = \beta^2 = 1$ and $\alpha\beta$ is transitive. Such a pair may be considered as a (combinatorial) map with exactly one face in the terminology of [**2**], [**4**], [**6**] and [**8**], Brenner and Lyndon computed the automorphism group of such a map (which is necessarily a cyclic group) for $n \leq 12$. The groups they find are 1, $C_2$, $C_3$ and $C_6$ and they conjectured that no other groups can arise.

In what follows we prove a more general result and show that if $\alpha\beta$ is transitive and if $p$ and $q$ are primes ($p \neq q$) such that $\alpha^p = \beta^q = \mathbf{1}$ then the automorphism group of $(\alpha, \beta)$ is one of $\mathbf{1}$, $C_p$, $C_q$, $C_{pq}$. It remains an open question to know whether $C_{pq}$ can be found for arbitrary large values of $n$ ($n \gg pq$)

Our main tool is the introduction of the concept of colorings of a hypermap. These colorings count in a certain way the number of fixed points of an automorphism of $(\alpha, \beta)$ when it acts on the set of cells (i.e. orbits of $\alpha$, $\beta$ and $\alpha\beta$). One step in the proof is to show that an automorphism of prime order cannot have exactly one fixed point in the set of cells: such a result is well known in the theory of Riemann surfaces ([**5**], p. 266).

All the permutations we consider act on a finite set $\Omega$ of $n$ elements. We will also use the following conventions:

The product $\alpha\beta$ of two permutations $\alpha$ and $\beta$ is the permutation defined by $\alpha\beta(x) = \alpha(\beta(x))$; for a subset $\Omega'$ of $\Omega$, $\alpha\Omega'$ denotes the set $\{\alpha x \mid x \in \Omega'\}$, which has the same cardinality as $\Omega'$; a permutation $\alpha$ is *regular* if all its orbits have the same length, which is also the order of $\alpha$; the number of orbits of the permutation $\theta$ will be denoted by $z(\theta)$; a permutation is transitive if $z(\theta) = 1$.

A hypermap is a pair $(\alpha, \beta)$ of permutations such that the group $\langle \alpha, \beta \rangle$ generated by them is transitive on $\Omega$. The orbits of $\alpha$, $\beta$ and $\alpha\beta$ are the cells of the hypermap.

An automorphism of $(\alpha, \beta)$ is an element $\varphi$ of $\mathrm{Sym}(\Omega)$ that commutes with $\alpha$ and $\beta$. By the transitivity of $\langle \alpha, \beta \rangle$ for any $x$ and $y$ in $\Omega$ there exists $\theta$ in $\langle \alpha, \beta \rangle$ such that $x = \theta y$ and as for any integer $k$, $\varphi^k(x) = \theta\varphi^k(x)$ we have

$$\varphi^k x = x \quad \text{if and only if} \quad \varphi^k y = y;$$

hence an automorphism of $(\alpha, \beta)$ is a regular permutation.

In order to study the automorphism group of a hypermap we are led to examine for a given permutation $\theta$ the set of regular permutations $\varphi$ commuting with $\theta$. This will be done in detail in the next paragraph.

**I. Commuting permutations.** We state here for later use some elementary facts about a pair of commuting permutations $\alpha$ and $\beta$ of a finite set. Throughout this section it will be assumed that $\alpha$, $\beta$ act on a finite set $\Omega$ of $n$ elements and that the group $\langle \alpha, \beta \rangle$ generated by $\alpha$ and $\beta$ is abelian.

We write $\Omega/\alpha$ for the set of $\alpha$-orbits. As $\alpha$ and $\beta$ commute, the actions of $\alpha$, $\beta$ on $\Omega$ induce actions of $\alpha$ on $\Omega/\beta$ and of $\beta$ on $\Omega/\alpha$.

LEMMA I.1. *If $G = \langle \alpha, \beta \rangle$ is transitive, then any element $\theta$ of $G$ is regular.*

*Proof*. For any $x$ and $y$ in $\Omega$ there exists $\varphi$ in $G$ such that $y = \varphi x$, since $\theta^m x = x$ and as $\langle \alpha, \beta \rangle$ is abelian, $\theta^m y = \varphi\theta^m x = y$.

