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AN ALGORITHM IN A COMPLEX FIELD AND ITS APPLICATION TO THE CALCULATION OF UNITS

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In the present paper the author develops a new periodic algorithm for certain algebraic number fields and calculates units in these fields from its periodicity. This algorithm (abbr. ACF) is a modification of the Jacobi-Perron Algorithm, where for the first time the expanding of the complex (not only real) numbers was possible. The ACF discloses new units of which the already known ones are special cases, and it has the advantage that many results in the theory of units can be derived by means of a unified periodic algorithm.

0. Introduction. Hilbert's dream and advice (Zahlbericht) of tailoring a universal algorithm by means of which the expansion of any real algebraic number (or any vector in E^{n-1}, $n \geq 2$) becomes periodic, thus enabling a complete system of fundamental units in the corresponding field to be calculated, still is, and probably will remain, far away from realization. Though C.G.J. Jacobi invented an algorithm for totally real cubic fields, generalizing ingeniously the Euclidean algorithm, he could not prove or disprove the periodicity of his new tool nor calculate units by its means. Jacobi's algorithm was later generalized by Perron [**11**] for any total real algebraic field, but even he, the master of continued fractions, did not succeed in proving or disproving the periodicity of his new algorithm.

In his last scientific work Perron [**12**], though not successfully, shows his efforts to solve the periodicity question of his algorithm. A partial answer to this question in the case of cubic irrationalities gave E. Dubois and R. Paysant-Le-Roux [**6**].

The title algorithm (abbr. ACF) can be interpreted as a modification of Jacobi-Perron Algorithm (abbr. JPA), where for the first time the employment of the complex (not only real) numbers was possible. The choice of $D_i - s$ to be complex numbers implies also a slight modification of the T-function in the companion vector.

This new algorithm ACF solves the problem of periodicity for infinitely many classes of algebraic numbers (but not for all of them) of any degree, and states explicitly some units in the corresponding fields. The periodicity of real algebraic numbers of degree $n \geq 3$ was first proved by Bernstein [**3**] for the JPA, and later for its modification [**3**, **1**].

The ACF discloses new units of which the already known ones by Bernstein [**3**], Halter-Koch [**7**], Halter-Koch and Stender [**8**] are special cases. The latter were obtained by the last two authors without an algorithm. This new algorithm ACF has the advantage that many results in the theory of units can be derived by means of a unified algorithm. The new units were also discovered by Neubrand [**9**], again without using an algorithm. Neubrand's method is quite different from the other methods, namely algebraic geometric or function theoretic oriented.

1. The ACF-definition, basic properties. We denote:

$$S_{n+1} = \{d, D_1, D_2, \ldots, D_n\}, \qquad n \geq 2, \tag{1.1}$$

a set of any $n + 1$ algebraic numbers;

$$\mathcal{K}_1 = Q(d, D_1, D_2, \ldots, D_n) \tag{1.2}$$

the field generated by adjoining S_{n+1} to Q;

$$\begin{cases} G(x) = \sum_{i=0}^{n} c_i x^{n-i}; \quad c_0 = 1;\ c_i \in \mathcal{K}_1,\ i = 1, \ldots, n \\ \text{an irreducible polynomial in } x \text{ over } \mathcal{K}_1; \end{cases} \tag{1.3}$$

$$\begin{cases} \mathcal{K}_2 = \mathcal{K}_1(w); \quad G(w) = 0 \\ \text{the field generated by a root of } G(x) \text{ in} \\ \text{adjunction with } \mathcal{K}_1. \end{cases} \tag{1.4}$$

Thus $\mathcal{K}_2 = Q(w, d, D_1, D_2, \ldots, D_n)$ is of degree n over $\mathcal{K}_1$ and of degree $n \cdot [\mathcal{K}_1 : Q]$ over Q.

DEFINITION 1. A vector $a^{(0)}$ in $\mathcal{K}_2^{n-1}$ $(n \geq 2)$ with components which are functions of w is called a starting or fixed vector; the vectors $a^{(v)} \in \mathcal{K}_2^{n-1}$ obtained by an algorithm from $a^{(0)}$,

$$a^{(v)} = \left(a_1^{(v)}(w), a_2^{(v)}(w), \ldots, a_{n-1}^{(v)}(w)\right), \qquad v = 0, 1, \ldots \tag{1.5}$$

are called the current vectors; the vectors

$$\begin{cases} b^{(v)} = \left(b_1^{(v)}, b_2^{(v)}, \ldots, b_{n-1}^{(v)}\right) \in \mathcal{K}_1^{n-1}, \quad \text{derived from} \\ a^{(v)} \text{ or given by any formula are called the} \\ \text{companion vectors of the } a^{(v)}. \end{cases} \tag{1.6}$$

□

We shall introduce the notation

$$a_0^{(v)}, b_0^{(v)} = 1, \qquad v = 0, 1, \ldots. \tag{1.7}$$

DEFINITION 2. The ACF of a starting vector $a^{(0)}$ is defined by the recurrence formula

$$(1.8)\qquad a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1}\left(a_2^{(v)} - b_2^{(v)},\ldots,a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1\right)$$
$$a_1^{(v)} - b_1^{(v)} \neq 0; \qquad v = 0, 1,\ldots. \qquad \square$$

From (1.8) we obtain

$$(1.9)\qquad a_i^{(v)} = b_i^{(v)} + \frac{a_{i-1}^{(v+1)}}{a_{n-1}^{(v+1)}}; \qquad i = 1,\ldots,n-1;\ v = 0, 1,\ldots.$$

We define the matrix entries

$$(1.10)\qquad \begin{cases} A_i^{(j)} = \delta_j^i; \qquad i,j = 0, 1,\ldots,n-1,\ \delta_j^i \text{ the Kronecker delta;} \\ A_i^{(v+n)} = \sum_{j=0}^{n-1} b_j^{(v)} A_i^{(v+j)}, \qquad i = 0, 1,\ldots,n-1;\ v = 0, 1,\ldots. \end{cases}$$

From (1.10) we obtain easily by induction

$$(1.11)\qquad \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}$$

for $v = 0, 1,\ldots.$

We obtain formula

$$(1.12)\qquad a_i^{(0)} = \frac{\sum_{j=0}^{n-1} a_j^{(v)} A_i^{(v+j)}}{\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}}, \qquad i = 1,\ldots,n-1;\ v = 0, 1,\ldots;$$

(1.12) is correct for $v = 0$, giving on the right side,

$$\left(0 + 0 + \cdots + a_i^{(0)} A_i^{(i)} + 0 + \cdots + 0\right)/A_0^{(0)} = a_i^{(0)}.$$

Then substitute inductively on the right the values of $a_j^{(v)}$ from (1.10), (1.9).

We shall need the formula

$$(1.13)\qquad \prod_{j=1}^{v} a_{n-1}^{(j)} = \sum_{i=0}^{n-1} a_i^{(v)} A_0^{(v+i)}, \qquad v = 1, 2,\ldots.$$

Proof by induction. (1.13) is correct for $v = 1$. Then proceed as in the proof of (1.12).

We shall need the basic formula

$$(1.14)\qquad \begin{cases} \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} \\ \qquad = (-1)^{v(n-1)}\left(\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}\right)^{-1}. \end{cases}$$

Proof. Start with (1.11). Multiply both sides by $\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}$, add to the first column the $a_j^{(v)}$ multiple of the $j+1$st column. The result is obtained by induction.

2. A periodic ACF—notations. In this section we begin preparation for the periodicity of the most general $a^{(0)} \in \mathcal{K}_2$. This $a^{(0)}$ will be specified later when we approach the central theorem. The choice of $a^{(0)}$ will result from specification of the numbers d, D_1, $D_2, \ldots, D_n \in S_{n+1}$ and the function $G(x)$ from (1.3). We first choose

$$(2.1)\qquad G_F(x) = -d + \prod_{j=1}^{n} (x - D_j), \qquad n \geq 2,$$

where $G_F(x)$ is irreducible over $\mathcal{K}_1$. This polynomial was treated by Stender in his dissertation, if the D_i are natural numbers.

$$(2.2)\qquad \begin{cases} G_F(w_i) = 0,\ i = 1, \ldots, n;\ \text{Let } w_1, w_2, \ldots, w_n \text{ be the roots} \\ \text{of } G_F(x) \text{ is an algebraic closure of } \mathcal{K}_2(w_1, \ldots, w_n). \\ w \in \{w_1, w_2, \ldots, w_n\}. \end{cases}$$

Thus w is chosen to be one fixed root of the n-roots of G_F. We introduce the notation

$$(2.3)\qquad \begin{cases} f_{i,i} = f_i = w - D_i, & i = 1, 2, \ldots, n; \\ f_{i,k} = \prod_{j=i}^{k} (w - D_j); & 1 \leq i \leq k \leq n; \end{cases}$$

In $\mathcal{K}_2$ we have of course

$$(2.4)\qquad f_{i,k} = f_{i,k}(w), \qquad 1 \leq i \leq k \leq n$$

and need not add $f_{i,k} = f(w, D_i, \ldots, D_k)$. We have, from (2.1)–(2.3)

$$\prod_{j=1}^{n} (w - D_j) = f_{1,n} = f_{1,n}(w) = d. \tag{2.5}$$

The following operations will be useful

$$\begin{cases} \dfrac{1}{f_{i,k}(w)} = \dfrac{1}{f_{i,k}} = \dfrac{f_{1,i-1} f_{k+1,n}}{d}; & i \geq 2, k \leq n-1, \\ \dfrac{1}{f_{1,k}} = \dfrac{f_{k+1,n}}{d}, \quad k \leq n-1; \quad \dfrac{1}{f_{1,n}} = \dfrac{f_{1,i-1}}{d}, & 2 \leq i \leq n. \end{cases} \tag{2.6}$$

For the fixed vector $a^{(0)}$ we now choose

$$a^{(0)} = (f_{1,n-1}, f_{1,n-2}, \ldots, f_{1,2}, f_{2,2}). \tag{2.7}$$

We shall in the sequel conduct the ACF of $a^{(0)}$ from (2.7), getting the current vectors $a^{(v)}$, $v = 1, 2, \ldots$. The companion vectors of the current vectors are derived from the current vectors by the formula, remembering (2.4);

$$\begin{cases} b_s^{(v)} = a_s^{(v)}(D_1); & s = 1, \ldots, n-1; \\ b^{(v)} \in \mathcal{K}_1^{n-1}, & v = 0, 1, \ldots. \end{cases} \tag{2.8}$$

The choice of $D_1 \in S_{n+1}$ for the derivation of $b^{(v)}$ from $a^{(v)}$ is, as we shall later see, completely arbitrary. Any $D_i \in S_{n+1}$, $i = 1, \ldots, n$, would do.

The reader should pay priority-attention to the formula

$$\begin{cases} b_s^{(0)} = f_{1,n-s}(D_1) = [(w - D_1) \cdots (w - D_{n-s})]_{w=D_1} = 0, \\ s = 1, \ldots, n-1. \end{cases} \tag{2.9}$$

We shall illustrate the first step in the ACF of $a^{(0)}$, working out all the necessary details of (1.8). In the sequel the current vectors' result will sometimes be enumerated directly without going into the details. We obtain from (2.7), in view of (2.9)

$$\begin{cases} b_i^{(0)} = f_{1,n-i}(D_1) = 0, & i = 1, 2, \ldots, n-2; \\ b_{n-1}^{(0)} = f_{2,2}(D_1) = f_2(D_1) = D_1 - D_2. \end{cases} \tag{2.10}$$

Thus

$$b^{(0)} = (0, 0, \ldots, 0, D_1 - D_2).$$

$$a_1^{(0)} - b_1^{(0)} = f_{1,n-1} - 0 = f_{1,n-1},$$

and by (2.6)

$$\left(a_1^{(0)} - b_1^{(0)}\right)^{-1} = \frac{1}{f_{1,n-1}} = \frac{f_{n,n}}{d} = \frac{f_n}{d},$$

$$a_{n-1}^{(0)} - b_{n-1}^{(0)} = f_{2,2} - (D_1 - D_2) = w - D_2 - (D_1 - D_2) = w - D_1 = f_1,$$

$$a_i^{(0)} - b_i^{(0)} = a_i^{(0)} - 0 = a_i = f_{1,n-1}, \qquad i = 2,\dots,n-2$$

and from (1.8);

$$a^{(1)} = d^{-1}f_n(f_{1,n-2}, f_{1,n-3},\dots,f_{1,2}, f_1, 1), \tag{2.11}$$

$$a^{(1)} = \left(d^{-1}f_{1,n-2}f_n, d^{-1}f_{1,n-3}f_n,\dots,d^{-1}f_1f_n, d^{-1}f_n\right).$$

3. The first fugue of the ACF of $a^{(0)}$.

DEFINITION 3. A sequence of $n-1$ current vectors (including the fixed vector), viz.