LEMMA I.2. *If $G = \langle \alpha, \beta \rangle$ is transitive on $\Omega$, then $\alpha$ is transitive on $\Omega/\beta$, and $G$ is also transitive on the set of all intersections $A \cap B$ for $A \in \Omega/\alpha$, $B \in \Omega/\beta$. Therefore these intersections all have the same cardinality.*

*Proof*. The first statement is clear. If $A, A' \in \Omega/\alpha$ and $B, B' \in \Omega/\beta$, then $A' = \beta^k A$ and $B' = \alpha^h B$ for some $h$ and $k$ in $Z$. Then

$$\alpha^h\beta^k(A \cap B) = \alpha^h(A' \cap B) = A' \cap B'.$$

LEMMA I.3. *Let $r$ be the common value of $|A \cap B|$, $n = |\Omega|$, let $a$, $b$ be the orders of $\alpha$ and $\beta$. Then there exist $a_1$, $b_1$ such that $n = a_1 b_1 r$, $a = a_1 r$, $b = b_1 r$. If $b$ is prime then $|\Omega/\alpha| = 1$ or $b$.*

*Proof.* As any $A$ and $B$ are both unions of $A_i \cap B_j$, $r$ divides $a$ and $b$, so that $a = a_1 r$, $b = b_1 r$. Since $\alpha$ and $\beta$ are regular $|\Omega/\alpha| = n/a$, $|\Omega/\beta| = n/b$ and there are $n^2/ab$ disjoint intersections $A \cap B$. Thus $n = r \cdot (n^2/ab)$ and $n = ab/r = a_1 b_1 r$. If $b$ is prime then $r = 1$ or $b$ and $n/a = b$ or 1.

LEMMA I.4. *If $\langle \alpha, \beta \rangle$ is transitive, and $a$, $b$, $r$ are as above, then there exists an integer $k$ relatively prime with $r$ such that $\alpha^{n/b} = \beta^{nk/a}$.*

*Proof.* Since $\alpha$ is transitive on $\Omega/\beta$, and $|\Omega/\beta| = n/b$ then $\alpha^{n/b}$ stabilizes each $B \in \Omega/\beta$; it also stabilizes each $A \cap B$ as $\alpha A = A$. As $\alpha$ is transitive on $A$ of length $a$, $\alpha^{n/b} = \alpha^{a/r}$ is transitive on $C = A \cap B$. Similarly $\beta^{n/a}$ is transitive on $C$. For a particular $C$ the restrictions of $\alpha^{n/b}$ and $\beta^{n/a}$ to $C$ generate the same cyclic group of order $r$, then for some $k$ such that $(k, r) = 1$, $\alpha^{n/b}$ and $\beta^{nK/a}$ have the same action on $C$. Thus the element $\alpha^{n/b}\beta^{-nk/a}$ of $\langle \alpha, \beta \rangle$ has at least one fixed point by I.1, it is the identity.

**II. Colorings.** Throughout this section we assume that $\varphi$ is a regular permutation of order $m$ acting on a finite set $\Omega$ of $n$ elements.

A *coloring* on the set $\Omega$ is a map $\lambda$ defined on $\Omega$ with values in an abelian group $R$. For any permutation $\alpha$ and any coloring $\lambda$ of $\Omega$ we define another coloring $D_\alpha\lambda$ by setting

$$D_\alpha\lambda(x) = \lambda(\alpha(x)) - \lambda(x).$$

A coloring is said to be *orthogonal* to $\alpha$ if $D_\alpha\lambda$ is constant on $\Omega$. In this case $\lambda(\alpha^k(x)) = \lambda(x) + k \cdot u$ where $u$ is the constant value of $D_\alpha\lambda$. The length $l$ of an orbit of $\alpha$ must verify $lu = 0$ in the abelian group. As we will only consider colorings orthogonal to $\varphi$, we will assume that $R$ is the additive group $Z/mZ$. Thus the relation $mu = 0$ is satisfied for any $u$.

We are now interested in the extension of a coloring vanishing on a transversal $T$ of $\Omega/\varphi$, and having a given value $v$ on an element $x$ not in $T$. For such an $x$ there exists a unique $\bar{x}$ in $T$ and an integer $h$ $(1 \leq h \leq m)$ such that $\varphi^h(\bar{x}) = x$.