$$a^{((n-1)v+j)}, \qquad v = 0,1,\dots; j = 0,1,\dots,n-2, \tag{3.1}$$

is called the $v+1$st fugue (of the current vector) of the ACF of $a^{(0)}$; the sequence of the corresponding companion vectors–the $v+1$st fugue of the companion vectors of the ACF of $a^{(0)}$, is

$$b^{(n-1)v+j}, \qquad v = 0,1,\dots; j = 0,1,\dots,n-2. \tag{3.2}$$

□

From (2.11) we obtain

$$b^{(1)} = \left(0,0,\dots,0, d^{-1}(D_1 - D_n)\right), \tag{3.3}$$

$$a_1^{(1)} - b_1^{(1)} = d^{-1}f_{1,n-2}f_n - 0; \qquad \left(a_1^{(1)} - b_1^{(1)}\right)^{-1} = f_{n-1},$$

$$a_{n-1}^{(1)} - b_{n-1}^{(1)} = d^{-1}f_n - d^{-1}(D_1 - D_n) = d^{-1}f_1,$$

$$a^{(2)} = f_{n-1}\left(d^{-1}f_{1,n-3}f_n,\dots,d^{-1}f_{1,2}f_n, d^{-1}f_1f_n, d^{-1}f_1, 1\right),$$

$$a^{(2)} = \left(d^{-1}f_{1,n-3}f_{n-1,n},\dots,d^{-1}f_{1,2}f_{n-1,n}, d^{-1}f_1f_{n-1,n}, d^{-1}f_1f_{n-1}, f_{n-1}\right). \tag{3.4}$$

We can now prove the important

LEMMA 1. *The $i+1$st current vector of the first fugue has the form*

$$\begin{cases} a^{(i)} = \left(d^{-1}f_{1,n-i-1}f_{n-i+1,n}, d^{-1}f_{1,n-i-2}f_{n-i+1,n},\dots, \right. \\ \qquad d^{-1}f_1f_{n-i+1,n}, d^{-1}f_1f_{n-i+1,n-1}, f_1f_{n-i+1,n-2},\dots, \\ \qquad\qquad \left. f_1f_{n-i+1}, f_{n-i+1}\right) \\ i = 2,\dots,n-2;\ n \geq 4. \end{cases} \tag{3.5}$$

Proof. (3.5) is correct for $i = 2$, as can be verified from (3.4). By the method applied in calculating $a^{(1)}$, $a^{(2)}$, the reader will now have no difficulty in proving Lemma 1 by induction. The special cases $n = 2, 3$ will be observed separately because of their importance.

$$\begin{aligned} a^{(0)} &= w - D_2; & b^{(0)} &= D_1 - D_2 \\ a^{(1)} &= d^{-1}(w - D_2); & b^{(1)} &= d^{-1}(D_1 - D_2) \\ a^{(2)} &= w - D_2 = a^{(0)}. \end{aligned}$$

Here we have a purely periodic continued fraction representation.

$$(w - D_1)(w - D_2) - d = 0; \quad w - D_2 = [\overline{D_1 - D_2}], \quad d = 1.$$

$$w - D_2 = [\overline{D_1 - D_2, d^{-1}(D_1 - D_2)}]; \quad d > 1; \quad d \mid D_1 - D_2.$$

For $n = 3$, we obtain, $d \neq 1$.

$$\begin{aligned} a^{(0)} &= ((w - D_1)(w - D_2), w - D_2); & b^{(0)} &= (0, D_1 - D_2); \\ a^{(1)} &= (d^{-1}(w - D_1)(w - D_3), d^{-1}(w - D_3)); & b^{(1)} &= (0, d^{-1}(D_1 - D_3)); \\ a^{(2)} &= (d^{-1}(w - D_1)(w - D_2), w - D_2); & b^{(2)} &= (0, D_1 - D_2); \\ a^{(3)} &= ((w - D_1)(w - D_3), w - D_3); & b^{(3)} &= (0, D_1 - D_3); \\ a^{(4)} &= (d^{-1}(w - D_1)(w - D_2), d^{-1}(w - D_2)); & b^{(4)} &= (0, d^{-1}(D_1 - D_2)); \\ a^{(5)} &= (d^{-1}(w - D_1)(w - D_3), w - D_3); & b^{(5)} &= (0, D_1 - D_3); \\ a^{(6)} &= ((w - D_1)(w - D_2), w - D_2) = a^{(0)}. \end{aligned}$$

The ACF of $a^{(0)}$ is purely periodic, and the length of its primitive period is $m = 6$.

For $d = 1$ we obtain

$$\begin{aligned} a^{(0)} &= ((w - D_1)(w - D_2), w - D_2); \quad b^{(0)} = (0, D_1 - D_2); \\ a^{(1)} &= ((w - D_1)(w - D_3), w - D_3); \quad b^{(1)} = (0, D_1 - D_3). \\ a^{(2)} &= a^{(0)}; \quad (w - D_1)(w - D_2)(w - D_3) - d = 0, \quad \text{generally.} \end{aligned}$$

As a numerical example in the case $n = 2$;

$$(w - 4)(w - 2) - 2 = 0; \quad \text{choose } w = 3 + \sqrt{3}; \quad D_1 = 4;$$

$$D_2 = 2; \quad d = 2. \quad \mathcal{K}_1 = Q; \quad \mathcal{K}_2 = Q(\sqrt{3});$$

$$a^{(0)} = w - 2 = [\overline{4 - 2, 1/2(4 - 2)}]$$

$$\sqrt{3} + 3 - 2 = [2, 1]; \quad \sqrt{3} = [1, \overline{1, 2}],$$

a very simple way of constructing the periodic continued fraction of $\sqrt{3}$.

4. The first fugue of the ACF of $a^{(0)}$, continued. From (2.7), (2.11) and (3.5) we have completed the calculations of the first fugue of the ACF of $a^{(0)}$, viz.

$$\left\langle a^{(0)}, a^{(1)}, \ldots, a^{(i)}, \ldots, a^{(n-2)} \right\rangle, \qquad i = 2, \ldots, n-2$$

altogether the fixed vector $a^{(0)}$, followed by $n-2$ current vectors. We obtain for the $n-1$st vector, substituting $i = n-2$ in (3.5),

$$a^{(n-2)} = \left(d^{-1} f_{1,1} f_{3,n},\, d^{-1} f_1 f_{3,n-1},\, f_1 f_{3,n-2}, \ldots, f_1 f_3,\, f_3\right). \tag{4.1}$$

From (4.1) we obtain, calculating (in detail because of its essential pattern) the next current vector $a^{(n-1)}$. For this purpose we have

$$b^{(n-2)} = (0, 0, 0, \ldots, 0,\, D_1 - D_3) \tag{4.2}$$

and from (4.1), (4.2)

$$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = d^{-1} f_{1,1} f_{3,n}; \quad \left(a_1^{(n-2)} - b_1^{(n-2)}\right)^{-1} = f_2; \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D_3 - (D_1 - D_3) = w - D_1 = f_1. \end{cases} \tag{4.3}$$