LEMMA II.1. *For $v$ in $Z/mZ$, there exists a coloring $\lambda$ orthogonal to $\varphi$, vanishing on $T$ and such that $\lambda(x) = v$ if and only if the equation in $u$, $hu \equiv v$, has a solution in $Z/mZ$.*

*Proof.* If $D_\alpha\lambda$ is a constant $u$, then $\lambda(x) = \lambda(\bar{x}) + hu$ so that $hu = v$. If this equation has a solution $u_0$ say, then for any $y$ in $\Omega$ there exists $\bar{y}$ in $T$ such that $y = \varphi^l(\bar{y})$; setting $\lambda(y) = lu_0$ we obtain the coloring $\lambda$. □

LEMMA II.2. *Let $\langle \varphi, \alpha \rangle$ be abelian and $\lambda$ be a coloring orthogonal to $\varphi$. Then $D_\alpha\lambda$ is constant on the orbits of $\varphi$.*

*Proof.* We have to show that $D_\alpha\lambda(\varphi x) = D_\alpha\lambda(x)$. But as $D_\alpha\lambda(\varphi(x)) = \lambda\alpha\varphi x - \lambda\varphi x$ and since $\alpha$ and $\varphi$ commute:

$$\begin{aligned} D_\alpha\lambda\varphi(x) &= \lambda\varphi\alpha x - \lambda\alpha x + \lambda\alpha x - \lambda x + \lambda x - \lambda\varphi x \\ &= D_\varphi\lambda(\alpha x) + D_\alpha\lambda(x) - D_\varphi\lambda(x). \end{aligned}$$

As $D_\varphi\lambda$ is constant, also the result follows. Remark that $D_\alpha\lambda$ defines a coloring on $\Omega/\varphi$. For $A$ in $\Omega/\varphi$, $D_\alpha\lambda(A)$ denotes the common value of $D_\alpha\lambda(x)$ for $x$ in $A$.

LEMMA II.3. *Let $\langle \varphi, \alpha \rangle$ be abelian and transitive on $\Omega$. Then there exists a coloring $\lambda$ orthogonal to $\varphi$, such that*

$$\sum_{A \in \Omega/\varphi} D_\alpha\lambda(A) \equiv z(\alpha) \quad \text{in } Z/mZ.$$

*Proof.* Let $|\Omega| = n$, $\alpha$ have order $a$, and let $r$ be the cardinality of the intersection of an orbit of $\alpha$ with one of $\varphi$. As $\alpha$ is transitive on $\Omega/\varphi$ there exists $x$ such that $T = \{x, \alpha x, \dots, \alpha^{n/m-1}x\}$ is a transversal of $\Omega/\varphi$. Let $y = \alpha^{n/m}x$; we claim that there exists $\lambda$ vanishing on $T$ and such that $\lambda(y) = z(\alpha) = n/a$.

By Lemma I.4 there exists $k$ such that $\varphi^{n/a \cdot k} = \alpha^{n/m}$; then $y = \varphi^{n/a \cdot k}(x)$. By II.1 such a $\lambda$ exists if the equation

$$nku/a \equiv n/a$$

has a solution in $Z/mZ$.

But since $(k, r) = 1$ there exist $u$, $v$, such that $uk + vr = 1$. Then

$$nku/a + nvr/a = n/a$$

and as $nr/a = m$ (I.3), we are done.

LEMMA II.4. *Let $G = \langle \varphi, \alpha \rangle$ be abelian. Then there exists a coloring $\lambda$ such that $D_\varphi\lambda$ is constant on G-orbits and such that*

$$\sum_{A \in \Omega/\varphi} D_\alpha\lambda(A) = z(\alpha).$$

*Moreover if $\varphi$ is of prime order and fixes only one orbit of $\alpha$ then $\lambda$ can be found orthogonal to $\varphi$.*

*Proof*. By Lemma II.3 for any $G$-orbit $\Omega_h$ there exists a coloring $\lambda_h$ such that $D_\varphi\lambda_h$ is constant on $\Omega_h$ and

$$\sum_{A\in\Omega_h/\varphi} D_\alpha\lambda_h(A) = z(\alpha_h)$$

where $\alpha_h$ is the restriction of $\alpha$ to $\Omega_h$. Taking for $\lambda$ the union of the $\lambda_h$ we have the result, since $z(\alpha) = \Sigma\, z(\alpha_h)$. If $\varphi$ is of prime order, then by I.3 $|\Omega_h/\alpha| = 1$ or $m$. In the first case, $\Omega_h$ is an $\alpha$ orbit fixed by $\varphi$. This occurs only once, for $h_0$ say; the equation to solve in $\Omega_{h_0}$ is $ku \equiv 1 \pmod m$ which gives $u = k^{-1}$ in $Z/mZ$. In the second case $|\Omega_h/\alpha| = m$ and the equation to solve is $mk'u \equiv m \pmod m$ which is satisfied by any $u$, in particular for $u = k^{-1}$. We thus can choose $\lambda$ such that $D_\alpha\lambda = u$ on any $\Omega_h \cdot D_\alpha\lambda$ is thus constant.