From (4.1)–(4.3) we obtain, by virtue of Definition 1,

$$\begin{aligned} a^{(n-1)} &= f_2\left(d^{-1} f_1 f_{3,n-1},\, f_1 f_{3,n-2}, \ldots, f_1 f_3,\, f_1, 1\right) \\ a^{n-1} &= \left(d^{-1} f_{1,n-1},\, f_{1,n-2}, \ldots, f_{1,2},\, f_2\right). \end{aligned} \tag{4.4}$$

(4.4) is important. If we compare the latter with (2.7) we obtain

$$a^{(0)} \overset{(d)}{=} a^{(n-1)} \tag{4.5}$$

which means to say that $a^{(0)} = a^{(n-1)}$ in the case $d = 1$. We have thus obtained the interesting

THEOREM 1. *The* ACF *of the fixed vector* $a^{(0)}$ *from* (2.7), *with the notations* (2.3) *and* (2.5), *is purely periodic in the case* $d = 1$, *and the length of the primitive period equals* $m = n - 1$.

The latter has the form

$$\left\langle a^{(0)}, a^{(1)}, \ldots, a^{(i)}, \ldots, a^{(n-2)} \right\rangle, \qquad i = 2, \ldots, n-2.$$

$a^{(0)}$ *from* (2.7), $a^{(1)}$ *from* (2.11), $a^{(i)}$ *from* (3.5) ($i = 2, \ldots, n-2$), *substituting in these formulas* $d = 1$. *The corresponding companion vectors have the form*:

$$\begin{cases} b^{(0)} = (0, 0, \ldots, 0,\, D_1 - D_2), \\ b^{(1)} = (0, 0, \ldots, 0,\, D_1 - D_n),, \ldots, \\ b^{(i)} = (0, 0, \ldots, 0,\, D_1 - D_{n-i+1}), \qquad i = 2, \ldots, n-2. \end{cases} \tag{4.6}$$

□

The reader is reminded that while $d = 1$ in Theorem 1, the other n elements $D_1, \ldots, D_n$ of the set S_{n+1} may still be any algebraic numbers. It is surprising that in the case $d = 1$ the ACF of the fixed vector $a^{(0)}$ from (2.7) indeed is purely periodic with the astonishingly short primitive period $m = n - 1$. In the case $d \neq 1$, the ACF of the same $a^{(0)}$ is also purely periodic, but the length of its primitive period equals $m = n(n-1)$. This will be proved in the next sections. But already, we have learned from the previous section, that in the case $n = 2$ the ACF of $a^{(0)}$ from (2.7) has primitive period of length $m = 2 = 2(2-1)$ when $d \neq 1$, and $m = 1 = 2 - 1$ when $d = 1$; and in the case $n = 3$, $m = 6 = 3(3-2)$ when $d \neq 1$, and $m = 2 = 3 - 1$ when $d = 1$.

5. Periodicity of the ACF of $a^{(0)}$—completed. We shall use a new manner of writing the vectors of the primitive period of the ACF of $a^{(0)}$ from Theorem 1 (with $d = 1$) and introduce

$$g^{(i)} = \left(g_1^{(i)}, g_2^{(i)}, \ldots, g_{n-1}^{(i)}\right), \qquad i = 0, 1, \ldots, n-2;\ n \geq 4. \tag{5.1}$$

It will be useful, for later purposes, to write out in full the values of $g_j^{(i)}$ from (2.7), (2.11), (3.5) where $d = 1$,

$$\begin{cases} g^{(0)} = \left(g_1^{(0)}, g_2^{(0)}, \ldots, g_{n-1}^{(0)}\right) = \left(f_{1,n-1}, f_{1,n-2}, \ldots, f_{1,2}, f_2\right); \\ g^{(1)} = \left(g_1^{(1)}, g_2^{(1)}, \ldots, g_{n-1}^{(1)}\right) \\ \qquad = \left(f_{1,n-2}f_n, f_{1,n-3}f_n, \ldots, f_{1,2}f_n, f_1 f_n, f_n\right); \\ g^{(i)} = \left(g_1^{(i)}, g_2^{(i)}, \ldots, g_{n-1}^{(i)}\right) \\ \qquad = \left(f_{1,n-i-1}f_{n-i+1,n}, f_{1,n-i-2}f_{n-i+1,n}, \ldots, \right. \\ \qquad\qquad \left. f_{1,2}f_{n-i+1,n}, f_1 f_{n-i+1,n}, f_1 f_{n-i+1,n-1}, \ldots, \right. \\ \qquad\qquad\qquad\qquad \left. f_1 f_{n-i+1}, f_{n-i+1}\right), \\ i = 2, 3, \ldots, n-2. \end{cases} \tag{5.2}$$

With the notation (5.2) we shall now write out in full a few of the fugues of the ACF of $a^{(0)}$ from (2.7) with $d \neq 1$.

$$\begin{cases} a^{(0)} = \left(g_1^{(0)}, g_2^{(0)}, \ldots, g_{n-1}^{(0)}\right), \\ a^{(1)} = \left(d^{-1}g_1^{(1)}, d^{-1}g_2^{(1)}, \ldots, d^{-1}g_{n-1}^{(1)}\right), \\ a^{(2)} = \left(d^{-1}g_1^{(2)}, d^{-1}g_2^{(2)}, \ldots, d^{-1}g_{n-1}^{(2)}\right), \\ \vdots \\ a^{(n-2)} = \left(d^{-1}g_1^{(n-2)}, d^{-1}g_2^{(n-2)}, g_3^{(n-2)}, \ldots, g_{n-1}^{(n-2)}\right). \end{cases} \tag{5.3}$$

(5.3) is the first fugue.

$$(5.4)\quad \begin{cases} a^{(1(n-1)+0)} = \left(d^{-1}g_1^{(0)}, g_2^{(0)}, g_3^{(0)},\ldots,g_{n-1}^{(0)}\right), \\ a^{(1(n-1)+1)} = \left(g_1^{(1)}, g_2^{(1)},\ldots,g_{n-1}^{(1)}\right), \\ a^{(1(n-1)+2)} = \left(d^{-1}g_1^{(2)}, d^{-1}g_2^{(2)},\ldots,d^{-1}g_{n-1}^{(2)}\right), \\ a^{(1(n-1)+3)} = \left(d^{-1}g_1^{(3)}, d^{-1}g_2^{(3)},\ldots,d^{-1}g_{n-2}^{(3)}, g_{n-1}^{(3)}\right), \\ \vdots \\ a^{(1(n-1)+n-2)} = \left(d^{-1}g_1^{(n-2)}, d^{-1}g_2^{(n-2)}, d^{-1}g_3^{(n-2)}, g_4^{(n-2)},\ldots,g_{n-1}^{(n-2)}\right). \end{cases}$$

(5.4) is the second fugue.

$$(5.5)\quad \begin{cases} a^{(2(n-1)+0)} = \left(d^{-1}g_1^{(0)}, d^{-1}g_2^{(0)}, g_3^{(0)},\ldots,g_{n-1}^{(0)}\right), \\ a^{(2(n-1)+1)} = \left(d^{-1}g_1^{(1)}, g_2^{(1)},\ldots,g_{n-1}^{(1)}\right), \\ a^{(2(n-1)+2)} = \left(g_1^{(2)}, g_2^{(2)},\ldots,g_{n-1}^{(2)}\right), \\ a^{(2(n-1)+3)} = \left(d^{-1}g_1^{(3)}, d^{-1}g_2^{(3)},\ldots,d^{-1}g_{n-1}^{(3)}\right), \\ a^{(2(n-1)+4)} = \left(d^{-1}g_1^{(4)}, d^{-1}g_2^{(4)},\ldots,d^{-1}g_{n-2}^{(4)}, g_{n-1}^{(4)}\right), \\ \vdots \\ a^{(2(n-1)+n-2)} = \left(d^{-1}g_1^{(n-2)},\ldots,d^{-1}g_4^{(n-2)}, g_5^{(n-2)},\ldots,g_{n-1}^{(n-2)}\right). \end{cases}$$

(5.5) is the third fugue.

The reader will now have no difficulty in proving

LEMMA 2. *The factors d^{-1} appearing before the $g_i^{(j)}$ in the current vectors of the* ACF *of $a^{(0)}$ from* (2.7) *with $d \neq 1$ appear subsequently one after the other starting with the first component of the vector; their frequency of occurrence in a vector is k, $0 \leq k \leq n-1$; if in a vector the frequency of d^{-1} is k, then in the next vector it is $k-1$; if its frequency in one vector is zero, then in the next vector, it is $n-1$.* □

For the jth vector of the $i+1$st fugue we shall use the notation

$$(5.6)\qquad a^{(i(n-1)+j)} = (k)g^{(j)}$$

where (k) denotes the frequency of the factor d^{-1} in this vector $g^{(j)}$. Of course, (5.6) does not say how to state explicitly any such vector $a^{(i(n-1)+j)}$. For this purpose a functional relation between i, j and k is necessary and most important. This is:

$$(5.7)\qquad \begin{cases} i - j \equiv k(n), & k = 0, 1,\ldots,n-1; \\ \text{For } k < 0, \text{ take } n-k. & j = 0, 1,\ldots,n-2. \end{cases}$$

The proof of (5.7) is entirely based on Lemma 2.

If in a certain fugue we have the vector $a^{i(n-i)+j}$, then in the next vector we have $j+1$, and since k decreases by one in this next vector, it becomes $k-1$, so that $i-(j+1) \equiv k-1(n)$ or $i-j \equiv k(n)$, as should be. If we look for the next fugue with the same row-vector j, viz. $a^{(i+1)(n+1)+j}$, then the frequency of k for the same j is one greater, namely $k+1$, so that we have $(i+1)-j \equiv k+1(n)$ or again, $i-j \equiv k(n)$. This proves (5.7) by induction, since it is correct for $i=j=k=0$.

We are now able to write down any current vector in the ACF of $a^{(0)}$. For example, let $n=9$, $m=9\cdot 8=72$, and we want to find $a^{(61)} = a^{7(9-1)+5}$; $i=7, j=5$; $7-2=2=k$. Hence

$$a^{(61)} = \left(d^{-1}g_1^{(5)}, d^{-1}g_2^{(5)}, g_3^{(5)}, \ldots, g_8^{(5)}\right).$$

We now ask the decisive question: Can ACF of $a^{(0)}$ become purely periodic, and what is the length m of the primitive period? (Every periodic ACF can be transformed into a purely periodic ACF.) In such a case we must have

$$j=k=0, \quad \text{as in } a^{(0)}. \tag{5.8}$$

(5.7), (5.8) result in chooseing min. i,

$$i \equiv 0\,(n), \qquad i=n. \tag{5.9}$$

We have obtained:

THEOREM 2. *The* ACF *of the fixed vector* $a^{(0)}$ *from* (2.7) *with the notation* (2.3), (2.5) *and* $d \neq 1$ *is purely periodic and the length of its primitive period is* $m = n(n-1)$, *consisting of* n *fugues.* □

The length of the primitive period of the ACF is indeed very large but, as it will be shown, it does not prevent us from calculating units.

6. Irreducibility and roots of polynomials. Bernstein [**3**, pp. 72–79] has proved the following result which we state here with some slight alterations:

THEOREM 3. *Let*

$$\begin{cases} P(x) = x^n + \left(\sum_{j=1}^{n-1} k_j x^{n-j}\right) - d, \\ d, k_1, \ldots, k_{n-1} \in \mathbf{Z}; \quad d \mid k_j \qquad (j=1,\ldots,n-1), \\ |k_{n-1}| \geq 2|d|(2+B); \quad B = 1 + \sum_{j=1}^{n-2} |k_j|. \end{cases} \tag{6.1}$$

Then $P(x)$ *is irreducible and has at least one real root.* □

If d is square free and $|d| > 1$, then irreducibility of $P(x)$ follows from Eisenstein's Criterion. But this would then exclude $|d| = 1$. Bernstein's theorem is valid also for any algebraic integers d, $k_1, \ldots, k_{n-1}$, $d \mid k_{n-1}$. This we could well use, but it would take us too long to prove it.

We introduce a polynomial which will be central to our investigation on units, viz.

$$(6.2)\qquad \begin{cases} T(x) = -d + \prod_{i=1}^{k} (x^{s_i} - D_i^{s_i}), \\ d, D_i \in \mathbf{Z}; \quad d \mid D_i; \qquad i = 1, 2, \ldots, k; \\ s_i \geq 1; \quad \text{if all } s_i = 1, \quad \text{then } k \geq 2; \\ 0 < D_1 < D_2 < \cdots < D_k. \end{cases}$$

We prove

LEMMA 3. *$T(x)$ is irreducible in infinitely many cases and has, in these cases, at least one real root.* □

We rearrange (6.2):

$$(6.3)\qquad \begin{cases} T(y) = -d + \prod_{i=1}^{k} \left[(y + D_k)^{s_i} - D_i^{s_i}\right], \\ y = x - D_k. \end{cases}$$