LEMMA II.5. *Let $G = \langle\varphi, \alpha\rangle$ be abelian and such that the intersection $A_i \cap B_j$ of an orbit of $\varphi$ with one of $\alpha$ contains at most one element. Then for any coloring $\lambda$ orthogonal to $\varphi$ we have*

$$\sum_{A\in\Omega/\varphi} D_\alpha\lambda(A) = 0.$$

*Proof*. It suffices to show that the sum vanishes on each $\alpha$ orbit in $\Omega/\varphi$. Let $C$ be such an orbit; under the hypothesis of the lemma, there exists an orbit $\Gamma$ in $\Omega$ of length $|C|$ and $\Sigma_{c\in C} D_\alpha\lambda(c) = \Sigma_{x\in\Gamma} D_\alpha\lambda(x)$. But $\Gamma = \{x, \alpha x, \dots, \alpha^k x\}$ and the last sum is $\Sigma_{i=0}^{k}(\lambda\alpha^{i+1}x - \lambda\alpha^i x)$ which vanishes as $\alpha^{k+1}x = x$.

LEMMA II.6. *Let $\alpha$ and $\beta$ be any two permutations commuting with $\varphi$, then for any coloring $\lambda$ orthogonal to $\varphi$ one has*

$$\sum_{A\in\Omega/\varphi} D_{\alpha\beta}\lambda(A) = \sum_{A\in\Omega/\varphi} D_\alpha\lambda(A) + \sum_{A\in\Omega/\varphi} D_\beta\lambda(A).$$

Let $\Gamma$ be any subset of $\Omega$ having exactly one element in each cycle of $\varphi$. Then

$$\sum_{A\in\Omega/\varphi} D_{\alpha\beta}\lambda(A) = \sum_{x\in\Gamma} D_{\alpha\beta}\lambda(x).$$

Since $D_{\alpha\beta}\lambda(a) = \lambda(\alpha\beta(a)) - \lambda(\beta(a)) + \lambda(\beta(a)) - \lambda(a)$ we have

$$\sum_{A\in\Omega/\varphi} D_{\alpha\beta}\lambda(A) = \sum_{x'\in\beta(\Gamma)} D_\alpha\lambda(x') + \sum_{x\in\gamma} D_\beta\lambda(x).$$

But $\beta(\Gamma)$ is also a subset of $\Omega$ having one element in each cycle of $\varphi$ and the result follows.

## III. The main theorems.

THEOREM 1. *Let $H = (\alpha, \beta)$ be a hypermap $\varphi$ an automorphism of $H$ of prime order $p$. Then the number of cells fixed by $\varphi$ is necessarily different from one.*

*Proof.* Let us show that the assumption that $\varphi$ fixes exactly one cell leads to a contradiction. Suppose that this cell is an orbit of $\alpha$ (a similar proof holds for an orbit of $\beta$ or $\alpha\beta$). If $\varphi$ fixes no other cycle of $\alpha$ then $z(\alpha) - 1$ is clearly divisible by $p$. Then by Lemma II.4 there exists a coloring orthogonal to $\varphi$ such that $\Sigma_{A \in \Omega/\varphi} D_\alpha \lambda(A) \equiv 1 \pmod{p}$, but by Lemma II.6.

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) = \sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) + \sum_{A \in \Omega/\varphi} D_\beta \lambda(A)$$

and Lemma II.5 insures the nullity of $\Sigma_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A)$ and $\Sigma_{A \in \Omega/\varphi} D_\beta \lambda(A)$. As no cycle of either $\beta$ or $\alpha\beta$ is fixed by $\varphi$, we have thus found a contradiction and Theorem II.1 is proved.

LEMMA III.1 *Let $\varphi$ be a permutation of order $p^2$ commuting with $\alpha$ of order $p$. Then for any $\lambda$ orthogonal to $\varphi$*

$$p \sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) \equiv 0 \pmod{p^2}.$$

*Proof.* We can assume that $\langle \varphi, \alpha \rangle$ acts transitively on $\Omega$; the general case is then obtained by summing over the orbits of $\langle \varphi, \alpha \rangle$.