From (6.3)

$$(6.4)\qquad \begin{cases} T(y) = -d + \left(y^{s_k} + \binom{s_k}{1} D_k y^{s_k - 1} + \cdots + \binom{s_k}{s_k - 1} D_k^{s_k - 1} y\right) \\ \cdot \prod_{i=1}^{k-1} \left[(y + D_k)^{s_i} - D_i^{s_i}\right] \\ = y^n + t_1 y^{n-1} + \cdots + t_{n-1} y - d, \\ d \mid t_i; \quad t_{n-1} = \binom{s_k}{s_k - 1} D_k^{s_k - 1} \prod_{i=1}^{k-1} (D_k^{s_i} - D_i^{s_i}), \\ n = s_1 + s_2 + \cdots + s_k; \qquad i = 1, \ldots, n-1 \quad \text{in } t_i, \end{cases}$$

where $d \mid t_i$ since all t_i are polynomials in $D_1, D_2, \ldots, D_k$ with rational coefficients; further t_{n-1} contains the highest power of D_k, viz. $s_k - 1 + s_1 + s_2 + \cdots + s_{k-1} = n - 1$. Hence we can choose sufficiently large D_k so that $|t_{n-1}| \geq 2|d|(B_1 + 2)$, $B_1 = 1 + \sum_{j=1}^{n-2} |t_j|$. Thus the polynomial $T(y)$, hence also $T(x)$, satisfies the conditions of the polynomial $P(x)$,

and Theorem 3, so that $T(x)$ is irreducible and has at least one real root in infinitely many cases.

In case $s_k = n$, $T(x)$ becomes

$$T(x) = x^n - D^n - d, d \mid D, n \geq 2; \qquad D \geq 1.$$

$$T(y) = y^n + \sum_{j=1}^{n-1} \binom{n}{j} D^j y^{n-j} \Big) - d; \qquad d, D \in \mathbf{Z},$$

and we have to choose

$$\binom{n}{1} D^{n-1} \geq 2|d| \left[3 + \binom{n}{1} D + \binom{n}{2} D^2 + \cdots + \binom{n}{n-2} D^{n-2}\right].$$

It suffices to choose $D \geq 2^{n+1} \mid d \mid$; $n \geq 2$. Concluding, we shall write $T(x)$ in the form of (2.1).

$$(6.5) \qquad \begin{cases} G_T(x) = -d + \prod_{i=1}^{k} \prod_{j=1}^{s_i} \left(x - \rho_j^{j-1} D_j\right) \\ \rho_j = \exp \dfrac{2\pi\sqrt{-1}}{s_i}. \quad \left(\text{We write } \sqrt{-1} \text{ to avoid} \right. \\ \left. \text{confusion with the index } i.\right) \end{cases}$$

$$(6.6) \qquad \begin{cases} \{D_1, \rho_1 D_1, \ldots, \rho_1^{s_1-1} D_1, D_2, \rho_2 D_2, \ldots, \rho_2^{s_2-1} D_2, \\ \qquad\qquad ,\ldots, D_k, \rho_k D_k, \ldots, \rho_k^{s_k-1} D_k\} \\ = \{\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n\}, \quad \text{for useful notation.} \end{cases}$$

Let

$$(6.7) \qquad \begin{cases} (s_1, s_2, \ldots, s_k) = s; \quad \bar{\rho} = \exp \dfrac{2\pi\sqrt{-1}}{s}. \\ G_T(\overline{w}) = 0; \quad \overline{w} \text{ real}; \quad Q(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_k, \overline{w}) = Q(\bar{\rho}, w). \end{cases}$$

$$(6.8) \qquad G_T(x) = -d + \prod_{j=1}^{n} \left(x - \overline{D}_j\right); \quad -d + \prod_{j=1}^{n} \left(\overline{w} - \overline{D}_j\right) = 0.$$

7. Units. the main result.

DEFINITION 4. Let

$$(7.1) \qquad \begin{cases} \mathcal{K} \text{ be an algebraic number field;} \\ a^{(0)} = \left(a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)}\right) \in \mathcal{K}^{n-1}, n \geq 2, \quad \text{and} \\ a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)} \quad \text{algebraic integers.} \end{cases}$$

Then $a^{(0)}$ is called an integral vector.

THEOREM 4. *Let*

$$(7.2)\qquad \begin{cases} \mathcal{K}_1, \mathcal{K}_2 \text{ be algebraic number fields;} \\ \mathcal{K}_1 \subset \mathcal{K}_2; \quad [\mathcal{K}_2 : \mathcal{K}_1] = g \geq n \geq 2; \\ a^{(0)} \in \mathcal{K}_2 \quad \text{an integral vector;} \\ \langle b^{(v)} \rangle_{v=0}^{\infty} \quad \text{a sequence of integral vectors} \in \mathcal{K}_1; \\ A_i^{(v)} \quad \text{the matrix entries from } (1.9). \end{cases}$$

If the ACF *of* $a^{(0)}$, *carried out with the companion vectors* $b^{(v)}$ *is purely periodic with* $m =$ *the length of the primitive period then the algebraic number*

$$(7.3)\qquad \prod_{i=1}^{m} a_{n-1}^{(i)} = A_0^{(m)} + a_1^{(0)} A_0^{(m+1)} + \cdots + a_{n-1}^{(0)} A_0^{(m+n-1)}$$

is a unit in $\mathcal{K}_2$ (*relative to* $\mathcal{K}_1$). □

Proof. Since, because of periodicity, we have $a_i^{(m)} = a_i^{(0)}$, $i = 1, \ldots, n-1$, we obtain (7.3) from (1.13). The $a_i^{(0)}$ are algebraic integers by hypothesis, so are the $A_0^{(m+j)}$, $j = 0, 1, \ldots, n-1$, since the $b^{(v)}$ are integral by hypothesis. Thus $\sum_{j=0}^{n-1} a_j^{(0)} A_0^{(m+j)}$ in (7.3) is an algebraic integer.

We further note that, by virtue of the conditions of the theorem, all entries in (1.14) are algebraic integers, so that $(\sum_{j=0}^{n-1} a_j^{(0)} A_0^{(m+j)})^{-1}$ is an integer. This proves the theorem.