Since $\alpha$ is of order $p$, by Lemma I.3 the cardinality of the intersection of a cycle of $\varphi$ and one of $\alpha$ is either 1 or $p$. If it is 1, then by Lemma II.4 we have

$$\sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) \equiv 0 \pmod{p^2}.$$

If it is $p$, then $\Omega/\varphi$ has only one element. Let $\varphi = (b_1, b_2, \dots, b_{p^2})$. The sum $\Sigma_{A \in \Omega/\varphi} D_\alpha \lambda(A)$ equals $D_\alpha \lambda(b_1)$ and we find

$$D_\alpha \lambda(b_1) = \lambda(\alpha(b_1)) - \lambda(b_1).$$

But as $\varphi$ and $\alpha$ commute and $\varphi$ is a cycle, $\alpha$ is a power of $\varphi$ and $\alpha = \varphi^{ip}$, $0 \le i \le p - 1$. Thus as $\lambda$ is orthogonal to $\varphi$, $\lambda(\alpha(b_1)) = \lambda(\varphi^{ip}(b_1)) = \lambda(b_1) + ipu$, so that $D_\alpha \lambda(b_1) = ipu$, as required.

We are now able to prove our main theorem.

THEOREM 2. *Let $p$ and $q$ be two distinct primes $\alpha$ and $\beta$ be two permutations such that*

(1) *$\alpha\beta$ is a cycle,*

(2) $\alpha^q = \beta^p = 1$.

*Then the automorphism group of $(\alpha, \beta)$ is either trivial or one of $C_p$, $C_q$, $C_{pq}$.*

*Proof.* It is clear that $\mathrm{Aut}\langle \alpha, \beta \rangle$ is cyclic.

Let now $\varphi$ be an automorphism of prime order, clearly $\varphi$ fixes one cell of the hypermap $(\alpha, \beta)$: the unique cycle of $\alpha\beta$. By Theorem 1 it fixes one more cell, if this cell is of length one then $\varphi$ is the identity, if it is of length $p$ or $q$ then clearly $\varphi$ has orbits of length dividing $p$ or $q$ and $\varphi$ is of order $p$, $q$ or 1. This proves that $\mathrm{Aut}\langle \alpha, \beta \rangle$ is of order $p^u q^v$. To obtain the complete result we will show that assuming the existence of an automorphism of order $m = p^2$ (or $m = q^2$ similarly) we have a contradiction. Let $\varphi$ be such an automorphism, let $\lambda$ be the coloring constructed in Lemma II.3 for $\alpha\beta$, we have

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) \equiv z(\alpha\beta) \quad (\equiv 1) \pmod{p^2}.$$

But by Lemma II.6:

$$1 \equiv \sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) \equiv \sum_{A \in \Omega/\varphi} D_{\alpha}\lambda(A) + \sum_{A \in \Omega/\varphi} D_{\beta}\lambda(A) \pmod{p^2}$$

and

$$\sum_{A \in \Omega/\varphi} D_{\beta}\lambda(A) \equiv 0 \pmod{p^2}$$

as the cardinality $r$ of the intersection of a cycle of $\varphi$ and one of $\beta$ is 0 or 1 ($r$ dividing $p^2$ and $q$).

We thus have using Lemma III.1 and multiplying by $p$ the above equality:

$$p \equiv p \sum_{A \in \Omega/\varphi} D_{\alpha}\lambda(A) \equiv 0 \pmod{p^2}.$$

Which is the contradiction we are looking for.

**Acknowledgement.** *The authors wish to thank R. C. Lyndon for many helpful comments on a preliminary version of this work.*

## REFERENCES

[1] J. L. Brenner and R. C. Lyndon, *Permutations and cubic graphs*, Pacific J. Math., **104** (1983), 285–315.
[2] R. Cori, *Un code pour les graphes planaires et ses applications*, Astérisque, Société Mathématique de France, **27** (1975).
[3] R. Cori, A. Machí, J. G. Penaud and B. Vauquelin, *On the automorphism group of a planar hypermap*, European J. Combin., **2** (1981), 331–334.
[4] J. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices Amer. Math. Soc., **7** (1960), 646.
[5] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag (1980).
[6] A. Jacques, *Sur le genre d'une paire de substitutions*, C. R. Acad. Sci. Paris, série A.B, **267** (1968), 625–627.
[7] V. A. Liskovets, *A census of non isomorphic planar maps*, Colloquia Matematica Societatis János Bolyai (1–25), (1978)-Algebraic method in graph theory, p. 479–494.
[8] T. R. S. Walsh, *Hypermap versus bipartite maps*, J. Combinatorial Theory-Ser. B, **18** (1975), 155–163.

Received December 9, 1981 and in revised form May 24, 1982.

UNIVERSITY OF BORDEAUX I
351, COURS DE LA LIBERATION
33405 TALENCE CEDEX, FRANCE