We now return to the vector $a^{(0)}$ from (2.7), substituting there $\bar{a}^{(0)}$ for $a^{(0)}$, $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n$ for $D_1, D_2, \ldots, D_n$ from (6.6), and now let d have the value from (6.2), viz. d a rational integer. Theorem 2 then becomes: The ACF of the fixed vector $\bar{a}^{(0)}$ with $d \neq 1$ is purely periodic and the length of its primitive period $m = n(n-1)$, consisting of n fugues. From (5.3)–(5.7) we learn a few important facts about the ACF of $\bar{a}^{(0)}$:

(i) for D_1 we substitute $\overline{D}_1$; altogether there are n such choices of substitution;

(ii) the numbers $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n$ are all algebraic integers, since $\overline{D}_j = \rho_j^t D_j$, $(t = 0, 1, \ldots, s_j - 1)$, and ρ_j and D_j are integers;

(iii) $d \mid \overline{D}_j$ $(j = 1, \ldots, n-1)$, since $D \mid D_j$;

(iv) the product of the last components of $n-1$ of the n fugues of the primitive period of ACF has the form:

$$(7.4)\qquad d^{-1} \prod_{j=2}^{n} \left(\bar{w} - \overline{D}_j\right);$$

(v) One fugue out of the n fugues does not have among its last components the factor d^{-1}. This fugue is that one whose first vector has $k = n - 2$, so that

$$i - 0 \equiv k - 2(n); \qquad i = n - 2.$$

Thus the product of the $n - 1$st $(i + 1)$ fugue equals (with $i = n - 2$)

$$\prod_{j=2}^{n} \left(\bar{w} - \bar{D}_j\right). \tag{7.5}$$

(vi) All the entries of the companion vectors of the n fugues are either zero or $\bar{D}_1 - \bar{D}_j$ or $d^{-1}(\bar{D}_1 - \bar{D}_j)$, $j = 2,\ldots,n$, $(j \neq n - 1)$. Hence they are all integers (and so are the $A_i^{(v)}$).

(vii) Because of (7.4), (7.5), we have

$$\prod_{j=0}^{n(n-1)-1} \bar{a}_{n-1}^{(j)} = d^{-(n-1)}\left(\prod_{j=2}^{n} \left(\bar{w} - \bar{D}_j\right)\right)^n. \tag{7.6}$$

But because of periodicity $a_{n-1}^{n(n-1)} = \bar{a}_{n-1}^{(0)}$, hence from (7.6)

$$\prod_{j=0}^{n(n-1)-1} \bar{a}_{n-1}^{(j)} = \prod_{j=1}^{n(n-1)} \bar{a}_{n-1}^{(j)}, \tag{7.7}$$

so that, by virtue of (7.6), (7.7) and Theorem 4, we have obtained the second main result in

THEOREM 5. *With the meanings of $\bar{w}$ and $\bar{\rho}$, $\bar{D}_j$, as explained before,*

$$e = d^{-(n-1)}\left(\prod_{j=2}^{n} \left(\bar{w} - \bar{D}_j\right)\right)^n \tag{7.8}$$

is a unit in $Q(\bar{w}, \bar{\rho})$. □

8. Units from Theorem 5. The unit e from (7.8) has a complicated form unsuitable to practicalities. We shall find simpler forms for units in $Q(\bar{w}, \bar{\rho})$, and also disclose units in $Q(\bar{w})$. Since we took $\bar{w} = w_0$ to be a real root of $G_T(x)$, we could have taken any of the roots $w_0, w_1,\ldots,w_{n-1}$ of $G_T(x)$ (some, even all of which may be real) and still have obtained the unit $d^{-(n-1)}(\prod_{j=2}^{n}(\bar{w}_t - \bar{D}_j))^n$, $t = 0, 1,\ldots,n - 1$. Also, since $\bar{D}_1$ may assume any of the values

$$\bar{D}_1 = \rho_i^u D_i, \qquad i = 1,\ldots,n; \quad u = 0, \quad s_1,\ldots,s_i - 1, \tag{8.1}$$

we can obtain altogether n units of the form (7.8) for $\bar{w}$ constant, $\bar{w} = w_0$, and altogether n^2 units in $Q(w_t, \bar{\rho})$ among them some units in $Q(\bar{w})$, $\bar{w}$

real, as we shall soon see. From (6.8) we obtain

$$\prod_{j=2}^{n} \left(\overline{w} - \overline{D}_j\right) = \frac{d}{\overline{w} - \overline{D}_1}. \tag{8.2}$$

Hence from (7.8), (8.2)

$$e = \frac{d}{\left(\overline{w} - \overline{D}_1\right)^n},$$

$$\begin{cases} e_1 = \dfrac{\left(\overline{w} - \overline{D}_1\right)^n}{d}, \\ \text{is a unit in } Q\left(\overline{w}, \overline{D}_1\right). \end{cases} \tag{8.3}$$

Since in (8.3) we can use any of the $\overline{D}_j$, $(j = 1,\ldots,n)$ instead of $\overline{D}_1$, we obtain

THEOREM 6. *All the numbers*

$$e_j = \frac{\left(\overline{w} - \overline{D}_j\right)^n}{d}, \qquad j = 1,\ldots,n, \tag{8.4}$$

are units in $Q(\overline{w}, \overline{\rho})$. *Among them*

$$e^{(i)} = \frac{\left(\overline{w} - D_i\right)^n}{d}, \qquad i = 1,\ldots,k; \quad k \textit{ from } (6.2) \tag{8.5}$$

are units in the real field $Q(\overline{w})$. □

The product

$$\prod_{u=0}^{s_i-1} \left(\frac{\left(\overline{w} - \rho_i^u D_i\right)^n}{d} \right) = \frac{\left(\overline{w}^{s_i} - D_i^{s_i}\right)^n}{d^{s_i}}$$

is a unit as a product of units. Hence

COROLLARY 1. *The numbers*

$$e^{(s_i)} = \frac{\left(\overline{w}^{s_i} - D_i^{s_i}\right)^n}{d^{s_i}}, \qquad i = 1,2,\ldots,k; \; k \textit{ from } (6.2), \; s_i \geq 1 \tag{8.6}$$

are units in the real field $Q(\overline{w})$. □

From (8.5) we obtain

$$\left(e^{(i)}\right)^{-s_i} = \frac{d^{s_i}}{\left[\left(\overline{w} - D_i\right)^{s_i}\right]^n},$$

so that

$$(e^{(i)})^{-s_i} e^{(s_i)} = \left(\frac{\overline{w}^{s_i} - D_i^{s_i}}{(\overline{w} - D_i)^{s_i}} \right)^n,$$

is a unit, hence

COROLLARY 2. *The numbers*

$$e_{ii} = \frac{\overline{w}^{s_i} - D_i^{s_i}}{(\overline{w} - D_i)^{s_i}}, \qquad i = 1,\ldots,k; \quad k \textit{ from } (6.2),\ s_i > 1 \tag{8.7}$$

are units in the real field $Q(\overline{w})$. □

Let

$$\begin{cases} n = s_i t_i, \quad s_i, t_i > 1, \\ \text{for some } i \leq k. \end{cases} \tag{8.8}$$

Substituting n from (8.8) in (8.6), we obtain

$$e^{(s_i)} = \frac{\left[(\overline{w}^{s_i} - D_i^{s_i})^{t_i} \right]^{s_i}}{d^{s_i}},$$

hence

COROLLARY 3. *Let* $n = s_i t_i$, s_i, $t_i > 1$; $s_i \in \{s_1, s_2, \ldots, s_k\}$. *Then*

$$e^{(t_i)} = \frac{(\overline{w}^{s_i} - D_i^{s_i})^{t_i}}{d} \tag{8.9}$$

are units in the real field $Q(\overline{w})$. □

Let $|d| = 1$. Then we have

COROLLARY 4. *The numbers*

$$\begin{cases} e_{j,1} = \overline{w} - \overline{D}_j, \qquad u = 1,\ldots,n \textit{ are units in } Q(\overline{w}, \overline{\rho}). \\ e_1^{(i)} = \overline{w} - D_i, \qquad i = 1,\ldots,k \textit{ are units in the real field} \\ Q(\overline{w}),\ k \textit{ from } (6.2) \\ e_1^{(s_i)} = \overline{w}^{s_i} - D_i^{s_i}, \qquad i = 1,\ldots,k;\ s_i > 1;\ k \textit{ from } (6.2) \\ \textit{are units in the real field } Q(\overline{w}). \end{cases} \tag{8.10}$$

9. Derivation of previously known units. We return to the polynomial $T(x)$ of (6.2). In the case $k = 1$, $T(x)$ takes the form, $s_1 = n$,

$$(9.1)\quad \begin{cases} T(x) = x^n - D^n - d; & d, D \in \mathbf{Z};\ D > 0;\ n \geq 2;\ d \mid D; \\ T(\bar{w}) = 0; & \bar{w} \text{ real};\ \bar{w}^n = D^n + d;\ \bar{w} \text{ of degree } n \text{ over } Q. \end{cases}$$

But even in this case the method leading to the periodic ACF of $\bar{a}^{(0)}(w)$, and the derivation of units from it is the same. We obtain from (9.1)

$$(9.2)\quad \begin{cases} -d + \prod_{j=1}^{n} \left(\bar{w} - \bar{D}_j\right) = 0. \\ \bar{D}_j = \rho^{j-1} D; \qquad j = 1,\ldots,n;\ \bar{D}_1 = D; \\ \rho = \exp \dfrac{2\pi\sqrt{-1}}{n}; \qquad d \mid \bar{D}_j \quad \text{since } d \mid D. \end{cases}$$

With the conditions of (9.2) those of Theorems 4 and 5 are satisfied, and we have

THEOREM 7. *Let conditions* (9.1) *and* (9.2) *hold. Then the n numbers*

$$(9.3)\qquad e_j = \frac{\left(\bar{w} - \rho^{j-1} D\right)^n}{d}; \qquad j = 1,\ldots,n$$

are units in $Q(\bar{w}, \rho)$. *The real number*

$$(9.4)\qquad e_1 = \frac{\left(\bar{w} - D\right)^n}{d}$$

is a unit in the real field $Q(\bar{w})$.

If $|d| = 1$

$$(9.5)\qquad e_{11} = \bar{w} - D \quad \text{is a unit in } Q(\bar{w}). \qquad \square$$

Let $st = n$, $s > 1$. We obtain from (9.1)

$$(9.6)\qquad (\bar{w}^s)^t = (D^s)^t + d.$$

Since $d \mid D$, we also have $d \mid D^s$, and all that was said in Theorem 7 about $\bar{w}$, D.

$$(9.7)\qquad e_1^{(s)} = \frac{\left(\bar{w}^s - D^s\right)^t}{d}$$

is a unit in $Q(\overline{w})$. From (9.4), (9.6) we obtain

$$e_1^{(s)}e_1^{-1} = \frac{(\overline{w}^s - D^s)^t}{d} \cdot \frac{d}{\left[(\overline{w} - D)^s\right]^t} = \left(\frac{\overline{w}^s - D^s}{(\overline{w} - D)^s}\right)^t,$$

a unit in $Q(w)$. Hence

COROLLARY 5. *The* $\tau(n) - 1$ *numbers*

$$e_1^{(s,t)} = \frac{\overline{w}^s - D^s}{(\overline{w} - D)^s}, \quad st = n, s > 1, \tag{9.8}$$

are units in $Q(\overline{w})$. □

If $n = p^v$, p prime, $v = 1, 2, \ldots$, then instead of d, we have pd which makes $d \mid D$ valid.

Bernstein and Hasse [**5**] obtained the units (9.4), (9.8) with the additional restrictions $D \geq (n-2)d$ for $d > 0$ and $D \geq 2d(n-1)$ for $d < 0$; $n \geq 2$. These were the restrictions Bernstein [**1**, **2**] needed for the periodicity of the Jacobi-Perron Algorithm with $w = \sqrt[n]{D^n \pm d}$, d, $D > 0$, $d \mid D$.

In his most remarkable Ph.D. thesis Stender [**13**] proved that in the cases $n = 3, 4, 6$ the possible 1, 2, 3 units from (9.4), (9.8) form a system of fundamental units in the respective fields. This was a breakthrough, but did not prove in general that the units gained from (periodic) Jacobi-Perron Algorithms are always fundamental.

The units are independent, whether they are fundamental or not, has not been proved for the general case. This fact is also true for the units obtained from ACF.

Later, Halter-Koch and Stender [**8**] proved that formulas (9.4), (9.8) hold for a much wider class of units.

Another breakthrough in units of this kind came with an important paper by Halter-Koch [**7**] who started with the polynomial

$$G_T(x) = \prod_{i=1}^{r_1} (x - D_i) \prod_{j=r_1+1}^{r_1+r_2} (x - D_j)(x - \overline{D}_j) - d,$$

$r_1 \geq 0, \quad r_2 \geq 0, \quad n = r_1 + 2r_2, \quad d, D_i \in Z;$

$D_1 > D_2 > \cdots > D_{r_1}; \quad D_j, \overline{D}_j$ complex conjugate,

d a divisor of pairs of D_i and the D_i,

D_i and D_j, D_j and $\overline{D}_j$.

Halter-Koch proves that $G_T(x)$ has exactly r_1 real roots. If w is one of them then

$$d^{-1}(w - D_i)^n \quad \text{and} \quad d^{-2}\left[(w - D_j)(w - \bar{D}_j)\right]^n$$

are units in $Q(w)$. Of course these units can also be obtained from our ACF of $\bar{a}^{(0)}$. Bernstein [4] used his so-called zero algorithm to gain these Halter-Koch units.

The zero algorithm (abbr. ZA) is also a modification of JPA, and it is different from ACF.

The author conjectures that many more known units can be calculated by ACF. This will be discussed in a further paper.

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